Mean Curvature Comparison for Tubular Hypersurfaces in Symmetric Spaces

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Dedicated to the Memory of Grigorios TSAGAS (1935-2003), President of Balkan Society of Geometers (1997-2003)

Abstract

We obtain comparison results for the mean curvature of tubular hypersurfaces, P_t, around a submanifold P of a riemannian manifold M, with bounded curvature, taking as a model tubular hypersurfaces around totally geodesic, curvature preserving submanifolds in symmetric spaces of arbitrary rank, and we give an application to get estimates for the relative volume.

Mathematics Subject Classification: 53C21, 53C35.
Key words: comparison theorems submanifolds, mean curvature, symmetric spaces, totally geodesic, tubes.

1 Introduction

The first purpose of this paper is to obtain comparison theorems for the mean curvature of tubular hypersurfaces, P_t, around a submanifold P of a Riemannian manifold M, with bounded curvature, taking as a model tubular hypersurfaces around totally geodesic, curvature preserving submanifolds in symmetric spaces of arbitrary rank. Results of this type have been widely studied in the literature considering different rank-one symmetric spaces as a model. Moreover, these results have been applied to obtain comparison results for geometric Riemannian invariants such as volume, mean exit time,.... (see, for instance, [10], [7], [4], [5], [12], [13], [11]).

The bounds imposed to the $q-$mean curvatures defined in [1, page 253] to obtain the comparison theorems are given from the restricted roots of the symmetric space. That is the reason because these bounds are constant for rank-one symmetric spaces, but they depend, in general, on the vector used to define the $q-$mean curvatures in arbitrary rank symmetric spaces. The above situation is closely related with the fact that the eigenvalues of the Weingarten map, $S(t)$, of a tubular hypersurface in a symmetric space, are constant for rank-one symmetric spaces but depend on the vector used to define $S(t)$ for arbitrary rank symmetric spaces ([8], [14]).

The second purpose of the paper is to obtain a comparison result for the relative volume using, as a model, totally geodesic submanifolds in the symmetric space for which the first conjugate locus and the cut-focal locus agree ([2]).

Balkan Journal of Geometry and Its Applications, Vol.8, No.1, 2003, pp. 53-62.

We will consider along the paper compact symmetric spaces, however using the duality between compact and noncompact symmetric spaces we will extend the results to noncompact symmetric spaces in the Appendix at the end of the paper.

2 Preliminaries

Let M be a Riemannian manifold of dimension n. Let P be a submanifold of M of dimension s. We shall denote by $\mathcal{NP}$ the normal bundle of P in M and by $\mathcal{N}_pP$ the fibre of $\mathcal{NP}$ over $p \in P$. Given a unitary vector $u \in \mathcal{N}_pP$, we consider an orthogonal decomposition

$$T_pM = \left(\bigoplus_{j=1}^{l_1} H_j\right) \oplus \left(\bigoplus_{i=1}^{l_2} V_i\right), \tag{2.1}$$

where

$$T_pP = \bigoplus_{j=1}^{l_1} H_j \quad \text{and} \quad (T_pP)^{\perp} = \bigoplus_{i=1}^{l_2} V_i$$

and such that $u \in V_1$, $\dim V_1 = r$, $\dim V_i = m_i$ $(i = 2, \ldots, l_2)$, $\dim H_1 = q - r$ and $\dim H_j = n_j$ $(j = 2, \ldots, l_1)$.

The m_i-Ricci curvature $K(u, V_i)$ of u at V_i (called the m_i-mean curvature in [1, page 253]) is defined as

$$K(u, V_i) = \sum_{k_2=1}^{m_i} R(u, X^{\perp}_{i_{k_2}}, u, X^{\perp}_{i_{k_2}}); \tag{2.2}$$

where $\{X^{\perp}_{i_{k_2}},\ k_2 = 1, \ldots, m_i\}$ is an orthonormal basis of V_i. This definition can be extended to any subspace of T_pM and, in particular, to H_j $(j = 1, \ldots, l_1)$.

Let $\gamma_u(t)$ be the geodesic such that $\gamma_u(0) = p \in P$ and $\gamma'_u(0) = u \in \mathcal{N}_{\surd}\mathcal{P}$, and τ_t the parallel transport along $\gamma_u(t)$ from 0 to t. Then, if $V_i^t = \tau_t V_i$ and $H_j^t = \tau_t H_j$, we have

$$T_{\gamma_u(t)}M = \left(\bigoplus_{j=1}^{l_1} H_j^t\right) \oplus \left(\bigoplus_{i=1}^{l_2} V_i^t\right).$$

Let P_t be the tubular hypersurface around P of radius t. We denote by $S(t)$ the Weingarten map of P_t and we consider in P_t the operator $R(t)X = R(\gamma'_u(t), X)\gamma'_u(t)$.

Lemma 1.1. [6]. *Let $A_P(t)$ denote the volume of P_t; then,*

$$S'(t) = S^2(t) + R(t), \tag{2.3}$$

$$\frac{\theta'_u(t)}{\theta_u(t)} = -\left(\frac{n-s-1}{t} + \operatorname{tr}(S(t))\right), \tag{2.4}$$

$$A_p(t) = t^{n-s-1} \int_P \int_{S^{n-s-1}(1)} \theta_u(t)\, \mathrm{d}u\, \mathrm{d}P, \tag{2.5}$$

where $\theta_u(t)$ is the infinitesimal change of volume function in the direction of u and $S^{n-s-1}(1)$ denotes the unit sphere in $\mathcal{N}_pP$.

Now, if $f(u) = \inf\{t > 0 \ / \ \gamma_u(t)$ is a conjugate point of $p\}$, $c(u) = \sup\{t > 0 \ / \ d(p, \gamma_u(t)) = t\}$ (d being the distance function in M) and $z(\theta_u(t))$ is the first positive zero of the function $\theta_u(t)$, we have

Lemma 1.2.

$$c(u) \leq f(u) = z(\theta_u(t)), \tag{2.6}$$

$$\mathrm{Vol}(M) = \int_P \int_{S^{n-s-1}(1)} \int_0^{c(u)} t^{n-s-1}\theta_u(t)\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}P. \tag{2.7}$$

To compare, in the next section, the trace of $S(t)$, we will use a pair $(\widetilde{M}, \widetilde{P})$ as a model, where $\widetilde{M} = G/K$ is a compact symmetric space of dimension n and $\widetilde{P}$ is a totally geodesic submanifold of $\widetilde{M}$ of dimension s, which satisfies the following properties.

Let $\mathbf{g} = \mathbf{k} + \mathbf{m}$ be the canonical decomposition of $\widetilde{M}$ ($\mathbf{m}$ is indetified with the tangent space of $\widetilde{M}$ at any point), and $\mathbf{h}$ a maximal abelian subspace of $\mathbf{m}$. Since $\widetilde{P}$ is totally geodesic, it is also a symmetric space $\widetilde{P} = U/L$. Let $\mathbf{u} = \mathbf{l} + \mathbf{p}$ be the canonical decomposition of $\widetilde{P}$, where $\mathbf{p}$ is identified with the tangent space of $\widetilde{P}$ at any point.

We assume that the orthogonal complement of $\mathbf{p}$ in $\mathbf{m}$, $\mathbf{p}^\perp$, is a Lie triple system, i.e.

$$[\mathbf{p}^\perp, [\mathbf{p}^\perp, \mathbf{p}^\perp]] \subset \mathbf{p}^\perp, \tag{2.8}$$

then, [9], $\widetilde{P}^\perp = \mathrm{Exp}(\mathbf{p}^\perp)$ is a totally geodesic submanifold of $\widetilde{M}$, and $\widetilde{P}^\perp = U'/L'$ is also a Riemannian globally symmetric space.

A list of pairs $(\widetilde{P}, \widetilde{P}^\perp)$ for all compact symmetric spaces can be found in [3].

We say that $\widetilde{P}$ is a totally geodesic curvature preserving submanifold of $\widetilde{M}$, because, for each vector $u \in \mathbf{p}^\perp$, the curvature operator R_u satisfies, from (2.8), the following condition of preserving the curvature,

$$R_u(\mathbf{p}) \subset \mathbf{p} \quad \text{and} \quad R_u(\mathbf{p}^\perp) \subset \mathbf{p}^\perp. \tag{2.9}$$

Let $\mathbf{a}$ be a maximal abelian subspace of $\mathbf{p}^\perp$ with $u \in \mathbf{a} \subset \mathbf{h}$, then, if $\mathrm{rank}(\widetilde{P}^\perp) = r$ and $\mathrm{rank}(\widetilde{M}) = q$, we have that $r \leq q$. Let $\mathbf{b} = \mathbf{h} \cap \mathbf{p}$.

Let α_i $(1 \leq i \leq l_2)$ be the positive restricted root system of $\mathbf{p}^\perp$ and β_j $(1 \leq j \leq l_1)$ that of $\mathbf{p}$. From (2.9), $\mathbf{m}$ can be decomposed as

$$\mathbf{m} = \left(\mathbf{a} \oplus \sum_{i=2}^{l_2} \mathbf{m}_i\right) \oplus \left(\mathbf{b} \oplus \sum_{j=2}^{l_1} \mathbf{n}_j\right), \tag{2.10}$$

where $\mathbf{m}_i$ is the root subspace of dimension m_i corresponding to α_i, $(\alpha_1(u) = 0,\ m_1 = r)$, and $\mathbf{n}_j$ is the root subspace of dimension n_j corresponding to β_j, $(\beta_1(u) = 0,\ n_1 = q - r)$.

Let $\{X^\perp_{i_k},\ k = 1, \ldots, l_2\}$ be an orthonormal basis of $\mathbf{p}^\perp$ such that $X^\perp_{1_1} = u$, $\{X^\perp_{1_k}, k = 1, \ldots, r\} \subset \mathbf{a}$ and $\{X^\perp_{i_k}, k = 1, \ldots, m_i\} \subset \mathbf{m}_i$. Let $\{X^\top_{j_k}, k = 1, \ldots, l_1\}$

be an orthonormal basis of $\mathbf{p}$ such that $\{X^{\top}_{1_k}, k = 1,\ldots,q-r\} \subset \mathbf{b}$ and $\{X^{\top}_{j_k}, k = 1,\ldots,n_j\} \subset \mathbf{n}_j$.

In this way we get a basis $\{X^{\top}_{j_{k_1}},\ X^{\perp}_{i_{k_2}}\}$, $(k_1 = 1,\ldots,n_j,\ j = 1,\ldots,l_1)$, $(k_2 = 1,\ldots,m_i,\ i = 1,\ldots,l_2)$, of $\mathbf{m}$ which diagonalizes R_u, with eigenvalues given by

$$\{\beta_j^2(u), \alpha_i^2(u)\}, \quad (j = 1,\ldots,l_1),\ (i = 1,\ldots,l_2).$$

The m_i-Ricci curvature of u at $\mathbf{m}_i$ and the n_j-Ricci curvature of u at $\mathbf{n}_j$ are, respectively,

$$K(u,\mathbf{m}_i) = m_i\alpha_i^2(u) \quad \text{and} \quad K(u,\mathbf{n}_j) = n_j\beta_j^2(u). \tag{2.11}$$

The Ricci curvature of $\widetilde{M}$ with respect to u is

$$\rho(u,u) = \sum_{i=2}^{l_2} m_i\alpha_i^2(u) + \sum_{j=2}^{l_1} n_j\beta_j^2(u). \tag{2.12}$$

Let $\{E^{\top}_{j_{k_1}}(t), E^{\perp}_{i_{k_2}}(t)\}$, $(k_1 = 1,\ldots,n_j,\ j = 1,\ldots,l_1)$, $(k_2 = 1,\ldots,m_i,\ i = 1,\ldots,l_2)$, be the parallel transport of $\{X^{\top}_{j_{k_1}}, X^{\perp}_{i_{k_2}}\}$ along the geodesic $\gamma_u(t)$. The operators $S(t)$ and $R(t)$ corresponding to $\widetilde{M}$ satisfy:

$$\begin{array}{rcl} \widetilde{R}(t)E^{\top}_{1_{k_1}}(t) &=& 0, \quad k_1 = 2,\ldots,q-r, \\ \widetilde{R}(t)E^{\top}_{j_{k_1}}(t) &=& \beta_j^2(u)E^{\top}_{j_{k_1}}(t), \quad k_1 = 1,\ldots,n_j,\ j = 2,\ldots,l_1, \\ \widetilde{R}(t)E^{\perp}_{1_{k_2}}(t) &=& 0, \quad k_2 = 1,\ldots,r, \\ \widetilde{R}(t)E^{\perp}_{i_{k_2}}(t) &=& \alpha_i^2(u)E^{\perp}_{i_{k_2}}(t), \quad k_2 = 1,\ldots,m_i,\ i = 2,\ldots,l_2, \end{array} \tag{2.13}$$

and

$$\begin{array}{rcl} \widetilde{S}(t)E^{\top}_{1_{k_1}}(t) &=& 0, \quad k_1 = 2,\ldots,q-r, \\ \widetilde{S}(t)E^{\top}_{j_{k_1}}(t) &=& \beta_j(u)\tan(t\beta_j(u))E^{\top}_{j_{k_1}}(t), \quad k_1 = 1,\ldots,n_j,\ j = 2,\ldots,l_1, \\ \widetilde{S}(t)E^{\perp}_{1_{k_2}}(t) &=& -1/t\,E^{\perp}_{1_{k_2}}(t), \quad k_2 = 1,\ldots,r, \\ \widetilde{S}(t)E^{\perp}_{i_{k_2}}(t) &=& -\alpha_i(u)\cot(t\alpha_i(u))E^{\perp}_{i_{k_2}}(t), \quad k_2 = 1,\ldots,m_i,\ i = 2,\ldots,l_2, \end{array} \tag{2.14}$$

Moreover, when the first conjugate locus of $\widetilde{P}$ and the cut-focal locus of $\widetilde{P}$ agree (see examples in [2]), from (2.6), the minimal focal distance of $\widetilde{P}$ in $\widetilde{M}$, $c(\widetilde{P}) = \min\{c(u)\ /\ u \in \mathcal{N}_p\widetilde{P}\}$, is given by

$$c(\widetilde{P}) = \inf\left\{\frac{\pi}{2\beta_j(u)},\ \frac{\pi}{\alpha_i(u)}\ /\ u \in \mathbf{a} \text{ and } \|u\| = 1\right\}. \tag{2.15}$$

In the following we will identify T_pM with $T_{p'}\widetilde{M}$ and T_pP with $T_{p'}\widetilde{P}$, and, given a unitary vector $u \in V_1 \subset \mathcal{N}_pP$, having in mind that $\mathbf{p}^{\perp} = \cup_k \mathrm{Ad}(k)\mathbf{a}$, we can suppose that $u \in \mathbf{a}$ with restricted roots $\{\alpha_i(u),\ \beta_j(u)\}$.

3 Mean curvature comparison

Now, we will obtain two comparison theorems when the pair $(\widetilde{M}, \widetilde{P})$ is considered as a model.

Without lost of generality we will suppose an arrangement of the roots $\{\alpha_i(u),\ \beta_j(u)\}$ with $u \in \mathbf{a} \subset \mathbf{h}$, such that

$$0 = \alpha_1(u) < \alpha_2(u) \leq \alpha_3(u) \leq \ldots \leq \alpha_{l_2}(u), \tag{3.1}$$

$$0 = \beta_1(u) < \beta_2(u) \leq \beta_3(u) \leq \ldots \leq \beta_{l_1}(u). \tag{3.2}$$

Theorem 2.1. *Let P and M be as in the preceding section and let P be a totally geodesic submanifold of M. Suppose that given a unitary vector $u \in \mathcal{N}_pP$, the following conditions are satisfied for each $t \in [0, t_0]$ with $t_0 < c(P)$,*

1. $K(\gamma_u'(t), H_1^t) \geq 0.$
2. $K(\gamma_u'(t), V_1^t) \geq 0.$
3. $K(\gamma_u'(t), H_j^t) \geq n_j\beta_j^2(u), \quad j = 3, \ldots, l_1.$
4. $K(\gamma_u'(t), V_i^t) \geq m_i\alpha_i^2(u), \quad i = 3, \ldots, l_2.$
5. $\displaystyle\sum_{j=2}^{l_1} K(\gamma_u'(t), H_j^t) \geq \sum_{j=2}^{l_1} n_j\beta_j^2(u).$
6. $\displaystyle\sum_{i=2}^{l_2} K(\gamma_u'(t), V_i^t) \geq \sum_{i=2}^{l_2} m_i\alpha_i^2(u).$

Then, $\operatorname{tr} S(t) \geq \operatorname{tr} \widetilde{S}(t)$ *for all* $t \in [0, t_0]$ *with* $t_0 < \inf\left\{\dfrac{\pi}{2\beta_j(u)}, \dfrac{\pi}{\alpha_i(u)}\right\}$.

Proof. Fix $t \in [0, t_0]$ and let $\{E_{j_{k_1}}^\top(s), E_{i_{k_2}}^\perp(s)\}$ defined, for $s \in [0, t]$, as in the preceding section. Let $\{Y_{j_{k_1}}^\top(s), Y_{i_{k_2}}^\perp(s)\}$ be the P-Jacobi fields along $\gamma_u(s)$ satisfying $Y_{j_{k_1}}^\top(t) = E_{j_{k_1}}^\top(t)$ and $Y_{i_{k_2}}^\perp(t) = E_{i_{k_2}}^\perp(t)$. We define vector fields along $\gamma_u(s)|_{[0,t]}$ by

$$\begin{aligned}
Z_{1_{k_2}}^\perp(s) &= \frac{s}{t} E_{1_{k_2}}^\perp(s), \quad k_2 = 2, \ldots, r.\\
Z_{1_{k_1}}^\top(s) &= E_{1_{k_1}}^\top(s), \quad k_1 = 2, \ldots, q - r.\\
Z_{i_{k_2}}^\perp(s) &= s_{\alpha_i} E_{i_{k_2}}^\perp(s), \quad k_2 = 1, \ldots, m_i,\ i = 2, \ldots, l_2.\\
Z_{j_{k_1}}^\top(s) &= c_{\beta_j} E_{j_{k_1}}^\top(s), \quad k_1 = 1, \ldots, n_j,\ j = 2, \ldots, l_1.
\end{aligned}$$

where

$$s_{\alpha_i} = \frac{\sin(s\alpha_i(u))}{\sin(t\alpha_i(u))}, \qquad c_{\beta_j} = \frac{\cos(s\beta_j(u))}{\cos(t\beta_j(u))}. \tag{3.3}$$

From the Index Lemma for submanifolds ([1, page 228]) we have

$$
\begin{aligned}
I_0^t(Y_{j_{k_1}}^{\top}) &\le I_0^t(Z_{j_{k_1}}^{\top}),\\
I_0^t(Y_{i_{k_2}}^{\perp}) &\le I_0^t(Z_{i_{k_2}}^{\perp}),
\end{aligned}
\tag{3.4}
$$

where I_0^t denotes the index form. Moreover,

$$
\begin{aligned}
\operatorname{tr} S(t) \quad = \quad & -\sum_{1_{k_2}}^{r} I_0^t(Y_{1_{k_2}}^{\perp}) - \sum_{1_{k_1}}^{q-r} I_0^t(Y_{1_{k_1}}^{\top})\\
& -\sum_{i=2}^{l_2}\sum_{k_2=1}^{m_i} I_0^t(Y_{i_{k_2}}^{\perp}) - \sum_{j=2}^{l_1}\sum_{k_1=1}^{n_j} I_0^t(Y_{j_{k_1}}^{\top}).
\end{aligned}
\tag{3.5}
$$

Therefore, from (3.4) and (3.5),

$$
\begin{aligned}
\operatorname{tr} S(t) \quad \ge \quad & -\sum_{1_{k_2}}^{r} I_0^t(Z_{1_{k_2}}^{\perp}) - \sum_{1_{k_1}}^{q-r} I_0^t(Z_{1_{k_1}}^{\top})\\
& -\sum_{i=2}^{l_2}\sum_{k_2=1}^{m_i} I_0^t(Z_{i_{k_2}}^{\perp}) - \sum_{j=2}^{l_1}\sum_{k_1=1}^{n_j} I_0^t(Z_{j_{k_1}}^{\top}).
\end{aligned}
$$

Since P is totally geodesic,

$$
\sum_{1_{k_1}}^{q-r} < Z_{1_{k_1}}^{\top}, L_u Z_{1_{k_1}}^{\top} > (0) + \sum_{j=2}^{l_1}\sum_{k_1=1}^{n_j} < Z_{j_{k_1}}^{\top}, L_u Z_{j_{k_1}}^{\top} > (0) = 0,
\tag{3.6}
$$

where L_u denotes the Weingarten map of P along the normal vector u; in fact, the Theorem is also true if each of the sums in (3.6) is zero. Therefore,

$$
\begin{aligned}
\operatorname{tr} S(t) \quad \ge \quad & -\int_0^t \left\{ (r-1)\frac{1}{t^2} + \sum_{i=2}^{l_2} m_i {s'}_{\alpha_i}^2 + \sum_{j=2}^{l_1} n_j {c'}_{\beta_j}^2 \right\} \mathrm{d}s\\
& + \int_0^t \left\{ \frac{s}{t} K(\gamma_u', V_1^t) + K(\gamma_u', H_1^t) \right\} \mathrm{d}s\\
& + \int_0^t \left\{ \sum_{i=2}^{l_2} s_{\alpha_i}^2 K(\gamma_u', V_i^t) + \sum_{j=2}^{l_1} c_{\beta_j}^2 K(\gamma_u', H_j^t) \right\} \mathrm{d}s\\
= \quad & -\int_0^t \left\{ (r-1)\frac{1}{t^2} + \sum_{i=2}^{l_2} m_i {s'}_{\alpha_i}^2 + \sum_{j=2}^{l_1} n_j {c'}_{\beta_j}^2 \right\} \mathrm{d}s\\
& + \int_0^t \left\{ \frac{s}{t} K(\gamma_u', V_1^t) + K(\gamma_u', H_1^t) \right\} \mathrm{d}s\\
& + \int_0^t \left\{ s_{\alpha_2}^2 \sum_{i=2}^{l_2} K(\gamma_u', V_i^t) + c_{\beta_2}^2 \sum_{j=2}^{l_1} K(\gamma_u', H_j^t) \right\} \mathrm{d}s\\
& + \int_0^t \left\{ \sum_{i=3}^{l_2} (s_{\alpha_i}^2 - s_{\alpha_2}^2) K(\gamma_u', V_i^t) + \sum_{j=3}^{l_1} (c_{\beta_j}^2 - c_{\beta_2}^2) K(\gamma_u', H_j^t) \right\} \mathrm{d}s.
\end{aligned}
$$

Finally, from conditions 1.-6. and having in mind that $s^2_{\alpha_i} \geq s^2_{\alpha_2}$ and $c^2_{\beta_j} \geq c^2_{\beta_2}$ for $0 < s \leq t < \inf\left\{\frac{\pi}{2\beta_j}, \frac{\pi}{\alpha_i}\right\}$, we have

$$\begin{aligned}
\operatorname{tr} S(t) \quad \geq \quad & -\int_0^t \left\{(r-1)\frac{1}{t^2} + \sum_{i=2}^{l_2} m_i {s'}^2_{\alpha_i} + \sum_{j=2}^{l_1} n_j {c'}^2_{\beta_j}\right\} \mathrm{d}s \\
& + \int_0^t \left\{ s^2_{\alpha_2} \sum_{i=2}^{l_2} m_i \alpha_i^2 + c^2_{\beta_2} \sum_{j=2}^{l_1} n_j \beta_j^2 \right\} \mathrm{d}s \\
& + \int_0^t \left\{ \sum_{i=3}^{l_2} (s^2_{\alpha_i} - s^2_{\alpha_2}) m_i \alpha_i^2 + \sum_{j=3}^{l_1} (c^2_{\beta_j} - c^2_{\beta_2}) n_j \beta_j^2 \right\} \mathrm{d}s = \operatorname{tr} \widetilde{S}(t).
\end{aligned}$$

□

Corollary 2.1. *Under the hypotheses of Theorem 2.1, replacing conditions 3.-4. by one of the following group of conditions*:
Group 1.
3. $K(\gamma_u'(t), H_j^t) \leq n_j \beta_j^2(u), \quad j = 2, \ldots, l_1 - 1,$
4. $K(\gamma_u'(t), V_i^t) \geq m_i \alpha_i^2(u), \quad i = 3, \ldots, l_2,$

Group 2.
3. $K(\gamma_u'(t), H_j^t) \geq n_j \beta_j^2(u), \quad j = 3, \ldots, l_1,$
4. $K(\gamma_u'(t), V_i^t) \leq m_i \alpha_i^2(u), \quad i = 2, \ldots, l_2 - 1,$

Group 3.
3. $K(\gamma_u'(t), H_j^t) \leq n_j \beta_j^2(u), \quad j = 2, \ldots, l_1 - 1,$
4. $K(\gamma_u'(t), V_i^t) \leq m_i \alpha_i^2(u), \quad i = 2, \ldots, l_2 - 1,$

we obtain the same comparison result for tr $S(t)$.

Theorem 2.2. *Let P and M be as in Theorem 2.1. Suppose that given a unitary vector $u \in \mathcal{N}_p P$, the following conditions are satisfied for each $t \in [0, t_0]$ with $t_0 < c(P)$,*

1. $K(\gamma_u'(t), H_1^t) \geq 0.$

2. $K(\gamma_u'(t), V_1^t) \geq 0.$

3. $K(\gamma_u'(t), H_j^t) \geq n_j \beta_j^2(u), \quad j = 2, \ldots, l_1.$

4. $K(\gamma_u'(t), V_i^t) \geq m_i \alpha_i^2(u), \quad i = 3, \ldots, l_2.$

5. $\displaystyle\sum_{j=2}^{l_1} K(\gamma_u'(t), H_j^t) + \sum_{i=2}^{l_2} K(\gamma_u'(t), V_i^t) \geq \sum_{j=2}^{l_1} n_j \beta_j^2(u) + \sum_{i=2}^{l_2} m_i \alpha_i^2(u).$

Then, $\operatorname{tr} S(t) \geq \operatorname{tr} \widetilde{S}(t)$ *forall* $t \in [0, t_0]$ *with* $t_0 < \inf\left\{\frac{\pi}{2\beta_j(u)}, \frac{\pi}{\alpha_i(u)}\right\}$.

Proof. As in Theorem 2.1, having into account that $c^2_{\beta_j} \geq s^2_{\alpha_2}$ and, from (3.1), $s^2_{\alpha_i} \geq s^2_{\alpha_2}$ for $0 < s \leq t < \inf\left\{\frac{\pi}{2\beta_j(u)}, \frac{\pi}{\alpha_i(u)}\right\}$.

□

Corollary 2.2. *Under the hypotheses of Theorem 2.2, replacing conditions 3.-4. by one of the following group of conditions*:

Group 1.

3. $K(\gamma'_u(t), H^t_j) \geq n_j\beta^2_j(u), \quad j = 3, \ldots, l_1.$
4. $K(\gamma'_u(t), V^t_i) \leq m_i\alpha^2_i(u), \quad i = 2, \ldots, l_2.$

Group 2.

3. $K(\gamma'_u(t), H^t_j) \geq n_j\beta^2_j(u), \quad j = 2, \ldots, l_1.$
4. $K(\gamma'_u(t), V^t_i) \leq m_i\alpha^2_i(u), \quad i = 2, \ldots, l_2 - 1.$

Group 3.

3. $K(\gamma'_u(t), H^t_j) \leq n_j\beta^2_j(u), \quad j = 2, \ldots, l_1 - 1.$
4. $K(\gamma'_u(t), V^t_i) \leq m_i\alpha^2_i(u), \quad i = 2, \ldots, l_2.$

we obtain the same comparison result for tr $S(t)$.

Remark 2.1. When the pair $(\widetilde{M}, \widetilde{P})$ is $(\mathbf{C}P^n(\lambda), \mathbf{C}P^s(\lambda))$, Theorem 2.2 and the group of conditions 1 and 2 of Corollary 2.2 give the cases b), a) and c) of Theorem 2.1 in [12], respectively.

Finally, from Theorem 2.1 or Theorem 2.2 we obtain the following result.

Corollary 2.3. *Let P and M be as in Theorem 2.1. Suppose that given a unitary vector $u \in \mathcal{N}_pP$, the following conditions are satisfied for each $t \in [0, t_0]$ with $t_0 < c(P)$,*

1. $K(\gamma'_u(t), H^t_1) \geq 0.$
2. $K(\gamma'_u(t), V^t_1) \geq 0.$
3. $K(\gamma'_u(t), H^t_j) \geq n_j\beta^2_j(u), \quad j = 2, \ldots, l_1.$
4. $K(\gamma'_u(t), V^t_i) \geq m_i\alpha^2_i(u), \quad i = 2, \ldots, l_2.$

Then, $\operatorname{tr} S(t) \geq \operatorname{tr} \widetilde{S}(t)$ *forall* $t \in [0, t_0]$ *with* $t_0 < \inf\left\{\frac{\pi}{2\beta_j(u)}, \frac{\pi}{\alpha_i(u)}\right\}$.

4 Application: Relative volume

Let (M, P) and $(\widetilde{M}, \widetilde{P})$ be as in the preceding sections and suppose that the first conjugate locus of $\widetilde{P}$ and the cut-focal locus of $\widetilde{P}$ agree (see examples in [2]). Then, if $\theta_u(t)$ and $\widetilde{\theta}_u(t)$ denote the infinitesimal change of volume functions of M and $\widetilde{M}$, respectively, from the hypotheses of Theorems 2.1 and 2.2 and Eq. (2.4), having in mind that $\theta_u(0) = \widetilde{\theta}_u(0) = 1$, we have $\theta_u(t) \leq \widetilde{\theta}_u(t)$ and, from (2.6) and (2.7), we have

$$c(u) \leq z(\theta_u(t)) \leq z(\widetilde{\theta}_u(t)) = \widetilde{c}(u), \tag{4.1}$$

and therefore,

$$\begin{aligned} \mathrm{Vol}(M) &= \int_P \int_{S^{n-s-1}(1)} \int_0^{c(u)} t^{n-s-1}\theta_u(t)\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}P \\ &\leq \int_P \int_{S^{n-s-1}(1)} \int_0^{\widetilde{c}(u)} t^{n-s-1}\widetilde{\theta}_u(t)\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}P. \\ &\leq (\mathrm{Vol}(P)/\mathrm{Vol}(\widetilde{P})) \int_{\widetilde{P}} \int_{S^{n-s-1}(1)} \int_0^{\widetilde{c}(u)} t^{n-s-1}\widetilde{\theta}_u(t)\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}\widetilde{P} \\ &= (\mathrm{Vol}(P)/\mathrm{Vol}(\widetilde{P}))\mathrm{Vol}(\widetilde{M}). \end{aligned} \tag{4.2}$$

So, we conclude the following inequality between the relative volumes:

Theorem 3.1. *Let* (M, P) *and* $(\widetilde{M}, \widetilde{P})$ *as before; then,*

$$\frac{\mathrm{Vol}(\widetilde{P})}{\mathrm{Vol}(\widetilde{M})} \leq \frac{\mathrm{Vol}(P)}{\mathrm{Vol}(M)}. \tag{4.3}$$

Appendix. Suppose now that $\widetilde{M}$ is a noncompact symmetric space; then, having into account the duality between compact and noncompact symmetric spaces, the main differences with the compact case are that the Ricci curvatures defined in (2.11) and (2.12) are negative and all the trigonometric functions which appear for the compact case have to be changed to hyperbolic functions. Therefore, the comparison results in Section 2 remain valid for noncompact symmetric spaces but some of the inequalities imposed in the different conditions change. For instance, in Theorem 2.1, Corollary 2.1 and Corollary 2.3, having in mind that $s^2_{\alpha_i} \leq s^2_{\alpha_2}$ and $c^2_{\beta_j} \leq c^2_{\beta_2}$ for $0 < s \leq t < \inf\left\{\frac{\pi}{2\beta_j}, \frac{\pi}{\alpha_i}\right\}$, when hyperbolic functions are considered, we have to change all the inequalities $\leq$ by $\geq$ and vice versa, to obtain the comparison results for $\operatorname{tr} S(t)$. But, in Theorem 2.2 and Corollary 2.2, some of the inequalities change but other remain valid because $c^2_{\beta_j} \leq s^2_{\alpha_i}$ when $0 < s \leq t < \inf\left\{\frac{\pi}{2\beta_j}, \frac{\pi}{\alpha_i}\right\}$, also for hyperbolic functions.

However, concerning the application in Section 4, when $\widetilde{M}$ is noncompact, the totally geodesic submanifold $\widetilde{P}$ of $\widetilde{M}$ is also a noncompact symmetric space, therefore, Theorem 3.1 has no sense because we can not obtain finite values for the different volumes, [8].

Acknowledgements.Work partially supported by a DGES Grant BSA2001-0803-C02-02.

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