On the Randers Spaces of Second Order

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Abstract

The geometry of Randers spaces of order one has been investigated since fifties by G. Randers [9], using Riemannian techniques, only. A Finsler approach of this Randers space was given by R. Ingarden [4]. A fully Finslerian treatment was developed by R. Miron in [6,7]. In this paper we define and study the geometrical theory of a Randers space of order two, using the theory of the bundle of accelerations of order two, T^2M.

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1 Introduction

The first examples of regular Lagrangians of order greater then one were given by R.Miron. These are the prolongations of order k, $(k > 1)$ of a Riemannian, or a Finslerian metric. Using the prolongation of order two of a Riemannian metric γ_{ij} and a globally defined covector field A_i on the base manifold, we define the fundamental function F of a Randers space $RF^{(2)} = (M, \alpha + \beta)$. Then we determine the fundamental tensor g_{ij} of $RF^{(2)}$ and the relation between d-tensors γ_{ij} and g_{ij}, that is similar with the k=1 case. As this Randers space is a Finsler space of order two, we use techniques of Finslerian geometries of higher order to determine the canonical nonlinear connection, the canonical metrical $N-$connection. We may notice here that all the geometrical objects depend on the electromagnetic tensor F_{ij} of the space $RF^{(2)}$.

2 The notion of Randers space of order two

Let M be an n-dimensional, smooth manifold and $\mathcal{R}^n = (M, \gamma_{ij}(x))$ be a Riemannian space. We denoted by $Prol^2\mathcal{R}^n = (\widetilde{T^2M, G})$ the prolongation of order two of the space $\mathcal{R}^n$, [8]. The nonlinear connection $\overset{\circ}{N}$ of the space $Prol^2\mathcal{R}^n$, [8], has the dual coefficients

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$$(1.1)\qquad \begin{cases} \underset{(1)}{\overset{\circ}{M}}{}^i_j = \gamma^i_{jh}(x) y^{(1)h}, \\ \underset{(2)}{\overset{\circ}{M}}{}^i_j = \dfrac{1}{2}\left(\Gamma \underset{(1)}{\overset{\circ}{M}}{}^i_j + \underset{(1)}{\overset{\circ}{M}}{}^i_m \underset{(1)}{\overset{\circ}{M}}{}^m_j\right), \end{cases}$$

where Γ is the operator

$$(1.2)\qquad \Gamma = y^{(1)i}\frac{\partial}{\partial x^i} + 2y^{(2)i}\frac{\partial}{\partial y^{(1)i}}$$

and $\gamma^i_{jk}(x)$ are the Christoffel symbols of the Riemannian space $\mathcal{R}^n$. Between the coefficients of the nonlinear connection $\underset{(1)}{\overset{\circ}{N}}{}^i_j, \underset{(2)}{\overset{\circ}{N}}{}^i_j$ and the dual coefficients $\underset{(1)}{\overset{\circ}{M}}{}^i_j, \underset{(2)}{\overset{\circ}{M}}{}^i_j$ we have the relations:

$$(1.3)\qquad \begin{cases} \underset{(1)}{\overset{\circ}{N}}{}^i_j = \underset{(1)}{\overset{\circ}{M}}{}^i_j, \\ \underset{(2)}{\overset{\circ}{N}}{}^i_j = \underset{(2)}{\overset{\circ}{M}}{}^i_j - \underset{(1)}{\overset{\circ}{M}}{}^i_m \underset{(1)}{\overset{\circ}{M}}{}^m_j . \end{cases}$$

Having the coefficients of the nonlinear connection $\overset{\circ}{N}$ we can write its adapted basis, $\left\{\dfrac{\overset{\circ}{\delta}}{\delta x^i}, \dfrac{\overset{\circ}{\delta}}{\delta y^{(1)i}}, \dfrac{\overset{\circ}{\delta}}{\delta y^{(2)i}}\right\}$, given by

$$(1.4)\qquad \begin{cases} \dfrac{\overset{\circ}{\delta}}{\delta x^i} = \dfrac{\partial}{\partial x^i} - \underset{(1)}{\overset{\circ}{N}}{}^j_i \dfrac{\partial}{\partial y^{(1)j}} - \underset{(2)}{\overset{\circ}{N}}{}^j_i \dfrac{\partial}{\partial y^{(2)j}}, \\ \dfrac{\overset{\circ}{\delta}}{\delta y^{(1)i}} = \dfrac{\partial}{\partial y^{(1)i}} - \underset{(1)}{\overset{\circ}{N}}{}^j_i \dfrac{\partial}{\partial y^{(2)j}} \\ \dfrac{\overset{\circ}{\delta}}{\delta y^{(2)i}} = \dfrac{\partial}{\partial y^{(2)i}}. \end{cases}$$

The dual basis of the previous adapted basis is given by $\{dx^i, \overset{\circ}{\delta}\, y^{(1)i}, \overset{\circ}{\delta}\, y^{(2)i}\}$, where:

$$\begin{cases} \overset{\circ}{\delta}\, y^{(1)i} = dy^{(1)i} + \underset{(1)}{\overset{\circ}{M}}{}^i_j\, dx^j, \\ \overset{\circ}{\delta}\, y^{(2)i} = dy^{(2)i} + \underset{(1)}{\overset{\circ}{M}}{}^i_j\, dy^{(1)j} + \underset{(2)}{\overset{\circ}{M}}{}^i_j\, dx^j. \end{cases}$$

We consider the Liouville d-vector fields, [8]

$$(1.5)\qquad \begin{aligned} z^{(1)m} &= y^{(1)m}, \\ z^{(2)m} &= y^{(2)m} + \frac{1}{2}\gamma^m_{ij}(x) y^{(1)i} y^{(1)j}. \end{aligned}$$

The following theorem is known

Theorem 2.1 *The function* $\alpha^2 = \gamma_{ij}(x) z^{(2)i} z^{(2)j}$ *is a differential Lagrangian. It has the properties:*

1) α^2 *is global defined on* $\widetilde{T^2M} = T^2M \setminus \{0\}$*;*
2) α^2 *is a regular Lagrangian;*
3) α^2 *depends on the metric* γ_{ij}*, only;*
4) The Lagrangian α^2 *is homogeneous of order* 4 *on the fibres of* T^2M*;*
5) The fundamental tensor field of α^2 *is given by*

$$\frac{1}{2}\frac{\partial^2 \alpha^2}{\partial y^{(2)i} \partial y^{(2)j}} = \gamma_{ij}(x). \tag{1.6}$$

Let us consider the functions

$$\beta(x, y^{(1)}, y^{(2)}) = A_i(x) z^{(2)i} \tag{1.7}$$

where $A_i(x)$ are the electromagnetic potentials on the base manifold M. Clearly, the function β has a physical means.

Let us consider the function $F : T^2M \to \mathbb{R}$ which is given by $F = \alpha + \beta$, i.e.

$$F(x, y^{(1)}, y^{(2)}) = \sqrt{\gamma_{ij}(x) z^{(2)i} z^{(2)j}} + A_i(x) z^{(2)i} \tag{1.8}$$

and the square of this function

$$L(x, y^{(1)}, y^{(2)}) = (\alpha + \beta)^2. \tag{1.8$'$}$$

We can formulate

Theorem 2.2 *a) The function* $L = (\alpha + \beta)^2$ *is a differentiable Lagrangian of order 2;*

b) F *is* 2*-homogeneous and* L *is* 4*-homogeneous on the fibers of* T^2M*;*
c) The fundamental tensor field of the Lagrangian $L = (\alpha + \beta)^2$ *is given by*

$$g_{ij} = (p\gamma_{ij} + l_i l_j) - p \overset{\circ}{l}_i \overset{\circ}{l}_j \tag{1.9}$$

where $\overset{\circ}{l}_i = \dfrac{\partial \alpha}{\partial y^{(2)i}},\ l_i = \overset{\circ}{l}_i + A_i,\ p = \dfrac{\alpha + \beta}{\alpha}.$

Proof. *a)* L is of C^∞-class on $\widetilde{T^2M}$ and continuous on the null section of the projection $\pi : T^2M \longrightarrow M$ because α and β have these properties.

b) It is known that $z^{(2)i}$ is 2-homogeneous of the fibres of $\widetilde{T^2M}$.

c) By a straightforward calculus we obtain the formula (1.9). □

Theorem 2.3 *The pair* $F^{(2)n} = (M, F = \alpha + \beta)$ *is a Finsler space of order 2.*

Proof. We must show the following properties:

1. F is of C^∞-class on $\widetilde{T^2M}$ and continuous on the null section;
2. F is positive on an open set, where $\beta \geq 0$;
3. F is 2-homogeneous on the fibres of T^2M;
4. The Hessian of F^2 with the elements:

$$g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^{(2)i}\partial y^{(2)j}}$$

is positively defined.

Since the properties 1,2,3 holds in virtue of the properties of α and β we have to prove the property 4. In order to do it, we calculate the contravariant g^{ij} of the fundamental tensor field of the Lagrangian L.

The contravariant g^{ij} of g_{ij} is given by:

$$g^{ij} = \frac{1}{p}\gamma^{ij} - \frac{1}{p^2}[\overset{\circ}{l^i}\overset{\circ}{l^j}(1-\widetilde{l^2}) + \overset{\circ}{l^i}A^j + \overset{\circ}{l^j}A^i], \tag{1.10}$$

where $\widetilde{l^2} = \frac{1}{p}\gamma^{ij}l_il_j$.

Moreover, $det||g_{ij}|| = p^{n+1}det||\gamma_{ij}||$ where $p = \dfrac{\alpha+\beta}{\beta} > 0$. □

The Finsler space $RF^{(2)n} = (M, F)$ is called *the Randers space of order two.*

Theorem 2.4 *The nonlinear connection* $\overset{\circ}{N}$ *of the space* $Prol^2\mathcal{R}^n$*, from (1.1), is a nonlinear connection for the Randers space of order 2* $RF^{(2)n}$ *determined only by the fundamental function* $F = \alpha + \beta$.

3 Nonlinear connection of the space $RF^{(2)n}$

Let us consider the mixed form $F^i_j(x) = \gamma^{im}(x)F_{mj}(x)$ of the electromagnetic tensor field $F_{mj}(x) = \dfrac{\partial A_j}{\partial x^m} - \dfrac{\partial A_m}{\partial x^j}$.

Theorem 3.1 *The Randers space* $RF^{(2)n} = (M, \alpha+\beta)$ *has a nonlinear connection* N*, whose dual coefficients are given by*

(2.1)

$$\begin{cases} \underset{(1)}{M^i_j} = \underset{(1)}{\overset{\circ}{M^i_j}} - F^i_j||y^{(1)}||, \\ \underset{(2)}{M^i_j} = \underset{(2)}{\overset{\circ}{M^i_j}} - \frac{1}{2}\Gamma(F^i_j||y^{(1)}||) + \frac{1}{2}(\underset{(1)}{\overset{\circ}{M^i_m}}F^m_j + \underset{(1)}{\overset{\circ}{M^m_j}}F^i_m)||y^{(1)}|| - \frac{1}{2}||y^{(1)}||^2F^i_mF^m_j. \end{cases}$$

where $||y^{(1)}|| = \sqrt{\gamma_{ij}(x)y^{(1)i}y^{(1)j}}$.

Indeed, $\underset{(1)}{M^i_j}$ has the same rule of transformation with respect to changes of coordinates on $\widetilde{T^2M}$ like $\underset{(1)}{\overset{\circ}{M^i_j}}$ and $\underset{(2)}{M^i_j} = \frac{1}{2}(\Gamma\underset{(1)}{M^i_j} + \underset{(1)}{M^i_m}\underset{(1)}{M^m_j})$.

The nonlinear connection (2.1) will be called *canonical nonlinear connection* of the Randers space $RF^{(2)n}$.

The relations between coefficients $(\underset{(1)}{N^i_j}, \underset{(2)}{N^i_j})$ and dual coefficients of the nonlinear connection N are given by:

$$(2.2)\qquad \begin{cases} \underset{(1)}{N^i_j} = \underset{(1)}{M^i_j}, \\ \underset{(2)}{N^i_j} = \frac{1}{2}(\Gamma\ \underset{(1)}{M^i_j} + \underset{(1)}{M^i_m}\underset{(1)}{M^m_j}) - \underset{(1)}{M^i_m}\underset{(1)}{M^m_j} = \frac{1}{2}(\Gamma\ \underset{(1)}{N^i_j} - \underset{(1)}{N^i_m}\underset{(1)}{N^m_j}). \end{cases}$$

We use this canonical nonlinear connection in study of the canonical metrical N-connection.

4 The canonical N-linear connection

Let us consider d-tensor field

$$(3.1)\quad T^j_i = \Gamma(F^j_i\|y^{(1)}\|) + F^i_j\|y^{(1)}\| - (\underset{(1)}{\overset{\circ}{M}{}^i_m}\ F^m_j + \underset{(1)}{\overset{\circ}{M}{}^m_j}\ F^i_m)\|y^{(1)}\| + \|y^{(1)}\|^2 F^i_m F^m_j$$

The adapted basis $\left\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \frac{\partial}{\partial y^{(2)i}}\right\}$ of the canonical nonlinear connection can be expressed in the following form

$$(3.2)\qquad \begin{cases} \frac{\delta}{\delta x^i} = \frac{\overset{\circ}{\delta}}{\delta x^i} + F^j_i\|y^{(1)}\|\frac{\overset{\circ}{\delta}}{\delta y^{(1)j}} + \frac{1}{2}T^j_i\frac{\partial}{\partial y^{(2)j}}, \\ \frac{\delta}{\delta y^{(1)i}} = \frac{\overset{\circ}{\delta}}{\delta y^{(1)i}} + F^j_i\|y^{(1)}\|\frac{\partial}{\partial y^{(2)j}} \\ \frac{\delta}{\delta y^{(2)i}} = \frac{\partial}{\partial y^{(2)i}}. \end{cases}$$

Of course, this adapted basis depend only on the fundamental function $F = \alpha + \beta$ of the Randers spaces of order two, $RF^{(2)n}$.

Using a very known method we can determine the canonical N-linear connection D on $\widetilde{T^2M}$ metrical with respect to fundamental tensor g_{ij}, which depend only on the fundamental function F of the space $RF^{(2)n}$.

We have:

Theorem 4.1 1) *There exists an unique N-linear connection D on $\widetilde{T^2M}$ which verifies the following axioms:*

1° *The nonlinear connection N is specified by (2.2).*

2° $g_{ij|k} = 0,\quad g_{ij\overset{(1)}{|}k} = 0,\quad g_{ij\overset{(2)}{|}k} = 0,$

3° $T^i_{jk} = \overset{(1)}{S}{}^i_{jk} = \overset{(2)}{S}{}^i_{jk} = 0.$

2) *This connection has the coefficients given by the generalized Christoffel symbols:*

$$(3.3)\qquad \begin{cases} L^i_{jm} &= \dfrac{1}{2} g^{is}\left\{\dfrac{\delta g_{sm}}{\delta x^j} + \dfrac{\delta g_{sj}}{\delta x^m} - \dfrac{\delta g_{jm}}{\delta x^s}\right\}, \\ \underset{(1)}{C}{}^i_{jm} &= \dfrac{1}{2} g^{is}\left\{\dfrac{\delta g_{sm}}{\delta y^{(1)j}} + \dfrac{\delta g_{sj}}{\delta y^{(1)m}} - \dfrac{\delta g_{jm}}{\delta y^{(1)s}}\right\}, \\ \underset{(2)}{C}{}^i_{jm} &= \dfrac{1}{2} g^{is}\left\{\dfrac{\partial g_{sm}}{\partial y^{(2)j}} + \dfrac{\partial g_{sj}}{\partial y^{(2)m}} - \dfrac{\partial g_{jm}}{\partial y^{(2)s}}\right\}. \end{cases}$$

3) *This connection depend only on Randers space* $RF^{(2)n}$.

The metrical connection from the previous Theorem is the canonical connection of the Randers space $RF^{(2)n} = (M, \alpha + \beta)$.

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