Semi-Invariant Submanifolds of Riemannian Product Manifold

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Dedicated to the Memory of Grigorios TSAGAS (1935-2003), President of Balkan Society of Geometers (1997-2003)

Abstract

In this paper, the geometry of submanifolds of a Riemannian product manifold is studied. Fundamental properties of these submanifolds are investigated such as integrability of distributions, totally umbilical semi-invariant submanifold. Finally, necessary and sufficient conditions are given on a semi-invariant submanifold of a Riemannian product manifold to be a locally Riemannian manifold.

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1 Introduction

The geometry of a submanifold (M, g) of a locally Riemannian product manifold $(\overline{M}_1 \times \overline{M}_2, \overline{g}_1 \otimes \overline{g}_2)$ was widely studied by many geometers. In particularly, K. Matsumoto has proved that (M, g) is a locally product Riemannian manifold of Riemannian manifolds (M_a, g_a) and (M_b, g_b), if it is an invariant submanifold of a Riemannian product manifold $(\overline{M}_1 \times \overline{M}_2, \overline{g}_1 \otimes \overline{g}_2)$(see [4]). After then Senlin, Xu., and Yilong, Ni., have updated theorem of Matsumoto and proved that $M_a \subset \overline{M}_1$ and $M_b \subset \overline{M}_2$. Moreover, they have proved that (M_a, g_a) and (M_b, g_b) are pseudo-umbilical submanifolds of $(\overline{M}_1, \overline{g}_1)$ and $(\overline{M}_2, \overline{g}_2)$, respectively, if (M, g) is a pseudo-umbilical submanifold of $(\overline{M}, \overline{g}) = (\overline{M}_1 \times \overline{M}_2, \overline{g}_1 \otimes \overline{g}_2)$. They have also demonstrated that M is isometric to the production of its two totally geodesic submanifolds (M_a, g_a) and (M_b, g_b) which are submanifolds of $(\overline{M}_1, \overline{g}_1)$ and $(\overline{M}_2, \overline{g}_2)$, respectively (see [5]).

In this work, we study the geometry of semi-invariant submanifolds of a Riemannian manifold and proved that a semi-invariant submanifold of a Riemannian product manifold is a locally Riemannian product manifold iff $A_{FD^{\perp}}D = 0$, which is equivalent to $\nabla f = 0$, or $Bh(X, Y) = 0$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$. Moreover, necessary and sufficient conditions are given on distributions D and $D^{\perp}$ of

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a semi-invariant submanifold M are integrable. Finally, we show that there exists no totally umbilical semi-invariant submanifold of positively or negatively curved Riemannian product manifold. Also we give an example for semi-invariant submanifold to illustrate the our results.

2 Preliminaries

In this section, we give some notations and terminology used througthout this paper. We recall some necessary facts and formulas from the theory of submanifolds. For an arbitrary submanifold M of a Riemannian manifold $\overline{M}$, Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$

and

$$\overline{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi, \tag{2.2}$$

respectively, where $\overline{\nabla}$, ∇ are Levi-Civita connections on the Riemannian manifolds $\overline{M}$ and its submanifold M, respectively, and X, Y are vector fields tangent to M, ξ is a vector field normal to M, $h : TM \times TM \longrightarrow TM^\perp$ is the second fundamental form of M, $\nabla^\perp$ is the normal connection in the normal vector bundle $TM^\perp$, and A_ξ is the shape operator of the second quadratic form for a normal vector ξ. Moreover, we have

$$g(A_\xi X, Y) = \overline{g}(h(X,Y), \xi), \tag{2.3}$$

where the symbols $\overline{g}$ and g mean the Riemannian metrics of $\overline{M}$ and its submanifold M, respectively.

We denote the Riemannian curvature tensors of the Levi-Civita connections $\overline{\nabla}$ and ∇ on $\overline{M}$ and M by $\overline{R}$ and R, respectively. The Gauss, Codazzi, and Ricci equations are given by

$$\begin{aligned} \overline{g}(\overline{R}(X,Y)Z,W) &= \overline{g}(R(X,Y)Z,W) + \overline{g}(h(X,W),h(Y,Z)) \\ &- \overline{g}(h(X,Z),h(Y,W)) \end{aligned} \tag{2.4}$$

$$(\overline{R}(X,Y)Z)^\perp = (\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_Y h)(X,Z), \tag{2.5}$$

$$\overline{g}(\overline{R}(X,Y)\xi,\eta) = \overline{g}(\overline{R}^\perp(X,Y)\xi,\eta) - \overline{g}([A_\xi, A_\eta]X,Y) \tag{2.6}$$

respectively, where the vector fields X, Y, Z, W are tangent to M, the vector fields ξ and η are orthogonal to M, $(\overline{R}(X,Y)Z)^\perp$ denotes the normal of $\overline{R}(X,Y)Z$ and the derivative $\overline{\nabla}h$ is defined by

$$(\overline{\nabla}_X h)(Y,Z) = (\nabla^\perp_X h)(Y,Z) - h(\nabla_X Y, Z) - h(\nabla_X Z, Y). \tag{2.7}$$

We recall that M is called a curvature-invariant submanifold, if it has

$$(\overline{R}(X,Y)Z)^{\perp} = 0, \tag{2.8}$$

which is equivalent to

$$(\overline{\nabla}_X h)(Y,Z) = (\overline{\nabla}_Y h)(X,Z),$$

for all $X, Y, Z \in \Gamma(TM)$ [3].

Definition 2.1 *For a submanifold $M \subseteq \overline{M}$ the mean-curvature vector field H is defined by the formula*

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \tag{2.9}$$

where $\{e_i\}$ is a local orthonormal basis in TM. If a submanifold $M \subseteq \overline{M}$ having one of the conditions

$$h = 0, h(X,Y) = g(X,Y)H, \quad g(h(X,Y),H) = \lambda g(X,Y),$$

$$H = 0, \lambda \in C^{\infty}(M,R), \tag{2.10}$$

then it is called totally geodesic, totally umbilical, pseudo-umbilical and minimal, respectively [2].

Let $(\overline{M}_1, \overline{g}_1)$ and $(\overline{M}_2, \overline{g}_2)$ be Riemannian manifolds with dimensions n_1 and n_2, respectively. Then $\overline{M} = \overline{M}_1 \times \overline{M}_2$ is the Riemannian product manifold of Riemannian manifolds $\overline{M}_1$ and $\overline{M}_2$. We denote the projection mappings of $T(\overline{M}_1 \times \overline{M}_2)$ to $T\overline{M}_1$ and $T\overline{M}_2$ by π_* and σ_*, respectively. Then we have

$$\pi_* + \sigma_* = I, \pi_*^2 = \pi_*, \sigma_*^2 = \sigma_*, \pi_* \times \sigma_* = \sigma_* \times \pi_* = 0. \tag{2.11}$$

Then the Riemannian metric of $\overline{M}_1 \times \overline{M}_2$ is given by

$$\overline{g}(X,Y) = \overline{g}_1(\pi_* X, \pi_* Y) + \overline{g}_2(\sigma_* X, \sigma_* Y) \tag{2.12}$$

for all $X, Y \in \Gamma(T(\overline{M}_1 \times \overline{M}_2))$. Set $F = \pi_* - \sigma_*$, then we can easily see that $F^2 = I$. It follows

$$\overline{g}(X,Y) = \overline{g}(FX, FY) \tag{2.13}$$

for all $X, Y \in \Gamma(T(\overline{M}_1 \times \overline{M}_2))$.

By the definition of $\overline{g}$, $\overline{M}_1$ and $\overline{M}_2$ are totally geodesic submanifolds of $\overline{M}_1 \times \overline{M}_2$. We denote the Levi-Civita connection of $\overline{M}_1 \times \overline{M}_2$ by $\overline{\nabla}$, we can easily see that

$$(\overline{\nabla}_X F)Y = 0. \tag{2.14}$$

for any $X, Y \in \Gamma(T(\overline{M}_1 \times \overline{M}_2))$(For the detail, we refer to[5]).

3 Semi-invariant submanifold of a Riemannian product manifold

We denote the Riemannian product manifold $(\overline{M}_1 \times \overline{M}_2, \overline{g}_1 \times \overline{g}_2)$ by $(\overline{M}, \overline{g})$ througthout this paper.

Definition 3.1 *let M be a submanifold of a Riemannian product manifold $\overline{M}$. We suppose that M has two the distributions such as D and $D^{\perp}$ such that $TM = D \oplus D^{\perp}$, $F(D) = D$ and $F(D^{\perp}) \subset TM^{\perp}$. In this case, M is called semi-invariant submanifold of $\overline{M}$.*

In the rest of this paper, we assume that M semi-invariant submanifold of $\overline{M}$. We denote the orthogonal complementary of $F(D^{\perp})$ in $TM^{\perp}$ by V, then we have direct sum

$$TM^{\perp} = F(D^{\perp}) \oplus V.$$

We denote the projection mappings of TM to D and $D^{\perp}$ by P and Q, respectively. Then for each X tangent to TM, we can write FX in the following way:

$$FX = fX + \omega X, \tag{3.1}$$

where $fX = FPX$ and $\omega X = FQX$ are respectively the tangent part and the normal part of FX. Also, for each vector field ξ normal to M, we put

$$F\xi = B\xi + C\xi, \tag{3.2}$$

where $B\xi$ and $C\xi$ are respectively the tangent part and the normal part of $F\xi$.

We denote dimensions of the distributions D and $D^{\perp}$ by p and q, respectively. Then for $q = 0$(resp. $p = 0$) a semi-invariant submanifold becomes an invariant submanifold(resp. an anti-invariant submanifold). A proper semi-invariant submanifold is a semi-invariant submanifold which is neither an invariant submanifold nor an anti-invariant submanifold.

Example 3.2 *We consider a submanifold M in R^6 given by the equations:*

$$X_1 = X_6 + \frac{1}{2}(X_3 + X_4)^2, X_2 = X_5.$$

It is easy check that M is a semi-invariant submanifold of $R^6 = R^3 \times R^3$. Then by direct calculation we obtain

$$TM = Span\{U_1 = \frac{\partial}{\partial X_2} + \frac{\partial}{\partial X_5}, U_2 = (X_3 + X_4)\frac{\partial}{\partial X_1} + \frac{\partial}{\partial X_3},$$
$$U_3 = (X_3 + X_4)\frac{\partial}{\partial X_1} + \frac{\partial}{\partial X_4}, U_4 = \frac{\partial}{\partial X_1} + \frac{\partial}{\partial X_6}\}$$

and

$$TM^{\perp} = Span\{\xi_1 = -\frac{\partial}{\partial X_1} + (X_3 + X_4)\frac{\partial}{\partial X_3} + (X_3 + X_4)\frac{\partial}{\partial X_4} + \frac{\partial}{\partial X_6},$$
$$\xi_2 = \frac{\partial}{\partial X_2} - \frac{\partial}{\partial X_5}\},$$

where $D = Span\{U_2, U_3, U_4\}$ and $D^{\perp} = Span\{U_1\}$.

Definition 3.3 *Let $\overline{M}$ be a Riemannian product manifold and M be a semi-invariant submanifold of $\overline{M}$. Then M is called mixed-geodesic semi-invariant submanifold if $h(X,Y) = 0$ for any $X \in \Gamma(D)$ and $Y \in \Gamma(D^{\perp})$.*

We denote the Levi-Civita connections on M and $\overline{M}$ by ∇ and $\overline{\nabla}$, respectively.

Proposition 3.4 *Let $\overline{M}$ be a Riemannian product manifold and M be a semi-invariant submanifold of $\overline{M}$. Then we have*

$$A_{FZ}W = -A_{FW}Z, \tag{3.3}$$

for all $Z, W \in \Gamma(D^{\perp})$

Proof. From (2.1), (2.2), (2.14) and (3.1) we have

$$-A_{FZ}X + \nabla^{\perp}_{X}FZ = F\nabla_X Z + Fh(X,Z)$$

for any $X \in \Gamma(TM)$ and $Z \in \Gamma(D^{\perp})$. Using (2.13) we obtain

$$-\overline{g}(A_{FZ}X, W) = \overline{g}(h(X,Z), FW),$$

for any $W \in \Gamma(D^{\perp})$. Since A is self adjoint, from (2.3) we get

$$-\overline{g}(A_{FZ}W, X) = \overline{g}(A_{FW}Z, X),$$

which proves our assertion.

Lemma 3.5 *Let $\overline{M}$ be a Riemannian product manifold and M be a semi-invariant submanifold of $\overline{M}$. Then we have*

$$A_{\xi}FX = A_{F\xi}X \tag{3.4}$$

for any $X \in \Gamma(D)$ and $\xi \in \Gamma(V)$.

Proof. Since $\overline{\nabla}$ is the Levi-Civita connection, from (2.14) we derive

$$\overline{g}(h(FX,Y),\xi) = -\overline{g}(\overline{\nabla}_Y F\xi, X),$$

for any $X \in \Gamma(D)$, $Y \in \Gamma(TM)$ and $\xi \in \Gamma(V)$. Using (2.2) and (2.3) we get

$$\overline{g}(A_{\xi}FX, Y) = \overline{g}(A_{F\xi}X, Y).$$

Thus proof is complete.

Lemma 3.6 *Let $\overline{M}$ be a Riemannian product manifold and M be a semi-invariant submanifold of $\overline{M}$. Then we have*

$$\nabla^{\perp}_{Z}FW - \nabla^{\perp}_{W}FZ \in \Gamma(D^{\perp}), \tag{3.5}$$

for any $Z, W \in \Gamma(D^{\perp})$.

Proof. From (2.1) and (2.2) we have

$$\overline{g}(A_{F\xi}Z,W)=\overline{g}(\nabla^{\perp}_{Z}FW,\xi) \tag{3.6}$$

for any $W,Z\in\Gamma(D^{\perp})$ and $\xi\in\Gamma(V)$. Since A is self adjoint, from (3.6) we get

$$\overline{g}(\nabla^{\perp}_{Z}FW-\nabla^{\perp}_{W}FZ,\xi)=0,$$

which gives (3.5).

Theorem 3.7 *Let $\overline{M}$ be a Riemannian product manifold and M be a semi-invariant submanifold of $\overline{M}$. Then $D^{\perp}$ is integrable if and only if*

$$h(X,W)\in\Gamma(V) \tag{3.7}$$

for any $X\in\Gamma(D)$ and $W\in\Gamma D^{\perp}$.

Proof. From (2.2), (2.14) and (3.3) we get

$$F[Z,W]=2A_{FZ}W+\nabla^{\perp}_{Z}FW-\nabla^{\perp}_{W}FZ$$

for any $Z\in\Gamma(D^{\perp})$. Thus from (2.3) and (2.13) we derive

$$\overline{g}([Z,W],FX)=2\overline{g}(h(W,X),FZ).$$

Hence the proof is complete.

Theorem 3.8 *Let $\overline{M}$ be a Riemannian product manifold and M be a semi-invariant submanifold of $\overline{M}$. Then D is integrable if and only if*

$$h(X,FY)=h(Y,FX) \tag{3.8}$$

for any $X,Y\in\Gamma(D)$.

Proof. By using (2.1), (2.2), (2.14) and (3.1) we derive

$$\nabla_{X}FY+h(X,FY)=P\nabla_{X}Y+\omega\nabla_{X}Y+Fh(X,Y),$$

where interchanging role of vector fields X and Y, we obtain

$$\nabla_{Y}FX+h(Y,FX)=P\nabla_{Y}X+\omega\nabla_{Y}X+Fh(Y,X).$$

Thus we have

$$h(X,FY)-h(FX,Y)=\omega([X,Y]).$$

This completes the proof of the theorem.

Lemma 3.9 *Let $\overline{M}$ be a Riemannian product manifold and M be a mixed-geodesic semi-invariant submanifold of $\overline{M}$. Then we have*

$$A_{F\xi}X=FA_{\xi}X \tag{3.9}$$

for any $X\in\Gamma(D)$ and $\xi\in\Gamma(V)$.

Proof. From (2.1) and (2.2) we have

$$\overline{g}(A_{F\xi}X - FA_{\xi}X, Y) = \overline{g}(A_{F\xi}X, Y) - \overline{g}(A_{\xi}X, FY)$$

for any $X \in \Gamma(D)$, $Y \in \Gamma(D^{\perp})$ and $\xi \in \Gamma(V)$. Since M is a mixed-geodesic submanifold, we have $A_{F\xi}X \in \Gamma(D)$. Thus using the equation (2.3) we obtain

$$\overline{g}(A_{F\xi}X - FA_{\xi}, Y) = 0.$$

On the other hand, from (2.3) we get

$$\overline{g}(A_{F\xi}X - FA_{\xi}X, Z) = \overline{g}(h(X, Z), F\xi) - \overline{g}(h(X, FZ), \xi)$$

for any $X, Z \in \Gamma(D)$. Thus from (2.14) we derive

$$\overline{g}(A_{F\xi}X - FA_{\xi}, Z) = 0,$$

which proves our assertion.

Theorem 3.10 *Let $\overline{M}$ be a Riemannian product manifold and M be a semi-invariant submanifold of $\overline{M}$. Then M is a locally Riemannian product manifold if and only if $A_{FZ}X = 0$ for all $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.*

Proof. Let M be a semi-invariant submanifold of a Riemannian product manifold $(\overline{M}, \overline{g})$. Then from (2.1) and (2.2) we have

$$\overline{g}(\nabla_X FY, Z) = \overline{g}(A_{FZ}X, Y) \tag{3.10}$$

and

$$\overline{g}(\nabla_W Z, FX) = -\overline{g}(A_{FZ}X, W) \tag{3.11}$$

for any $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(D^{\perp})$. Now, we suppose that M is a locally Riemannian product manifold. Then the distributions D and $D^{\perp}$ are parallel. From (3.10) and (3.11) we have $A_{FZ}X \in \Gamma(D)$ and $A_{FZ}X \in \Gamma(D^{\perp})$. Since $D \cap D^{\perp} = \{0\}$ we obtain $A_{FZ}X = 0$.

Conversely, if $A_{FZ}X = 0$, then from (3.10) and (3.11) we have the distributions D and $D^{\perp}$ are integrable and leaves of them are parallel. This completes the proof of the theorem.

Proposition 3.11 *Any pseudo umbilical proper semi-invariant submanifold of a Riemannian product manifold is a mixed-geodesic submanifold.*

Proof. We suppose that M is a pseudo-umbilical proper semi-invariant submanifold of a Riemannian product manifold $(\overline{M}, \overline{g})$. Then we have

$$\overline{g}(h(X, Z), H) = \overline{g}(H, H)g(X, Z) = 0,$$

for all $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$, which implies that $h(X, Z) = 0$.

Theorem 3.12 *Let $\overline{M}$ be a Riemannian product manifold and M be a semi-invariant submanifold of $\overline{M}$. Then M is a locally Riemannian product manifold if and only if $\nabla f = 0$.*

Proof. Let M be a locally Riemannian product semi-invariant submanifold of $\overline{M}$. Then we have $\nabla_U Y \in \Gamma(D)$ for all $U \in \Gamma(TM)$ and $Y \in \Gamma(D)$. Thus from (2.2) and (3.1) we obtain

$$h(U, FY) = F\nabla_U Y + Bh(U, FY) + Ch(U, FY) - \nabla_U FY$$

for any $U \in \Gamma(TM)$ and $Y \in \Gamma(D)$. Hence we get

$$\begin{aligned} h(U, FY) &= Ch(U, FY) \\ (\nabla_U f)Y &= 0 \\ Bh(U, FY) &= 0. \end{aligned} \tag{3.12}$$

In the similar way, we obtain $(\nabla_U f)Z = 0$ for any $Z \in \Gamma(D^{\perp})$.

Conversely, we suppose that $\nabla f = 0$. Then we have $\nabla_X fY = f\nabla_X Y$, for any $X, Y \in \Gamma(D)$. It follows that $\nabla_X Y \in \Gamma(D)$. In the similar way $\nabla_Z W \in \Gamma(D^{\perp})$ for any $Z, W \in \Gamma(D^{\perp})$. Thus M is a locally Riemannian product manifold.

Theorem 3.13 *Let M be a semi-invariant submanifold of a Riemannian product manifold $\overline{M}$. Then M is a locally Riemannian product manifold if and only if $Bh(X, Y) = 0$ for all $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$.*

Proof. We assume that M is a locally Riemannian product manifold. Then from (3.12) we have $Bh(X, Y) = 0$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$.

Conversely, we assume that $Bh(X, Y) = 0$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$. Then from (2.1), (2.14), (3.1) and (3.2) we get

$$\nabla_X fY + h(X, FY) = f\nabla_X Y + \omega\nabla_X Y + Bh(X, Y) + Ch(X, Y)$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$. Thus we derive $(\nabla_X f)Y = 0$, that is, $\nabla_X Y \in \Gamma(D)$. On the other hand, making use of (2.1), (2.2), (2.14), (3.1) and (3.2) we obtain

$$-A_{FZ}X + \nabla^{\perp}_X FZ = f\nabla_X Z + \omega\nabla_X Z + Bh(X, Z) + Ch(X, Z)$$

for any $X \in \Gamma(TM)$ and $Z \in \Gamma(D^{\perp})$. Thus we obtain

$$-A_{FZ}X = f\nabla_X Z \tag{3.13}$$

for any $X \in \Gamma(TM)$ and $Z \in \Gamma(D^{\perp})$. By the using (2.3), (2.13) and (3.2) we derive

$$\overline{g}(f\nabla_X Z, Y) = -\overline{g}(Ch(X, Y), Z) = 0,$$

for all $X \in \Gamma(TM)$, $Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$. Hence we have $\nabla_X Z \in \Gamma(D^{\perp})$. Thus proof is complete.

In case $F(D^{\perp}) = TM^{\perp}$, we can give the following theorem.

Theorem 3.14 *Let $\overline{M}$ be a Riemannian product manifold and M be a semi-invariant submanifold of $\overline{M}$ such that $F(D^{\perp}) = TM^{\perp}$. Then M is a locally Riemannian product manifold if and only if $h(X,Y) = 0$ for all $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$.*

Theorem 3.15 *Let M be a totally umbilical proper semi-invariant submanifold of a Riemannian product manifold $\overline{M}$. Then one only of the following assertions are valid:*
1) $dim D^{\perp} = 1$
2) M is a totally geodesic submanifold.

Proof. We suppose that M is a totally umbilical submanifold of a Riemannian product manifold $\overline{M}$. Then from (2.1) and (2.2) and (3.1) we have

$$-\overline{g}(A_{FW}Z, Z) = \overline{g}(Fh(Z,W), Z)$$

for all $Z, W \in \Gamma(D^{\perp})$. Since M is a totally umbilical submanifold, from (2.3) and (2.13) we obtain

$$-\overline{g}(Z,Z)\overline{g}(FH, W) = \overline{g}(Z,Z)\overline{g}(FH, Z), \tag{3.14}$$

where H is the mean curvature vector field of M in $\overline{M}$. Interchanging role of Z and W in (3.14) we get

$$-\overline{g}(W,W)\overline{g}(FH, Z) = \overline{g}(W,W)\overline{g}(FH, W). \tag{3.15}$$

Thus from (3.14) and (3.15) we obtain

$$\overline{g}(FH, Z) = \frac{\overline{g}(Z,W)^2}{\|Z\|^2\|W\|^2}\overline{g}(FH, Z). \tag{3.16}$$

Hence, either $\overline{g}(FH, Z) = 0$ or Z and W are linearly dependent. If Z and W are linearly dependent, then $\dim D^{\perp} = 1$.

We suppose that $\dim D^{\perp} > 1$. Then from (3.3) we have

$$A_{FBH}Z = -A_{FZ}BH$$

for any $Z \in \Gamma(D^{\perp})$. By the using (2.3) we get

$$-\overline{g}(Z,W)\overline{g}(BH, BH) = \overline{g}(BH, W)\overline{g}(H, FZ).$$

Since $\dim D^{\perp} > 1$, we can choose W orthogonal to BH. Then $BH = 0$, that is, $H \in \Gamma(V)$. Now we assume that $H \neq 0$. From (2.3) and (3.4) we derive

$$\overline{g}(FH, H)\overline{g}(X,Y) = \overline{g}(H,H)\overline{g}(FX, Y) \tag{3.17}$$

for any $X \in \Gamma(D)$. We note that the leaf of D is an invariant submanifold of Riemannian product manifold $\overline{M}$. We denote the leaf of D by N. Since N is an invariant submanifold of $\overline{M}$, it is a product manifold. Set $N = N_1 \times N_2$. Then we have

$$TN_1 = \{X \in \Gamma(TN) | FX = X\}$$

and

$$TN_2 = \{X \in \Gamma(TN) | FX = -X\}.$$

From (3.17) we obtain

$$\overline{g}(X, FX) = \overline{g}(X, X)$$

for any $X \in \Gamma(D)$. Thus for $X = X_2 \in \Gamma(TN_2)$, we have

$$-\overline{g}(X_2, X_2) = \overline{g}(X_2, X_2),$$

i.e.,

$$\|X_2\| = 0 \Longrightarrow X_2 = 0.$$

This is a contradiction.

Theorem 3.16 *There exists no any totally umbilical proper semi-invariant submanifold of positively or negatively curved Riemannian product manifold $\overline{M}$.*

Proof. We assume that Riemannian product manifold $\overline{M}$ has constant sectional curvature $c \neq 0$ and let M be a totally umbilical proper semi-invariant submanifold of $\overline{M}$. Then form the equations Gauss and Codazzi, we have

$$\begin{aligned} \overline{K}(X, Y, X, Y) &= \overline{K}(X, Y, FX, FY) = -\overline{g}(X, FX)\overline{g}(\nabla^{\perp}_{Y} H, FY) \\ \overline{K}(X \wedge Y) &= -\overline{g}(X, FX)\overline{g}(\nabla^{\perp}_{Y} H, FY). \end{aligned}$$

Since the vector fields X and FX are linearly independent, we can choose X orthogonal to FX. In this case, we obtain

$$\overline{K}(X \wedge Y) = 0.$$

This is a contradiction, where $\overline{K}$ denotes the Riemannian-Christoffel curvature tensor of $\overline{M}$.

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