Linearized geometric dynamics of Tobin-Benhabib-Miyao economic flow

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Abstract

The aim of this paper is to study the influence of the Euclidean-Lagrangian structure of the state space on the generalized Tobin flow in economics, confirming the existence of optimal economic fluctuations (small oscillations).

Section 1 reviews the generalized Tobin economic flow as formulated by Benhabib and Miyao. Section 2 recalls some tools of single-time geometric dynamics which describe a geodesic motion under a gyroscopic field of forces. Section 3 studies the linearized geometric dynamics produced by the Tobin-Benhabib-Miyao flow and by the Euclidean-Lagrangian structure of the economic state space.

In this way we estimate the time economic evolution for spotlighting the expectations of the agents.

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1 Tobin-Benhabib-Miyao economic flow

The Tobin model [4] regarding the role of money on economic growth has been extended by Benhabib and Miyao [1] to incorporate the role of expectation parameters, and to show that the variation of this parameter produces a Hopf Bifurcation in a three sector economy (see also [5],[6]).

We introduce the state space by the following variables:

- k = the capital labour ratio;
- m = the money stock per head;
- q = the expected rate of inflation.

Then, the model is an ODEs system

$$\dot{k} = sf(k) - (1-s)(\theta - q)m - nk, \dot{m} = m(\theta - \bar{p} - n), \dot{q} = \mu(\bar{p} - q), \tag{1.1}$$

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where

$$\bar{p} = \varepsilon\Big(m - L(k,q)\Big) + q, \tag{1.2}$$

is the actual rate of inflation. The real functions $f(k)$ and $L(k,q)$ are differentiable, and $s, \theta, n, \mu, \varepsilon$ are parameters (s = saving ratio; θ = rate of money expansion, n = population growth rate, μ = speed of adjustment of expectations, ε = speed of adjustment of price level). Keeping the parameters $s, \theta, n, \varepsilon$ like constants, we obtain an ODEs system with one parameter μ.

An equilibrium point $(k^*(\mu), m^*(\mu), q^*(\mu))$, at which $\dot{k} = 0 = \dot{m} = \dot{q}$, is the solution of the algebraic system

$$\theta = q + n, L(k,q) = m, s\,f(k)\, - (1-s)m\,n - k\,n = 0. \tag{1.3}$$

(suppose we have isolated solutions). Denoting $x = (x_1, x_2, x_3) = (k - k^*, m - m^*, q - q^*)$, the linearization about the equilibrium point $(0,0,0)$ is

$$\dot{x} = A(\mu)\,x, \tag{1.4}$$

where $A(\mu)$ is the Jacobian matrix of the function

$$\Big(sf(k) - (1-s)(\theta - q)m - nk\ , m(\theta - \bar{p} - n)\ , \mu(\bar{p} - q) \Big), \tag{1.5}$$

computed at $(k = k^*, m = m^*, q = q^*)$, i.e.,

$$\begin{pmatrix} s\,f' - n & -(1-s)n & (1-s)m \\ \varepsilon m L_1 & -2\varepsilon m & m(\varepsilon L_2 - 1) \\ -\mu\varepsilon L_1 & \mu\varepsilon & -\mu\varepsilon L_2 \end{pmatrix}_{(k^*, m^*, q^*)} \tag{1.6}$$

The characteristic equation is

$$\det\Big(A(\mu) - \lambda(\mu) I\Big) = -\lambda^3 + c_1\lambda^2 - c_2\lambda + c_3 = 0\,, \tag{1.7}$$

where $c_1 = \mathrm{tr}A(\mu)$, c_2 = sum of principal minors of order two, $c_3 = \det A(\mu)$. If $(-1)^i c_i > 0$ $(i = 1,2,3)$ and $c_1 c_2 < c_3$, then we have solutions of type $\lambda_1(\mu) < 0$, $\lambda_{2,3}(\mu) = \alpha(\mu) \pm i\,\beta(\mu)$, $\alpha(\mu) < 0$, and consequently the equilibrium point is asymptotically stable. In the hypothesis $c_1 c_2 = c_3$, we can find μ_0 such that $\lambda_1(\mu_0) < 0$, and $\alpha(\mu_0) = 0$, $\frac{d\alpha}{d\mu}(\mu_0) \neq 0$. By the Hopf Bifurcation Theorem, there exist periodic solutions

$$\Big(k(t,\mu), m(t,\mu), q(t,\mu)\Big),\ t \in R$$

around the equilibrium point (k^*, m^*, q^*).

Medio [2],[3] generalized the previous model and studied the birth of limit cycles given by Hopf Bifurcation, in the framework of λ - matrices and gyroscopic models.

2 Geometric dynamics produced by a flow and a Riemannian metric

Our theory [7]-[10] can be applied equally to any kind of flow on a manifold endowed with a geometric structure capable to produce *"square of the length"* (density of energy) and *"derivatives"*. This geometric structure transforms a flow into a geodesic motion in a gyroscopic field of forces. As an example, we can use a Euclidean-Lagrangian structure associated to the metric δ_{ij} or a Riemannian-Lagrangian structure associated to the metric g_{ij}.

Let us start with an arbitrary flow

$$\dot{x}_i = X_i(x)\,, \quad i = 1, 2, 3 \tag{2.8}$$

on the Euclidean space $(R^3,\ \delta_{ij})$. Modifying the prolongation by derivation in a suitable way, we obtain a *gyroscopic prolongation*

$$\frac{d^2 x_i}{dt^2} = \delta^{jk}\Big(\frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i}\Big)\frac{dx_k}{dt} + \frac{\partial f}{\partial x_i}\,, \tag{2.9}$$

where

$$f = 0.5\ \delta^{ij}\, X_i\, X_j \tag{2.10}$$

is the density of economic energy. This new prolongation determines a geometric dynamics, i.e., a geodesic motion in a gyroscopic field of forces. Another way to realize a geometric dynamics is to consider the least squares Lagrangian

$$L(x, \dot{x}) = 0.5\, \delta^{ij}\, \Big(\dot{x}_i - X_i(x)\Big)\Big(\dot{x}_j - X_j(x)\Big) \tag{2.11}$$

and to write the Euler-Lagrange equations which are just (9). Automatically, the geometric dynamics conserves the Hamiltonian

$$H(x, \dot{x}) = 0.5\, \delta^{ij}\, \Big(\dot{x}_i - X_i(x)\Big)\Big(\dot{x}_j + X_j(x)\Big). \tag{2.12}$$

The linearization of the first order ODEs system (8) around an equilibrium point $(0, 0, 0)$ is of the form

$$\dot{x} = Ax. \tag{2.13}$$

Then the linearization of the second order ODEs system (9), around the same equilibrium point, is

$$\frac{d^2 x}{dt^2} = (A - A^T)\,\frac{dx}{dt} + A^T A\, x\ . \tag{2.14}$$

For this second order system, the equilibrium point is

$$x(t) = 0, \frac{dx}{dt}(t) = 0.$$

Of course, the second order ODEs system (9) can be linearized also around a nonzero critical point of the density of energy f, but this study will be made in a further paper.

3 Linearized geometric dynamics around Tobin-Benhabib-Miyao flow

The Tobin-Benhabib-Miyao flow and the Euclidean-Lagrangian structure of the state space determine a geometric dynamics. We shall analyse its linearization around the equilibrium point which is described by a second order ODEs system of type (14). The solution of this system is of the form $x = ue^{\lambda t}, t \in R$, where u is a nonzero solution of the linear system

$$(3.15) \qquad \Big(-A^T A - \lambda(A - A^T) + \lambda^2 I\Big)u = 0.$$

In other words, λ is a latent value of the λ-matrix in paranthesis, i.e., solution of the equation

$$(3.16) \qquad \det\Big(-A^T A - \lambda(A - A^T) + \lambda^2 I\Big) = 0\,,$$

and u is a latent vector (nonzero solution of the equation 15). On the other hand, the following proposition is true.

Theorem. *If λ is a proper value of a real matrix A, then λ and $-\lambda$ are latent values of the previuos λ-matrix. Consequently the latent values satisfy $\sum \lambda = 0$.*

Proof. We use the decomposition

$$(3.17) \qquad 0 = det\Big(-A^T A - \lambda(A - A^T) + \lambda^2 I\Big) = det(\lambda I - A)det(\lambda I + A^T).$$

In order to obtain informations about the nature of the latent values λ, associated to the latent vector u, we build an equation of degree two satisfied by λ.

Let λ be a real latent value and u be the corresponding real latent vector. Premultiplying by u^T we find

$$(3.18) \qquad \lambda^2 = \frac{u^T A^T A u}{u^T u}.$$

Let λ be a complex latent value and u be the associated complex latent vector. Premultiplying by $\bar{u}^T$ (conjugate transpose of u) gives

$$(3.19) \qquad m\,\lambda^2 \,+\, i\,g\,\lambda \,+\, n = 0\,,$$

where

$$m = \bar{u}^T\,u > 0\,,\, i\,g = -\bar{u}^T\,(A - A^T)\,u\,,\, n = -\bar{u}^T\,A^T\,A\,u < 0\,.$$

The discriminant of the equation (19) is

$$\Delta\,(\mu) = -g^2 \,-\, 4\,m\,n\,.$$

Theorem. *1) The ODEs system (14) has saddle point properties around the equilibrium point iff $\Delta(\mu) > 0$.*

2) If $\Delta(\mu) < 0$, no saddle point properties exist.

Let μ_0 be such that $\Delta(\mu_0) = 0$ and $\Delta(\mu) > 0$ or < 0 if $\mu < \mu_0$ respectively $\mu > \mu_0$ with $\frac{d\,\Delta}{d\,\mu}(\mu_0) \,<\, 0$. These are just the conditions in the *"Hopf Bifurcation Theorem* *for second order systems*". Consequently, when μ passes μ_0, the system will undergo a bifurcation and lose its saddle point properties. Assuming simple latent values around μ_0, we find:

Corollary. 1) *if* $g(\mu) = 0$ *for* $\mu \in N_\varepsilon(\mu_0)$*, then certain latent value lying on the real axis crosses imaginary axis from left to right, causing "total instability";*

2) if $g(\mu) \neq 0$*, then the loss of stability is of the "flutter type", i.e., a pair of complex conjugate latent values crosses the imaginary axis from the left, causing Hopf Bifurcation and giving birth to closed orbits around the equilibrium point.*

Remark. This limit cycle is optimal because it fulfils all optimality requirements, including the transversality condition

$$lim_{t\to\infty} k(t)\, q(t)\, e^{-\mu\, t} = 0. \tag{3.20}$$

Thus an economy satisfying all standard neo-classical competitive conditions such as perfect foresight, zero profit, market clearing, can exhibit permanent small oscillations in prices and capital stocks. Therefore we recovered the concept of optimal economic fluctuations.

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