Variational Solutions of Stationary Hamilton-Jacobi Equations

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Abstract

This work deals with the stationary Hamilton-Jacobi equation $F(B^*\psi_y) - (Ay, \psi_y) = g$ in the class of convex continuous functions ψ on a real Hilbert space H. After obtaining an symptotic result, the existence, the uniqueness and a Galerkin approximation of the solution to the above equation are established.

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Let H and U be two real Hilbert spaces with the scalar products $(\cdot,\cdot)$ and $< \cdot,\cdot >$, respectively. The norms of H and U will be denoted by $|\cdot|$ and $|\cdot|_U$, respectively. We define the function $\psi^\infty : H \to R$,

(1)

$$\psi^\infty(y) = \inf\left\{\int_0^\infty (g(x(t)) + h(u(t)))dt;\ x' = Ax + Bu,\ x(0) = y;\ u \in L^1(R^+;U)\right\}$$

in the following hypotheses:

(i) A is the infinitesimal generator of a C_0-semigroup of contractions on H i.e., A is m-dissipative on H;
B is a linear continuous operator from U to H.

(ii) The function $g : H \to R$ belongs to $K \cap C^1_{Lip}(H)$ and satisfies the following two conditions:

$$\text{if } g(y_n) \to 0 \text{ for some sequences } \{y_n\} \text{ then } y_n \to 0; \tag{2}$$

$$g(e^{-(\lambda I - A)t}y) \in L^1(R^+) \text{ for } \lambda > 0 \text{ and all } y \in H. \tag{3}$$

Here $e^{-(\lambda I-A)t}$ is the semigroup generated by the operator $-\lambda I + A$, K is the closed convex cone of $C(H)$ consisting of all convex functions $\varphi : H \to R$ which satisfy the condition $\varphi(y) \geq \varphi(0) = 0$ for all $y \in H$ and $C^1_{Lip}(H)$ is the space of all $\varphi \in C^1(H)$ such that the Fréchet derivative $\varphi^{(1)}$ is Lipschitz on H.

(iii) $F : U \to R$ is a convex function which is bounded on bounded subsets and belongs to $C^1(U)$. We associate the function $h : U \to \bar{R}$, defined by

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$$h(u) = \sup\{- < p, u > -F(p); p \in U\} \text{ i.e., } h(u) = F^*(-u),$$

to the function F. Then the assumption (iii) implies $\lim\limits_{|u|_U \to \infty} \dfrac{h(u)}{|u|_U} = +\infty$. Further assume that F and F^* are strictly convex.

(iv) $\psi_0 : H \to R$ is a function which belongs to $K \cap C^1_{Lip}(H)$.

Under the above hypotheses, we see that $\Psi^\infty < +\infty$ on H and for every $y \in H$ the infimum defining $\Psi^\infty(y)$ is attained in a unique pair (x^*, u^*). Moreover, the function Ψ^∞ is convex and lower semicontinuous on H. Since it is everywhere finite we conclude that it is continuous on H.

Now we shall prove the following asymptotic result.

Theorem 1. *Let assumptions* (i), (ii), (iii) *and* (iv) *be satisfied. Then the solution* Ψ *to the problem*

$$\begin{cases} \psi_t + F(B^*\psi_y) - (Ay, \psi_y) = g; \ t \geq 0, \ y \in D(A) \\ \psi(0, y) = \psi_0(y), \ y \in H \end{cases} \tag{4}$$

satisfies

$$\lim_{t \to \infty} \psi(t, y) = \psi^\infty(y) \text{ for all } y \in H. \tag{5}$$

We have denoted by Ψ_t and Ψ_y the partial derivatives (in a generalized sense) of the function Ψ with respect to t and y, respectively. Here B^* is the adjoint of B.

Proof. The solution Ψ to the Cauchy problem (4) is given by (see [1], p.54)

$$\begin{aligned} \psi(t, y) = \inf \Big\{ \int_0^t (g(x(s)) + h(u(s)))ds + \psi_0(x(t)); \ x' = Ax + Bu, \\ x(0) = y; \ u \in L^1(0, t; U) \Big\} \end{aligned} \tag{6}$$

for $t \geq 0$, $y \in H$. This yields

$$\begin{aligned} \int_0^t (g(x^t(s)) + h(u^t(s)))ds + \psi_0(x^t(t)) &= \psi(t, y) \leq \int_0^t (g(x^*(s)) + \\ &+ h(u^*(s)))ds + \psi_0(x^*(t)) \end{aligned} \tag{7}$$

where (x^*, u^*) is the optimal pair in Problem (1) and (x^t, u^t) in (6), i.e.,

$$\begin{cases} (x^t)' = Ax^t + Bu^t \text{ on } [0, t] \\ x^t(0) = y. \end{cases}$$

Letting $t \to +\infty$, we see by (7) that

$$\limsup_{t \to \infty} \psi(t, y) \leq \psi^\infty(y). \tag{8}$$

We set

$$\tilde{u}^t(s) = \begin{cases} u^t(s) & \text{for} \quad 0 \leq s \leq t \\ 0 & \text{for} \quad s > t \end{cases}$$

and

$$\tilde{x}^t(s) = e^{As}y + \int_0^s e^{A(s-\tau)}B\tilde{u}^t(\tau)d\tau.$$

Let some sequence $t_n \to \infty$ and ley $\{\tilde{x}^{t_n}\}$ and $\{\tilde{u}^{t_n}\}$ be such that

(9) $$\psi^\infty(y) \le \int_0^\infty \left(g(\tilde{x}^{t_n}(s)) + h(\tilde{u}^{t_n}(s))\right) ds \le \psi^\infty(y) + \frac{1}{n},$$

$$(\tilde{x}^{t_n})' = A\tilde{x}^{t_n} + B\tilde{u}^{t_n} \quad \text{on} \quad R^+$$

$$\tilde{x}^{t_n}(0) = y.$$

Since $|\tilde{x}^{t_n}(s)| \le C\left(|y| + \int_0^s \left\|\tilde{u}^{t_n}(\tau)\right\| d\tau\right)$, $\forall s \ge 0$ and $g(\tilde{x}^{t_n}(s)) \ge g(0) - |\partial g(0)|\,|\tilde{x}^{t_n}(s)|$, $\forall s \ge 0$ by (9) it follows that $\int_0^\infty h(\tilde{u}^{t_n}(s))ds \le C\left(\int_0^\infty \left\|\tilde{u}^{t_n}(s)\right\| ds + 1\right)$ where C is independent of n. The latter implies that $\{\tilde{u}^{t_n}\}$ is weakly compact in $L^1(R^+;U)$ (by virtue of the Dunfort-Pettis theorem). Hence on some sequence $t_n \to \infty$, $\tilde{u}^{t_n} \to \tilde{u}$ weakly in $L^1(R^+;U)$ and $\tilde{x}^{t_n}(s) \to \tilde{x}(s)$ weakly in H for every $s \ge 0$, where $\tilde{x}$ is the mild solution to

$$\begin{cases} x' = Ax + B\tilde{u} \quad \text{on} \quad R^+ \\ x(0) = y. \end{cases}$$

Since the functional $(x,u) \to \int_0^T (g(x) + h(u))dt$ is weakly lower semicontinuous on every $C([0,T];H) \times L^1(0,T;U)$ (because it is convex and lower semicontinuous), it follows by (7) and (8) that

$$\int_0^\infty (g(\tilde{x}^t(s)) + h(\tilde{u}^t(s)))ds \le \lim\sup_{t\to\infty} \psi(t,y) \le \psi^\infty(y)$$

which obviously implies (5) as claimed. □

Now let us consider the stationary Hamilton-Jacobi equation

(10) $$F(B^*\psi_y) - (Ay, \psi_y) = g, \quad \forall y \in D(A)$$

in the real Hilbert space H.

By a *solution* to (10) we shall mean a function $\psi : H \to R$ which is Gâteaux differentiable and staisfies Eq. (10) for all $y \in D(A)$.

Equation (10) is relevant in the calculus of variations with infinite horizon and in nonlinear analysis. The main result is

Theorem 2. *Let assumptions* (i), (ii) *part* (3) *and* (iii) *be satisfied. Then Eq.* (10) *has at least one solution* $\psi^\infty \in K$. *If also condition* (2) *holds then the solution* ψ^∞ *to Eq.* (10) *is unique.*

Proof. For every $t \ge 0$, the unique optimal pair (x^*, u^*) in Problem (1) is also an optimal pair for the finite horizon control problem

(11)

$$\begin{aligned} \psi^\infty(y) &= \inf\left\{\int_0^t (g(x(s)) + h(u(s)))ds + \psi^\infty(x(t));\ x' = Ax + Bu,\ x(0) = y\right\} \\ &= \int_0^t (g(x^*(s)) + h(u^*(s)))ds + \psi^\infty(x^*(t)). \end{aligned}$$

By the maximum principle ([1], p. 13), for every $t \geq 0$ there exists $p^t \in C([0,t];H)$ which satisfies the system

$$\begin{cases} (p^t)' + A^* p^t \in \partial g(x^*), & 0 \leq s \leq t \\ B^* p^t \in \partial h(u^*), & 0 \leq s \leq t \\ p^t(t) \in -\partial \psi^\infty (x^*(t)) & . \end{cases}$$

(Here ∂ is the subdifferentiation symbol). Next by (11) we see that

$$\psi^\infty(x^*(t)) = \int_t^\infty (g(x^*(s)) + h(u^*(s)))ds, \quad \forall t \geq 0. \tag{12}$$

Assume that $y \in D(A)$. Then x^* is right differentiable at $t = 0$ and $\frac{d^+ x^*}{dt}(0) = Ay + Bu^*(0)$. Since

$$\frac{\psi^\infty(x^*(t)) - \psi^\infty(y)}{t} \overset{t \to 0}{\to} \left(\eta, \frac{d^+ x^*}{dt}(0) \right)$$

where η is an element of $\partial \Psi^\infty(y)$, it follows by (12) that

$$(\eta, Ay + Bu^*(0)) + g(y) + h(u^*(0)) = 0 \tag{13}$$

where $u^*(0) = \nabla h^*(-B^*\eta)$ and $\eta \in \partial \psi^\infty(y)$ (we may take $\eta = -p^t(0)$). By using (13) and the conjugacy formula, we get

$$F(B^*\eta) - (Ay, \eta) = g(y).$$

To conclude the proof of existence it sufficies to show that Ψ^∞ is Gâteaux differentiable, i.e., $\partial \Psi^\infty$ is single valued.

To this end we define the operator $\Gamma : H \to H$, $\Gamma y = -p(0)$ where $p \in C([0,T];H)$ is the solution to the system (T is fixed)

$$\begin{cases} x' = Ax + Bu, & 0 \leq t \leq T \\ p' + A^* p \in \partial g(x), & 0 \leq t \leq T \\ x(0) = y, \ p(T) \in -\partial \Psi^\infty & (x(T)) \end{cases} \tag{14}$$

which, by virtue of the maximum principles is equivalent with the control problem

$$\inf \left\{ \int_0^T (g(x) + h(u))dt + \Psi^\infty(x(T)); \ x' = Ax + Bu, \ x(0) = y \right\}. \tag{15}$$

Since F and F^* are strictly convex, so is h^* and therefore there exists a unique optimal pair (x,u) for the problem (15). Since F is Gâteaux differentiable, ∂F^* and therefore ∂h is single valued. Thus there exists a unique $B^* p = \partial h(u)$ which satisfies the system (14) and therefore Γ is single valued. On the other hand, we see that $\Gamma y \in \partial \Psi^\infty(y)$. To prove that $\Gamma = \partial \Psi^\infty$ it suffices to show that the range $R(I + \Gamma)$ is all of H. To this end we consider the equation $y + \Gamma y = w$ which is equivalent to

$$\left\{\begin{array}{ll} x' = Ax + Bu, & 0 \le t \le T \\ p' + A^* p \in \partial g(x), & 0 \le t \le T \\ x(0) = y, \ p(0) = y - w & \\ p(T) \in -\partial \Psi^\infty(x(T)) & \\ B^* p \in \partial h(u), & 0 \le t \le T \end{array}\right. \tag{16}$$

and again by virtue of the maximum principle it is equivalent to the control problem

$$\inf \left\{ \int_0^T (g(x) + h(u))dt + \Psi^\infty(x(T)) + \frac{1}{2}|x(0) - w|^2; \ x' = Ax + Bu \right\}$$

which clearly admits at least one solution which is also a solution to (16). Hence $\Gamma = \partial \Psi^\infty$, and $\partial \Psi^\infty = (\Psi^\infty)'$ is single valued as claimed.

As for uniqueness we consider the differential equation

$$\left\{\begin{array}{ll} x' = Ax - B\nabla h^*(B^* \Psi_x^0(x)), & t \ge 0 \\ x(0) = y & \end{array}\right. \tag{17}$$

where Ψ^0 is any solution to Eq. (10). For each $y \in D(A)$, Eq. (17) has a unique solution $\tilde{x} \in W_{loc}^{1,\infty}(R^+; H)$. We set $\tilde{u} = -\nabla h^*(B^* \Psi_x^0(x))$ and take the inner product of (17) by $-\Psi_x^0(\tilde{x})$. Since Ψ^0 is a solution to Eq. (10) we get

$$\frac{d}{dt}\Psi^0(\tilde{x}(t)) - F(B^* \Psi_x^0(\tilde{x}(t))) + g(\tilde{x}(t)) + (\tilde{u}(t), -B^* \Psi_x^0(x^*(t))) = 0$$

a.e. $t \ge 0$. By using the conjugacy formula, the latter becomes

$$\frac{d}{dt}\Psi^0(\tilde{x}(t)) + g(\tilde{x}(t)) + h(\tilde{u}(t)) = 0 \quad \text{a.e.} \quad t \ge 0$$

and therefore

$$\Psi^0(y) = \Psi^0(\tilde{x}(t)) + \int_0^t (g(\tilde{x}(s)) + h(\tilde{u}(s)))ds, \quad \forall t \ge 0. \tag{18}$$

Finally

$$\Psi^0(y) = \int_0^\infty (g(\tilde{x}(s)) + h(\tilde{u}(s)))ds \ge \Psi^\infty(y) \tag{19}$$

where Ψ^∞ is given by (1). (To get (19) we have used the fact that $\tilde{x}(t_n) \to 0$ for some $t_n \to \infty$ because $g(\tilde{x}) \in L^1(R^+)$ and g satisfies condition (2)).

Now let (x^*, u^*) be the optimal pair for Problem (1) ($y \in D(A)$). Again by Eq. (10) we have

$$\begin{aligned} \frac{d}{dt}\Psi^0(x^*(t)) &= (x'^*(t), \Psi_x^0(x^*(t))) = (Ax^*(t) + Bu^*(t), \Psi_x^0(x^*(t))) = \\ &= F(B^* \Psi_x^0(x^*(t))) - g(x^*(t)) - (u^*(t), -B^* \Psi_x^0(x^*(t))) \ge \\ &\ge -h(u^*(t)) - g(x^*(t)) \text{ a.e. } t > 0 \end{aligned}$$

and therefore

$$\Psi^0(x^*(t)) + \int_0^t (h(u^*(s)) + g^*(x^*(s)))ds \geq \Psi^0(y), \quad \forall t \geq 0.$$

Since $g(x^*) \in L^1(R^+)$ it follows that $\lim_{t_n \to \infty} x^*(t_n) \to 0$ for some sequence $t_n \to \infty$ and therefore $\lim_{t_n \to \infty} \Psi^0(x^*(t_n)) = \Psi^0(0) = 0$. Hence $\Psi^0(y) \leq \Psi^\infty(y)$ for all $y \in D(A)$. Along with (19) the latter implies that $\Psi^\infty = \Psi^0$ as claimed. □

Now we shall briefly discuss a Galerkin approximation of the stationary Hamilton-Jacobi equation (10). Assume that the hypotheses (i)-(iii) hold. It will be more convenient to regard A as a linear continuous operator from the space $V \subset H$ to its dual $V' \supset H$. Following a well-known scheme (see for instance [1], [2], [3]), the internal approximations of the spaces V and U, of the operators A and B^* and of the functions F and g will be defined. Thus on the finite dimensional space V_h, Eq. (10) becomes

(20) $$F_h(B_h^* \Psi_y^h(y_h)) - (A_h y_h, \Psi_y^h(y_h))_h = g_h(y_h) \text{ for all } y_h \in V_h.$$

By Theorem 2, Eq. (20) has a generalized solution $\Psi^h : V_h \to R$ given by

$$\Psi^h(y_h) = \inf \left\{ \int_0^\infty (g_h(x_h(t)) + F_h^*(-u_h(t)))dt \right\}$$

where x_h, u_h satisfy the system

$$\begin{cases} x_h' = A_h x_h + B_h u_h \text{ a.e. } t > 0 \\ x_h(0) = y_h. \end{cases}$$

As regards the convergence of the solutions Ψ^h to a solution Ψ to Eq. (10), we have the following result

Theorem 3. *For $h \to 0$ we have $\Psi^h(y_h) \to \Psi(y)$ for all $y \in V$, where Ψ is the variational solution to Eq.* (10) *given by* (1).

The proof is similar (making obvious changes) to the proofs of the analogous results in [1] and [3].

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References

[1] V. Barbu and G. Da Prato, *Hamilton-Jacobi Equations in Hilbert Spaces*, Pitman, Research Notes in Mathematics, 86 (1983).

[2] R. Glowinski, J.L. Lions an R. Trémolières, *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, 1981.

[3] V. Iftode, *Approximation of dual Hamilton-Jacobi equations*, Rivista di Mat. Pura Appl., 15 (1994), 57-75.

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