Cooperative Games With a Simplicial Core

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Abstract

In this paper n-person cooperative games having the property that the core is a subsimplex of the imputation set are characterized. Also a characterization of games where the core is a subsimplex of the dual imputation set is given by using some duality relations for games. We also give a geometric characterization of games with a non–empty core, which follows easily from the well-known Bondareva–Shapley theorem.

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1 Introduction

A *cooperative n-person game* is a pair $< N, v >$, where $N = \{1, 2, ..., n\}$ is the *set of players* and $v : 2^N \to IR$ is the *characteristic function* with domain the family of subsets of N. Such subsets are called *coalitions* and $v(S)$ is called the *value* of coalition $S \in 2^N$. Such a game models a situation where a group N of persons can cooperate and also subgroups. For each subgroup S the value $v(S)$ indicates the amount of money which they can obtain when cooperating. There is only one restriction on the characteristic function, namely $v(\phi) = 0$, the value of the empty coalition is 0. This implies that the set of characteristic functions of n-person games forms with the obvious operations a $(2^n - 1)$-dimensional linear space G^N. Often v and $< N, v >$ will be identified. The question: 'How to divide $v(N)$, if all the players in N are cooperating?' has given rise to many proposals called solution concepts. Of the one-point solution concepts we mention only the Shapley value [6], the τ-value [9] and the nucleolus [5]. Sometimes subsets of payoff distributions of $v(N)$ are assigned to games as solutions; such a subset consists of points which are from a certain point of view better than the points outside. Three of such subsets, namely the imputation set, the dual imputation set and the core [4] will play a role in this paper and we describe them now.

The *imputation set* $I(v)$ of a game $< N, v >$ is defined by

$$I(v) = \left\{ x \in IR^n \mid \sum_{i=1}^{n} x_i = v(N),\ x_i \geq v(\{i\}) \text{ for each } i \in N \right\}$$

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and consists of those payoff distributions of $v(N)$, which are individual rational i.e. player i obtains an amount x_i which is at least as large as his individual value $v(\{i\})$, which he can obtain by staying alone. From the geometric point of view, the imputation set $I(v)$ is equal to the intersection of the efficiency hyperplane $\left\{x \in I\!R^n \mid \sum_{i=1}^{n} x_i = v(N)\right\}$ and the orthant $\{x \in I\!R^n \mid x \geq i(v)\}$ of individual rational payoff vectors.

The imputation set is non–empty iff $v(N) \geq \sum_{i=1}^{n} v(\{i\})$. If $v(N) > \sum_{i=1}^{n} v(\{i\})$, then $I(v)$ is an $(n-1)$-dimensional simplex with extreme points $f^1(v), f^2(v), ..., f^n(v)$, where the k-th coordinate $(f^i(v))_k$ of $f^i(v)$ equals $v(\{k\})$ if $k \neq i$ and $(f^i(v))_i = v(N) - \sum_{k \in N \setminus \{i\}} v(\{k\})$.

For an n-person game $< N.v >$ and $S \in 2^N$ we define the *dual value* $v^*(S)$ *of* S as $v^*(S) = v(N) - v(N \setminus S)$.

The amount $v^*(S)$ can be seen as the marginal contribution of S to the grand coalition or also as a sort of blocking power of S. The *dual imputation set* $I^*(v)$ of the game v is given by

$$I^*(v) = \left\{x \in I\!R^n \mid \sum_{i=1}^{n} x_i = v(N),\ x_i \leq v^*(\{i\}) \text{ for each } i \in N\right\}.$$

It consists of distributions of $v(N)$, where no player gets more than his marginal contribution to the grand coalition. From the geometric point of view $I^*(v)$ is equal to the intersection of the efficiency hyperplane and the orthant $\{x \in I\!R^n \mid x \leq u(v)\}$ of subutopic vectors.

Note that $I^*(v) \neq \emptyset$ iff $\sum_{i=1}^{n} v^*(\{i\}) \geq v(N)$. In the case of strict inequality, $I^*(v)$ is an $(n-1)$-dimensional simplex with extreme points $g^1(v), g^2(v), ...g^n(v)$, where

$$(g^i(v))_k = v^*(\{k\}) \text{ if } k \neq i, \text{ and } (g^i(v))_i = v(N) - \sum_{k \in N \setminus \{i\}} v^*(\{k\}).$$

The *core* $C(v)$ of a game $< N, v >$ is a subset of $I(v) \cap I^*(v)$ defined by

$$C(v) = \left\{x \in I\!R^n \mid \sum_{i=1}^{n} x_i = v(N),\ \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N\right\}.$$

Note that $v(\{i\}) \leq x_i \leq v^*(\{i\})$ for all $i \in N$ and $x \in C(v)$. So, the core is the bounded solution set of a set of linear inequalities, which means that the core is a polytope i.e. the convex hull of a finite set of vectors in $I\!R^n$. When proposing a core element for the division of $v(N)$ among the players, no subgroup $S \subset N$ will have an incentive to split off. However, the core may be empty. Independently, Bondareva in [2] and Shapley in [8] gave necessary and sufficient conditions for the non-emptyness of the core: let $< N, v >$ be an n-person game; then $C(v) \neq \emptyset$ iff $v(N) \geq \sum_{S \in 2^N \setminus \{\emptyset\}} \lambda_S v(S)$ in case $\lambda_S \geq 0$ for all $S \in 2^N \setminus \{\emptyset\}$ and $\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda_S e_S = e^N$.

Here $e^S \in IR^N$ is the characteristic vector of the coalition S, with $(e^S)_i = 0$ if $i \notin S$, and $(e^S)_i = 1$ otherwise.

In Section 2 we like to reformulate this Bondareva–Shapley result in geometric terms. In Section 3 we characterize n-person simplex games where the core is a subsimplex of the imputation set, and in Section 4 duality results for games lead to a characterization of dual simplex games, where the core is a subsimplex of the dual imputation set. Section 5 concludes.

2 Geometric characterization of games with a non-empty core

We define the *per capita value* $\bar{v}(S)$ of coalition $S \neq \emptyset$ by $\frac{1}{|S|}v(S)$ where $|S|$ is the cardinality of S. Further, let for a subsimplex $\Delta(S,v) = conv\{f^i(v) \mid i \in S\}$ of $I(v) = \Delta(N,v)$, the barycenter $\frac{1}{|S|}\sum_{i\in S} f^i(v)$ be denoted by $b(S,v)$. Then we obtain the following characterization of games with a non-empty core.

Theorem 2.1. *The game $< N,v >$ has a non-empty core iff* $\sum_{S\in 2^N\setminus\{\emptyset\}} \mu_S b(S,v) = b(N,v)$ *with* $\mu_S \geq 0$, $\sum_{S\in 2^N\setminus\{\emptyset\}} \mu_S = 1$, *implies that* $\sum_{S\in 2^N\setminus\{\emptyset\}} \mu_S \bar{v}(S) \leq \bar{v}(N)$.

The theorem tells that $< N,v >$ has a non-empty core iff for each way of writing the barycenter of the imputation set as a convex combination of barycenters of subsimplices, the per capita value of N is at least as large as the corresponding convex combination of per capita values of the subcoalitions.

Proof of Theorem 2.1. For $\lambda = (\lambda_S)_{S\in 2^N\setminus\{\emptyset\}}$, let $\mu = (\mu_S)_{S\in 2^N\setminus\{\emptyset\}}$ be defined by $\mu_S = n^{-1}|S|\lambda_S$. Then

$$\lambda_S \geq 0, \sum_{S\in 2^N\setminus\{\emptyset\}} \lambda_S e^S = e^N \text{ iff } \sum_{S\in 2^N\setminus\{\emptyset\}} \mu_S \frac{e^S}{|S|} = \frac{e^N}{|N|}, \mu_S \geq 0, \sum_{S\in 2^N\setminus\{\emptyset\}} \mu_S = 1.$$

This implies

(i) $\lambda_S \geq 0, \sum_{S\in 2^N\setminus\{\emptyset\}} \lambda_S e^S = e^N$ iff $\sum_{S\in 2^N\setminus\{\emptyset\}} \mu_S b(S,v) = b(N,v)$
since $b(S,v) = (v(\{1\}), v(\{2\}), ..., v(\{n\})) + \alpha|S|^{-1}e^S$ for each $S \in 2^N \setminus \{\emptyset\}$ where $\alpha = v(N) - \sum_{i\in N} v(\{i\})$.

(ii) $\sum_{S\in 2^N\setminus\{\emptyset\}} \lambda_S v(S) \leq v(N)$ iff $\sum_{S\in 2^N\setminus\{\emptyset\}} \mu_S \bar{v}(S) \leq \bar{v}(N)$.

From these observations the proof of Theorem 2.1 follows easily. □

3 Characterization of simplex games

Let us call a game $<N, v>$ a *T-simplex game*, where $\phi \neq T \subset N$, if $v(N) > \sum_{i=1}^{n} v(\{i\})$ are $C(v) = conv\{f^i(v) \mid i \in T\} = \Delta(T, v)$. Note that for a T-simplex game the imputation set is an $(n-1)$-dimensional simplex with $f^1(v), f^2(v), ..., f^n(v)$ as extreme points and the core is a $(|T|-1)$-dimensional subsimplex. In [1] N-simplex games (and also dual N-simplex games) were introduced and the family of these games was denoted by SI^N (and SI_*^N). The main results obtained were:

(i) $SI^N = \left\{ v \in G^N \mid v(S) \leq \sum_{i \in S} v(\{i\}) \text{ for all } S \neq N, \sum_{i \in N} v(\{i\}) < v(N) \right\}$ is a cone and the core correspondence is additive of SI^N;

(ii) For $v \in SI^N : CIS(v) = ENSR(v) = \tau(v)$, $CC(v) = C(v)$, where $CIS(v)$ is the center of the imputation set and $ENSR(v)$ is the center of the dual imputation set. For the definition of the τ-value we refer to [9], [1] or to [3].

In the next Theorem 3.1 we give some properties for games $v \in SI^T$, the set of T-simplex games. In Theorem 3.2 it turns out that these properties are characterizing properties. Then we show in Example 3.4 that SI^T is not necessarily a cone of games if $T \neq N$.

For a game $< N, v >$ the zero-normalization is the game $< N, 0 >$ with

$$v_0(S) = v(S) - \sum_{i \in S} v(\{i\}) \text{ for each } S \in 2^N.$$

Theorem 3.1. *Let $v \in SI^T$ for $\emptyset \neq T \subset N$ and let v_0 be the corresponding zero–normalization. Then*

(i) (Losing property) $v_0(S) \leq 0$ for each $S \in 2^N$ with $T \setminus S \neq \emptyset$

(ii) (Veto player property) $T = \cap\{S \in 2^N \mid v_0(S) = v_0(N)\}$

(iii) $(N, 0)$-monotonicity property) $v_0(S) \leq v_0(N)$ for all $S \in 2^N$.

Remarks. In the spirit of [7] we call S with $v_0(S) \leq 0$ *losing–coalitions* and those with $v_0(S) = v_0(N)$ *winning coalitions*. Players who are in each winning coalition are called *veto players*. Property (ii) says then that the set of veto players is equal to T.

Proof of Theorem 3.1.

(i) Take $S \in 2^N$ such that there is a $k \in T \setminus S$. Then $f^k(v) \in C(v)$ which implies that $v(S) \leq \sum_{i \in S} (f^k(v))_i = \sum_{i \in S} v(\{i\})$, $v_0(S) \leq 0$.

(iii) By (i), for $S \in 2^N$ with $T \setminus S \neq \emptyset : v_0(S) \leq 0 \leq v_0(N)$. For $S \in 2^N$ with $T \subset S$ and each $x \in C(v) = conv\{f^i(v) \mid i \in T\}$ we have

$$v(S) \leq \sum_{i \in S \setminus T} x_i + \sum_{i \in T} x_i = \sum_{i \in S \setminus T} v(\{i\}) + \left(v(N) - \sum_{i \in N \setminus T} v(\{i\}) \right),$$

which is equivalent with $v_0(S) \leq v_0(N)$.

(ii) From (i) it follows that $v_0(S) = v_0(N) > 0$ implies that $T \setminus S = \emptyset$, $T \subset S$. So $T \subset \cap\{S \in 2^N \mid v_0(S) = v_0(N)\}$. For the converse inclusion we have to prove that

$$(\cap\{S \in 2^N \mid v_0(S) = v_0(N)\}) \setminus T = \emptyset.$$

Suppose that this set is non–empty and that r is an element of it. We will deduce a contradiction. For each $U \in 2^N$ with $r \notin U$, U is not winning. This implies that $\max\{v_0(U) \mid r \notin U\} < v_0(N)$. Take $\varepsilon \in (0,1)$ such that $(1-\varepsilon)v_0(N) > v_0(U)$ for all U with $r \notin U$. Then we claim that for each $t \in T$ the element $z = (1-\varepsilon)f^t(v) + \varepsilon f^r(v)$ is a core element, which is in contradiction with the fact that $C(v) = \Delta(T, v) = conv\{f^i(v) \mid i \in T\}$. To prove the claim note that for $S \in 2^N$ with

(a) $t \notin S, r \notin S : \sum_{i \in S} z_i = (1-\varepsilon)\sum_{i \in S} v(\{i\}) + \varepsilon \sum_{i \in S} v(\{i\}) \geq v(S)$ by (i).

(b) $t \notin S, r \in S : \sum_{i \in S} z_i = (1-\varepsilon)\sum_{i \in S} v(\{i\}) + \varepsilon \left(\sum_{i \in S} v(\{i\}) + v_0(N)\right) > \sum_{i \in S} v(\{i\}) \geq$ $v(S)$ by (i).

(c) $t \in S, r \notin S : \sum_{i \in S} z_i = (1-\varepsilon)\sum_{i \in S} v(\{i\}) + (1-\varepsilon)v_0(N) + \varepsilon \sum_{i \in S} v(\{i\}) = \sum_{i \in S} v(\{i\}) +$ $(1-\varepsilon)v_0(N) > \sum_{i \in S} v(\{i\}) + v_0(S) = v(S)$ in view of the choice of ε.

(d) $t \in S, r \in S : \sum_{i \in S} z_i = \sum_{i \in S} v(\{i\}) + v_0(N) \geq \sum_{i \in S} v(\{i\}) + v_0(S) = v(S)$, where the inequality follows from (iii). □

Example 3.1. For $T \subset N$, the unanimity game $u_T : 2^N \to I\!R$ is defined by $u_T(S) = 1$ if $T \subset S$ and $u_T(S) = 0$, otherwise. The game u_T is a T-simplex game with $C(u_T) = conv\{e^i \mid i \in T\}$ and $I(u_T) = conv\{e^i \mid i \in N\}$, where e^i is the i-th standard basis element in $I\!R^n$.

Example 3.2. A game is called *simple* [7] if $v(S) \in \{0,1\}$ for all $S \in 2^N$ and $v(N) = 1$. The United Nations Security Council Game $< N, v >$ with $N = \{1, 2, ..., 15\}$ and

$$\begin{array}{ll} v(S) = 1 & \text{if } \{1,2,3,4,5\} \subset S \text{ and } |S| \geq 9, \\ v(S) = 0 & \text{otherwise} \end{array}$$

is a $\{1, 2, 3, 4, 5\}$-simplex game.

It corresponds to the situation when a bill can pass only if at least nine members agree with, among them the five veto players 1,2,3,4 and 5 are. In fact, all simple games with a non–empty core and with $v(\{i\}) = 0$ for each $i \in N$ are T–simplex games (see Corollary 3.1), where T is the non–empty set of veto players.

Example 3.3. Let $N = \{1, 2, 3, 4\}, v(\{1, 2\}) = v(\{1, 2, 3\}) = v(N) = 1, v(\{1, 2, 4\}) = \frac{1}{2}$, $v(S) = 0$ otherwise. Then $< N, v >$ is a $\{1, 2\}$-simplex game, with $C(v) = conv\{e^1, e^2\}$.

Example 3.4. Now we show that T-simplex games do not necessarily form a cone by considering the two 5-person $\{1,2\}$-simplex games $< N, v >$ and $< N, w >$ with

$$\begin{array}{ll} v(\{1,2,3\}) = v(\{1,2,4\}) = 1 = v(N),\ v(S) = 0 & \text{otherwise} \\ w(\{1,2,3\}) = w(\{1,2,5\}) = 1 = w(N),\ w(S) = 0 & \text{otherwise.} \end{array}$$

Then

$$\begin{array}{l} C(v) = conv\{f^1(v), f^2(v)\} = conv\{e^1, e^2\}, \\ C(w) = conv\{f^1(w), f^2(w)\} = conv\{e^1, e^2\}. \end{array}$$

For the sum game $u = v + w$ we have

$$u(\{1,2,3\}) = 2 = u(N), u(\{1,2,4\}) = u(\{1,2,5\}) = 1, u(S) = 0 \text{ otherwise.}$$

Note that u is not a simplex game, so it is certainly not an element of $SI^{\{1,2\}}$.

Theorem 3.2. *Let $< N, v >$ be a game with $v_0(N) > 0$. Suppose that*

(i) $v_0(S) \le v_0(N)$ *for each* $S \in 2^N$

(ii) $T := \cap\{S \mid v_0(S) = v_0(N)\} \neq \emptyset$

(iii) $v_0(S) \le 0$ *for all* S *with* $T \setminus S \neq \emptyset$.

Then $C(v) = \Delta(T, v),\ v \in SI^T$.

Proof. We have to show that $C(v) = \Delta(T, v)$.

(a) Suppose $x \in \Delta(T, v)$. Then for each $i \in N$ there is $\alpha_i \ge 0$ such that $x_i = v(\{i\}) + \alpha_i v_0(N)$ and $\sum_{i=1}^{n} \alpha_i = 1$, $\alpha_i = 0$ for $i \in N \setminus T$. Then for

$$S \in 2^N : \sum_{i \in S} x_i = \sum_{i \in S} v(\{i\}) + v_0(N) \sum_{i \in S} \alpha_i = \sum_{i \in S} v(\{i\}) + v_0(N) \sum_{i \in T \cap S} \alpha_i.$$

In case

$$T \subset S : \sum_{i \in S \cap T} \alpha_i = \sum_{i \in T} \alpha_i = \sum_{i \in N} \alpha_i = 1,$$

so

$$\sum_{i \in S} x_i = \sum_{i \in S} v(\{i\}) + v_0(N) \ge \sum_{i \in S} v(\{i\}) + v_0(S) = v(S),$$

where the inequality follows from (i).

In case

$$T \setminus S \neq \emptyset : \sum_{i \in S} x_i = \sum_{i \in S} v(\{i\}) + v_0(N) \sum_{i \in T \cap S} \alpha_i \ge \sum_{i \in S} v(\{i\}) \ge v(S),$$

where the last inequality follows from (iii). So $x \in C(v)$. We have proved that $\Delta(T, v) \subset C(v)$.

(b) For the converse inclusion, we show that $x \in I(v) \setminus \Delta(T, v)$ implies that $x \notin C(v)$. Take $x \in I(v) \setminus \Delta(T, v)$. Then there is a $k \in N \setminus T$ with $x_k = v(\{k\}) + \varepsilon$

and $\varepsilon > 0$. Further $x_i \geq v(\{i\})$ for all $i \in N$. By (ii) there is an S with $v_0(S) = v_0(N)$ and $k \notin S$. This implies

$$x(S) = v(N) - \sum_{i \in N \setminus S} x_i \leq v(N) - \sum_{i \in N \setminus S} (v\{i\}) - \varepsilon =$$

$$= v_0(N) + \sum_{i \in S} v(\{i\}) - \varepsilon = v_0(S) + \sum_{i \in S} v(\{i\}) - \varepsilon = v(S) - \varepsilon.$$

So we have proved that $\sum_{i \in S} x_i \leq v(S) - \varepsilon$, hence $x \notin C(v)$. □

As a corollary of Theorem 3.2 we obtain the following well–known fact about simple games.

Corollary 3.1. *Let $< N, v >$ be a game with the properties:*

(i) $v(S) \in \{0,1\}$ *for each* $S \in 2^N$,

(ii) $\Delta(N,v) = I(v) = conv\{e^1, e^2, ..., e^n\}$,

(iii) The set of veto players $T = \cap\{S \mid v(S) = 1\}$ *is non–empty.*

Then $C(v) = \Delta(T, v)$.

4 Characterization of dual simplex games

Now, we focus on characterizing all games with the property that the core is a non–empty subsimplex of the dual imputation set $I^*(v)$. Let us denote by SI_*^T the set of n-person games with $\emptyset \neq T \subset N$, $v^*(N) < \sum_{i=1}^{n} v^*(i)$ and $C(v)$=$conv\{g^i(v) \mid i \in T\}$= $\Delta^*(T, v)$.

Example 4.1. Let $< N, v >$ be the 3-person game with $v(\{i\}) = 0$ for each $i \in N$ $v(\{1,2\}) = 1$, $v(\{1,3\}) = 2$, $v(\{2,3\}) = v(N) = 6$. Then $v^*(N \setminus \{i\}) = v^*(N) = 6$ for each $i \in N$, $v^*(\{1\}) = 0$, $v^*(\{2\}) = 4$ and $v^*(\{3\}) = 5$. Here $C(v) = I(v) \cap I^*(v) = conv\{6e^1, 6e^2, 6e^3\} \cap conv\{(-3,4,5),(0,1,5),(0,4,2)\} = conv\{(0,1,5),(0,4,2)\} = \Delta^*(\{2,3\}, v)$, so $v \in SI_*^{\{2,3\}}$.

Example 4.2. Let $< N, v >$ be the 3-person unanimity game based on $\{1,2\}$, so $v(\{1,2\}) = v(\{1,2,3\}) = 1$, $v(S) = 0$ otherwise. Then $v^*(\{3\}) = 0$ and $v^*(S) = 1$ otherwise. The core $C(v)$ equals $conv\{e^1, e^2\} = conv\{f^1(v), f^2(v)\} = conv\{g^2(v), g^1(v)\}$. So $C(v) = \Delta(\{1,2\}, v) = \Delta^*(\{1,2\}, v)$, hence $v \in SI^{\{1,2\}}$ and $v \in SI_*^{\{1,2\}}$.

To solve the characterization problem for dual simplex games, we can use our characterization result in Section 3 for simplex games. For that purpose we develop some duality relations for cooperative games in the next lemma.

Lemma 4.1. *For each* $v \in G^N$ *and all* $k \in N$, $T \subset N$, $T \neq \emptyset$ *we have* *(i)* $(v^*)^* = v$

(ii) $-f^k(v) = g^k(-v^*)$

(iii) $\Delta^*(T,v) = -\Delta(T,-v^*)$

(iv) $C(-v^*) = -C(v)$

(v) $C(-v^*) = \Delta(T,-v^*)$ *iff* $C(v) = \Delta^*(T,v)$,
which is equivalent to $-v^* \in SI^T$ *iff* $v \in SI^T_*$.

Proof. We only prove (iv) and leave the other proofs to the readers.

$$\begin{aligned}
&C(-v^*) = \\
&= \left\{ x \in IR^n \mid \sum_{i=1}^n x_i = -v^*(N), \sum_{i\in S} x_i \geq -v^*(S) \text{ for each } S \in 2^N \right\} = \\
&= \left\{ x \in IR^n \mid \sum_{i=1}^n x_i = -v(N), \sum_{i\in N\setminus S} x_i \leq -v(N\setminus S) \text{ for each } S \in 2^N \right\} = \\
&= -\left\{ y \in IR^n \mid \sum_{i=1}^n y_i = v(N), \sum_{i\in N\setminus S} y_i \geq v(N\setminus S) \text{ for each } S \in 2^N \right\} = \\
&= -\left\{ y \in IR^n \mid \sum_{i=1}^n y_i = v(N), \sum_{i\in T} y_i \geq v(T) \text{ for each } T \in 2^N \right\} = \\
&= -C(v).
\end{aligned}$$

□

The key of finding the characterization of dual simplex games lies now in Lemma 4.1 (v): $v \in SI^T_*$ iff $-v^* \in SI^T$. So we can use the characterization of $v \in SI^T$ of Section 3 but with $-v^*$ in the role of v and obtain

Theorem 4.2. *Let* $\emptyset \neq T \subset N$ *and let* $v_0(N) > 0$. *Then* $v \in SI^T_*$ *iff the following three properties hold:*

(i) Dual $(N,0)$*-monotonicity property:* $(v^*)_0(S) \geq (v^*)_0(N)$ *for all* $S \in 2^N$

(ii) Dual veto player property: $\cap\{S \in 2^N \mid (v^*)_0(S) = (v^*)_0(N)\} = T \neq \emptyset$

(iii) Dual losing property: $(v^*)_0(S) \geq 0$ *for all* $S \in 2^N$ *with* $T \setminus S \neq \emptyset$.

5 Concluding remark

It could be interesting to study for simplex games and also for dual simplex games the relations between different existing solution concepts such as the τ-value, the nucleolus, the Shapley value, CIS etc.

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