Integral inequalities for maximal space-like submanifolds in the indefinite space form

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Abstract

In this note, we give two intrinsic integral inequalities for compact maximal space-like submanifolds in the indefinite space form and a sufficent and necessary condition for such submanifolds to be totally geodesic.

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1 Introduction

Let $M_p^{n+p}(c)$ be an $(n+p)$-dimensional connected semi-Riemannian manifold of constant curvature c whose index is p. It is called an indefinite space form of index p and simply a space form when $p = 0$. If $c > 0$, we call it as a de Sitter space of index p. Let M^n be an n-dimensional Riemannian manifold immersed in $M_p^{n+p}(c)$. As the semi-Riemannian metric of $M_p^{n+p}(c)$ induces the Riemannian metric of M^n, M^n is called a space-like submanifold. A space-like submanifold with vanishing mean curvature is called a maximal space-like submanifold. Kobayashi [5] gave the Weierstrass-Enneper representation formulas for maximal space-like surfaces in 3-dimensional Minkowski space and exhibited various examples. In particular, he determined the maximal space-like surfaces which are rotation surfaces or ruled surfaces. Montiel [6] give an integral inequality for compact space-like hypersurfaces in the de Sitter space and by use of this integral inequality, he studied the constant mean curvature space-like hypersurfaces. Also, Akutagawa [1] and Ramanathan [8] investigated space-like hypersurfaces in a de Sitter space and proved independently that a complete space-like hypersurface in a de Sitter space with constant mean curvature is totally umbilical if the mean curvature H satisfies $H^2 \leq c$ when $n = 2$ and $n^2H^2 < 4(n-1)c$ when $n \geq 3$. Later, Cheng [3] generalized this result to general submanifolds in a de Sitter space.

In this paper, we study compact maximal space-like submanifolds in the indefinite space form with flat normal bundle and obtain two intrinsic integral inequalities for such submanifolds. We also give a sufficent and necessary condition for such submanifolds to be totally geodesic. We will prove the following

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Theorem 1. *Let M^n be an n-dimensional compact maximal space-like submanifold in $M_p^{n+p}(c)$ with flat normal bundle, then*

$$\int_{M^n}\left\{\frac{1}{2}\sum R_{mijk}^2+\sum R_{mj}^2-ncR\right\}*1\le 0.$$

Theorem 2. *Let M^n be an n-dimensional compact maximal space-like submanifold in $M_p^{n+p}(c)$ with flat normal bundle, then*

$$\int_{M^n}\left\{\frac{1}{2}\sum R_{mijk}^2+\frac{1}{n}S^2+(n-2)cS-n(n-1)c^2\right\}*1\le 0.$$

Theorem 3. *Let M^n be an n-dimensional compact maximal space-like submanifold in $M_p^{n+p}(c)$ with flat normal bundle, then M^n is totally geodesic if and only if*

$$\int_{M^n}\left\{\frac{1}{2}\sum R_{mijk}^2+(n-2)cS-n(n-1)c^2\right\}*1=0.$$

In the above theorems, $\sum R_{mijk}^2$ is the square length of the Riemannian curvature tensor, $\sum R_{mj}^2$ the square length of the Ricci curvature tensor, S the square length of the second fundamental form, R the scalar curvature. All these are intrinsic properties of M^n.

2 Preliminaries

Let $M_p^{n+p}(c)$ be an $(n+p)$-dimensional semi-Riemannian manifold of constant curvature c whose index is p. Let M^n be an n-dimensional Riemannian manifold immersed in $M_p^{n+p}(c)$. We choose a local field of semi-Riemannian orthonormal frames $e_1,\dots,e_{n+p}$ in $M_p^{n+p}(c)$ such that at each point of M^n, $e_1,\dots,e_n$ span the tangent space of M^n and form an orthonormal frame there. We use the following convention on the range of indices:

$$1\le A,B,C,\dots\le n+p;\quad 1\le i,j,k,\dots\le n;\quad n+1\le\alpha,\beta,\gamma\le n+p.$$

Let $\omega_1,\dots,\omega_{n+p}$ be its dual frame field so that the semi-Riemannian metric of $M_p^{n+p}(c)$ is given by $d\bar{s}^2=\sum_i\omega_i^2-\sum_\alpha\omega_\alpha^2=\sum_A\epsilon_A\omega_A^2$, where $\epsilon_i=1$ and $\epsilon_\alpha=-1$. Then the structure equations of $M_p^{n+p}(c)$ are given by

$$d\omega_A=\sum_B\epsilon_B\omega_{AB}\wedge\omega_B,\quad \omega_{AB}+\omega_{BA}=0,\tag{1}$$

$$d\omega_{AB}=\sum_C\epsilon_C\omega_{AC}\wedge\omega_{CB}-\frac{1}{2}\sum_{C,D}K_{ABCD}\omega_C\wedge\omega_D,\tag{2}$$

$$K_{ABCD}=c\,\epsilon_A\epsilon_B(\delta_{AC}\delta_{BD}-\delta_{AD}\delta_{BC}).\tag{3}$$

By restricting these forms to M^n, we have

$$\omega_\alpha = 0, \quad n+1 \le \alpha \le n+p, \tag{4}$$

the Riemannian metric of M^n is written as $ds^2 = \sum_i \omega_i^2$. From Cartan's lemma we can write

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha. \tag{5}$$

From these formulas, we obtain the structure equations of M^n:

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{6}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} K_{ijkl} \omega_k \wedge \omega_l, \tag{7}$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \tag{8}$$

where R_{ijkl} are the components of the curvature tensor of M^n.

For indefinite Riemannian manifolds in detail, refer to O'Neill [7].

We call

$$h = \sum_\alpha h_\alpha e_\alpha = \sum_{i,j,\alpha} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha \tag{9}$$

the second fundamental form of M^n and the square length of the second fundamental form is defined by

$$S = \sum_\alpha tr(h_\alpha)^2 = \sum_{\alpha,i,j} (h_{ij}^\alpha)^2. \tag{10}$$

The mean curvature vector N of M^n is defined by

$$N = \frac{1}{n} \sum_\alpha tr(h_\alpha) e_\alpha = \frac{1}{n} \sum_\alpha (\sum_i h_{ii}^\alpha) e_\alpha, \tag{11}$$

and it is well known that N is independent of the choice of unit normal vectors $e_{n+1}, \ldots, e_{n+p}$ to M^n. The length of the mean curvature vector is called the mean curvature of M^n, denoted by H.

If M^n is maximal, then

$$\sum_i h_{ii}^\alpha = 0, \quad \alpha = n+1, \ldots, n+p. \tag{12}$$

Define the first and the second covariant derivatives of $\{h_{ij}^\alpha\}$, say $\{h_{ijk}^\alpha\}$ and $\{h_{ijkl}^\alpha\}$ by

$$\sum_k h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha + \sum_k h_{kj}^\alpha \omega_{ki} + \sum_k h_{ik}^\alpha \omega_{kj} + \sum_\beta h_{ij}^\beta \omega_{\beta\alpha}, \tag{13}$$

$$\begin{aligned} \sum_l h_{ijkl}^\alpha \omega_l &= dh_{ijk}^\alpha + \sum_m h_{mjk}^\alpha \omega_{mi} + \sum_m h_{imk}^\alpha \omega_{mj} \\ &+ \sum_m h_{ijm}^\alpha \omega_{mk} + \sum_\beta h_{ijk}^\beta \omega_{\beta\alpha}. \end{aligned} \tag{14}$$

Then we have

$$h^{\alpha}_{ijk} = h^{\alpha}_{ikj}, \tag{15}$$

$$h^{\alpha}_{ijkl} - h^{\alpha}_{ijlk} = \sum_m h^{\alpha}_{im} R_{mjkl} + \sum_m h^{\alpha}_{jm} R_{mikl} + \sum_\beta h^{\beta}_{ij} R_{\alpha\beta kl}, \tag{16}$$

where $R_{\alpha\beta kl}$ are the components of the normal curvature tensor of M^n, that is

$$R_{\alpha\beta kl} = \sum_i (h^{\alpha}_{ik} h^{\beta}_{il} - h^{\alpha}_{il} h^{\beta}_{ik}). \tag{17}$$

If $R_{\alpha\beta kl} = 0$ at point x of M^n we say that the normal connection of M^n is flat at x and it is well known [2] that $R_{\alpha\beta kl} = 0$ at x if and only if h_α are simultaneously diagonalizable at x.

The Laplacian Δh^{α}_{ij} of the fundamental form h^{α}_{ij} is defined to be $\sum_k h^{\alpha}_{ijkk}$, and hence, if M^n has flat normal bundle, from (15) and (16) we have

$$\begin{aligned}\Delta h^{\alpha}_{ij} &= \sum_k (h^{\alpha}_{ijkk} - h^{\alpha}_{ikjk}) + \sum_k (h^{\alpha}_{ikjk} - h^{\alpha}_{ikkj}) + \sum_k (h^{\alpha}_{ikkj} - h^{\alpha}_{kkij}) \\ &= \sum_{m,k} h^{\alpha}_{im} R_{mkjk} + \sum_{m,k} h^{\alpha}_{mk} R_{mijk}\end{aligned} \tag{18}$$

3 Proofs of the Theorems

Proof of Theorem 1. *From (8), (12) and (18), we have*

$$\begin{aligned}\sum h^{\alpha}_{ij}\Delta h^{\alpha}_{ij} &= \sum h^{\alpha}_{ij} h^{\alpha}_{mk} R_{mijk} + \sum h^{\alpha}_{ij} h^{\alpha}_{im} R_{mkjk} \\ &= \frac{1}{2}\sum (h^{\alpha}_{ij} h^{\alpha}_{mk} - h^{\alpha}_{mj} h^{\alpha}_{ik}) R_{mijk} + \sum (h^{\alpha}_{ij} h^{\alpha}_{im} - h^{\alpha}_{ii} h^{\alpha}_{jm}) R_{mj} \\ &= \frac{1}{2}\sum [c(\delta_{ij}\delta_{mk} - \delta_{mj}\delta_{ik}) - R_{imjk}] R_{mijk} \\ &\quad + \sum [c(\delta_{ij}\delta_{im} - \delta_{ii}\delta_{jm}) + R_{ijim}] R_{mj} \\ &= \frac{1}{2}\sum R^2_{mijk} + \sum R^2_{mj} - ncR.\end{aligned} \tag{19}$$

Since $\int_{M^n} \{\sum h^{\alpha}_{ij}\Delta h^{\alpha}_{ij}\} * 1 \leq 0$, *we have*

$$\int_{M^n} \left\{\frac{1}{2}\sum R^2_{mijk} + \sum R^2_{mj} - ncR\right\} * 1 \leq 0. \tag{20}$$

Theorem 1 is proved.

In order to prove Theorem 2, we need the following algebraic lemma

Lemma. *Let* $a_1, \ldots, a_n$ *be real numbers, then*

$$\sum (a_i)^2 \geq \frac{1}{n}(\sum a_i)^2, \tag{21}$$

and the equality holds if and only if $a_1 = \cdots = a_n$.

In fact,

$$n\sum(a_i)^2-(\sum a_i)^2=\sum(a_i-a_j)^2, \tag{22}$$

then Lemma follows immediately from (22).

Proof of Theorem 2. From (8), we have

$$R_{mj}=(n-1)c\delta_{mj}+\sum h^\alpha_{mi}h^\alpha_{ij}, \tag{23}$$

$$R=n(n-1)c+S. \tag{24}$$

Since M^n has flat normal bundle, so we can diagonalize the second fundamental form simultaneously so that $h^\alpha_{ij}=\lambda^\alpha_i\delta_{ij}$, then from (21), we have

$$R_{mj}=(n-1)c\delta_{mj}+\sum(\lambda^\alpha_j)^2\delta_{mj}, \tag{25}$$

$$\begin{aligned}\sum R^2_{mj} &= n(n-1)^2c^2+2(n-1)cS+\sum(\lambda^\alpha_j)^4\\ &\geq n(n-1)^2c^2+2(n-1)cS+\frac{1}{n}\{\sum(\lambda^\alpha_j)^2\}^2\\ &= n(n-1)^2c^2+2(n-1)cS+\frac{1}{n}S^2,\end{aligned} \tag{26}$$

therefore from (20), we have

$$\int_{M^n}\left\{\frac{1}{2}\sum R^2_{mijk}+\frac{1}{n}S^2+(n-2)cS-n(n-1)c^2\right\}*1\leq 0. \tag{27}$$

Theorem 2 is proved.

Proof of Theorem 3. From (27), we have

$$\int_{M^n}\left\{\frac{1}{2}\sum R^2_{mijk}+(n-2)cS-n(n-1)c^2\right\}*1\leq 0. \tag{28}$$

If M^n is totally geodesic, i.e., $S=0$, $h^\alpha_{ij}=0$, then from (8), we have

$$R_{mijk}=c(\delta_{mj}\delta_{ik}-\delta_{mk}\delta_{ij}),\quad \sum R^2_{mijk}=2n(n-1)c^2, \tag{29}$$

in this case, (28) becomes an equality.

Inversely, if (28) becomes an equality, then $S=0$, M^n is totally geodesic.

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