A class of locally symmetric Kähler Einstein structures on the nonzero cotangent bundle of a space form

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Abstract

We obtain a class of locally symetric Kähler Einstein structures on the nonzero cotangent bundle of a Riemannian manifold of positive constant sectional curvature. The obtained class of Kähler Einstein structures depends on one essential parameter and cannot have constant holomorphic sectional curvature.

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1 Introduction

In the study of the differential geometry of the cotangent bundle T^*M of a Riemannian manifold (M, g) one uses several Riemannian and semi-Riemannian metrics, induced from the Riemannian metric g on M. Next, one can get from g some natural almost complex structures on T^*M. The study of the almost Hermitian structures induced from g on T^*M is an interesting problem in the differential geometry of the cotangent bundle.

In [9] the authors have obtained a class of natural Kähler Einstein structures (G, J) of diagonal type induced on T^*M from the Riemannian metric g. The obtained Kähler structures on T^*M depend on two essential parameters a_1 and λ, which are smooth functions depending on the energy density t on T^*M. In the case where the considered Kähler structures are Einstein they get several situations in which the parameters a_1, λ are related by some algebraic relations. In the general case, (T^*M, G, J) has constant holomorphic curvature.

In this paper we study the singular case where the parameter $a_1 = At\lambda,\ A \in \mathbf{R}$. The class of the natural almost complex structures J on the nonzero cotangent bundle T_0^*M that interchange the vertical and horizontal distributions depends on two essential parameters λ and b_1. These parameters are smooth real functions depending on the energy density t on T_0^*M. From the integrability condition for J it follows that

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the base manifold M must have constant curvature c and the second parameter b_1 must be expressed as a rational function depending on the first parameter λ and its derivative. Of course, in the obtained formula there are involved the constant c and the energy density t.

A class of natural Riemannian metrics G of diagonal type on T_0^*M is defined by four parameters c_1, c_2, d_1, d_2 which are smooth functions of t. From the condition for G to be Hermitian with respect to J we get two sets of proportionality relations, from which we can get the parameters c_1, c_2, d_1, d_2 as functions depending on one new parameter μ and the parameter λ involved in the expression of J.

In the case where the fundamental 2-form ϕ, associated to the class of complex structures (G, J) is closed, one finds that $\mu = \lambda'$.

Thus, we get a class of Kähler structures (G, J) on T_0^*M, depending on one essential parameter λ.

Finally, we prove that the obtained class of Kähler structures on T_0^*M is locally symmetric, Einstein and cannot have constant holomorphic sectional curvature.

The manifolds, tensor fields and geometric objects we consider in this paper, are assumed to be differentiable of class C^∞ (i.e. smooth). We use the computations in local coordinates but many results from this paper may be expressed in an invariant form. The well known summation convention is used throughout this paper, the range for the indices h, i, j, k, l, r, s being always$\{1, ..., n\}$ (see [4], [6], [7]). We shall denote by $\Gamma(T_0^*M)$ the module of smooth vector fields on T_0^*M.

2 Some geometric properties of T^*M

Let (M, g) be a smooth n-dimensional Riemannian manifold and denote its cotangent bundle by $\pi : T^*M \longrightarrow M$. Recall that there is a structure of a $2n$-dimensional smooth manifold on T^*M, induced from the structure of smooth n-dimensional manifold of M. From every local chart $(U, \varphi) = (U, x^1, \dots, x^n)$ on M, it is induced a local chart $(\pi^{-1}(U), \Phi) = (\pi^{-1}(U), q^1, \dots, q^n,\ p_1, \dots, p_n)$, on T^*M, as follows. For a cotangent vector $p \in \pi^{-1}(U) \subset T^*M$, the first n local coordinates $q^1, \dots, q^n$ are the local coordinates $x^1, \dots, x^n$ of its base point $x = \pi(p)$ in the local chart (U, φ) (in fact we have $q^i = \pi^* x^i = x^i \circ \pi,\ i = 1, \dots n$). The last n local coordinates $p_1, \dots, p_n$ of $p \in \pi^{-1}(U)$ are the vector space coordinates of p with respect to the natural basis $(dx^1_{\pi(p)}, \dots, dx^n_{\pi(p)})$, defined by the local chart (U, φ), i.e. $p = p_i dx^i_{\pi(p)}$.

An M-tensor field of type (r, s) on T^*M is defined by sets of n^{r+s} components (functions depending on q^i and p_i), with r upper indices and s lower indices, assigned to induced local charts $(\pi^{-1}(U), \Phi)$ on T^*M, such that the local coordinate change rule is that of the local coordinate components of a tensor field of type (r, s) on the base manifold M (see [2] for further details in the case of the tangent bundle). An usual tensor field of type (r, s) on M may be thought of as an M-tensor field of type (r, s) on T^*M. If the considered tensor field on M is covariant only, the corresponding M-tensor field on T^*M may be identified with the induced (pullback by π) tensor field on T^*M.

Some useful M-tensor fields on T^*M may be obtained as follows. Let $u, v : [0, \infty) \longrightarrow \mathbf{R}$ be a smooth functions and let $\|p\|^2 = g^{-1}_{\pi(p)}(p, p)$ be the square of the norm of the cotangent vector $p \in \pi^{-1}(U)$ (g^{-1} is the tensor field of type (2,0) having the components $(g^{kl}(x))$ which are the entries of the inverse of the matrix $(g_{ij}(x))$ defined by the components of g in the local chart (U,φ)). The components $u(\|p\|^2)g_{ij}(\pi(p))$, p_i, $v(\|p\|^2)p_ip_j$ define M-tensor fields of types $(0,2)$, $(0,1)$, $(0,2)$ on T^*M, respectively. Similarly, the components $u(\|p\|^2)g^{kl}(\pi(p))$, $g^{0i}=p_hg^{hi}$, $v(\|p\|^2)g^{0k}g^{0l}$ define M-tensor fields of type $(2,0)$, $(1,0)$, $(2,0)$ on T^*M, respectively. Of course, all the components considered above are in the induced local chart $(\pi^{-1}(U),\Phi)$.

The Levi Civita connection $\dot\nabla$ of g defines a direct sum decomposition

$$TT^*M = VT^*M \oplus HT^*M. \tag{2.1}$$

of the tangent bundle to T^*M into vertical distributions $VT^*M=\mathrm{Ker}\ \pi_*$ and the horizontal distribution HT^*M.

If $(\pi^{-1}(U),\Phi)=(\pi^{-1}(U),q^1,\dots,q^n,p_1,\dots,p_n)$ is a local chart on T^*M, induced from the local chart $(U,\varphi)=(U,x^1,\dots,x^n)$, the local vector fields $\frac{\partial}{\partial p_1},\dots,\frac{\partial}{\partial p_n}$ on $\pi^{-1}(U)$ define a local frame for VT^*M over $\pi^{-1}(U)$ and the local vector fields $\frac{\delta}{\delta q^1},\dots,\frac{\delta}{\delta q^n}$ define a local frame for HT^*M over $\pi^{-1}(U)$, where

$$\frac{\delta}{\delta q^i}=\frac{\partial}{\partial q^i}+\Gamma^0_{ih}\frac{\partial}{\partial p_h},\quad \Gamma^0_{ih}=p_k\Gamma^k_{ih}$$

and $\Gamma^k_{ih}(\pi(p))$ are the Christoffel symbols of g.

The set of vector fields $(\frac{\partial}{\partial p_1},\dots,\frac{\partial}{\partial p_n},\frac{\delta}{\delta q^1},\dots,\frac{\delta}{\delta q^n})$ defines a local frame on T^*M, adapted to the direct sum decomposition (1).

We consider

$$t=\frac{1}{2}\|p\|^2=\frac{1}{2}g^{-1}_{\pi(p)}(p,p)=\frac{1}{2}g^{ik}(x)p_ip_k,\quad p\in\pi^{-1}(U) \tag{2.2}$$

the energy density defined by g in the cotangent vector p. We have $t\in[0,\infty)$ for all $p\in T^*M$.

From now on we shall work in a fixed local chart (U,φ) on M and in the induced local chart $(\pi^{-1}(U),\Phi)$ on T^*M.

Now we shall present the following auxiliary result.

Lemma 1. *If $n>1$ and u,v are smooth functions on T^*M such that*

$$ug_{ij}+vp_ip_j=0,\ p\in\pi^{-1}(U)$$

on the domain of any induced local chart on T^*M, then $u=0,\ v=0$.

The proof is obtained easily by transvecting the given relation with the components g^{ij} of the tensor field g^{-1} and g^{0j}.

Remark. From the relations of the type

$$ug^{ij}+vg^{0i}g^{0j}=0,\ p\in\pi^{-1}(U),$$

$$u\delta^i_j+vg^{0i}p_j=0,\ p\in\pi^{-1}(U),$$

it is obtained, in a similar way, $u=v=0$.

3 A class of natural complex structures of diagonal type on T_0^*M

The nonzero cotangent bundle T_0^*M of Riemannian manifold (M, g) is defined by the formula: T^*M minus zero section. Consider the real valued smooth functions $\lambda, a_1, a_2, b_1, b_2$ defined on $(0, \infty)$. We define a class of natural almost complex structures J of diagonal type on T_0^*M , expressed in the adapted local frame by

$$J\frac{\delta}{\delta q^i} = J^{(1)}_{ij}(p)\frac{\partial}{\partial p_j}, \qquad J\frac{\partial}{\partial p_i} = -J^{ij}_{(2)}(p)\frac{\delta}{\delta q^j}. \tag{3.3}$$

where,

$$J^{(1)}_{ij}(p) = a_1(t)g_{ij} + b_1(t)p_ip_j, \quad J^{ij}_{(2)}(p) = a_2(t)g^{ij} + b_2(t)g^{0i}g^{0j}, \quad A \in \mathbf{R}^*. \tag{3.4}$$

In this paper we study the singular case where

$$a_1(t) = At\lambda(t). \tag{3.5}$$

The components $J^{(1)}_{ij}, J^{ij}_{(2)}$ define symmetric M-tensor fields of types $(0, 2), (2, 0)$ on T^*M, respectively.

Proposition 2. *The operator J defines an almost complex structure on T^*M if and only if*

$$a_1a_2 = 1, \quad (a_1 + 2tb_1)(a_2 + 2tb_2) = 1. \tag{3.6}$$

Proof. The relations are obtained easily from the property $J^2 = -I$ of J and Lemma 1.

From the relations (5), (6) we can obtain the explicit expression of the parameter a_2, b_2

$$a_2 = \frac{1}{At\lambda}, \qquad b_2 = \frac{-b_1}{At^2\lambda(A\lambda + 2b_1)}. \tag{3.7}$$

The obtained class of almost complex structures defined by the tensor field J on T_0^*M is called *class of natural almost complex structures of diagonal type,* obtained from the Riemannian metric g, by using the parameters λ, b_1. We use the word diagonal for these almost complex structures, since the $2n\times 2n$-matrix associated to J, with respect to the adapted local frame $(\frac{\delta}{\delta q^1}, \dots, \frac{\delta}{\delta q^n}, \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n})$ has two $n \times n$-blocks on the second diagonal

$$J = \begin{pmatrix} 0 & -J^{ij}_{(2)} \\ J^{(1)}_{ij} & 0 \end{pmatrix}.$$

Remark. From the conditions (6) it follows that $a_1 = At\lambda$ and $a_2 = \frac{1}{At\lambda}$ cannot vanish and have the same sign. We assume that

$$\lambda(t) > 0 \ \ \forall t > 0, \ A > 0. \tag{3.8}$$

Similarly, from the conditions (6) it follows that $a_1 + 2tb_1$ and $a_2 + 2tb_2$ cannot vanish and have the same sign. We assume that $a_1 + 2tb_1 > 0, \ a_2 + 2tb_2 > 0 \ \forall t > 0$, i.e.

$$A\lambda + 2b_1 > 0 \ \ \forall t > 0. \tag{3.9}$$

Now we shall study the integrability of the class of natural almost complex structures defined by J on T_0^*M. To do this we need the following well known formulas for the brackets of the vector fields $\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^i}, \ i = 1, ..., n$

$$[\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}] = 0, \quad [\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}] = \Gamma^i_{jk}\frac{\partial}{\partial p_k}, \quad [\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}] = R^0_{kij}\frac{\partial}{\partial p_k}, \tag{3.10}$$

where $R^h_{kij}(\pi(p))$ are the local coordinate components of the curvature tensor field of $\dot\nabla$ on M and $R^0_{kij}(p) = p_h R^h_{kij}$. Of course, the components R^h_{kij}, R^0_{kij} define M-tensor fields of types (1,3), (0,3) on T_0^*M, respectively.

Recall that the Nijenhuis tensor field N defined by J is given by

$$N(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y], \ \ \forall \ \ X, Y \in \Gamma(T_0^*M).$$

Then, we have $\frac{\delta}{\delta q^k}t = 0, \ \frac{\partial}{\partial p_k}t = g^{0k}$. The expressions for the components of N can be obtained by a quite long, straightforward computation, as follows

Theorem 3. *The Nijenhuis tensor field of the almost complex structure J on T_0^*M is given by*

$$\begin{cases} N(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}) = \{At(\lambda + 2t\lambda')(b_1 + A\lambda)(\delta^h_i g_{jk} - \delta^h_j g_{ik}) - R^h_{kij}\}p_h\frac{\partial}{\partial p_k}, \\ N(\frac{\delta}{\delta q^i}, \frac{\partial}{\partial p_j}) = J^{kl}_{(2)}J^{jr}_{(2)}\{At(\lambda + 2t\lambda')(b_1 + A\lambda)(\delta^h_i g_{rl} - \delta^h_r g_{il}) - R^h_{lir}\}p_h\frac{\delta}{\delta q^k}, \\ N(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}) = J^{ir}_{(2)}J^{jl}_{(2)}\{At(\lambda + 2t\lambda')(b_1 + A\lambda)(\delta^h_l g_{rk} - \delta^h_r g_{lk}) - R^h_{klr}\}p_h\frac{\partial}{\partial p_k}. \end{cases}$$

Theorem 4. *Assume that exist*

$$\lim_{t\to 0} At(\lambda+2t\lambda')(b_1+A\lambda) \in \mathbf{R}, \quad \lim_{t\to 0}\frac{\partial}{\partial p_l}[At(\lambda+2t\lambda')(b_1+A\lambda)] = 0, \quad \forall\, l \in \{1, 2, \dots, n\}.$$

The almost complex structure J on T_0^*M is integrable if and only if (M, g) has constant sectional curvature c and the function b_1 is given by

$$b_1 = \frac{c - A^2t\lambda(\lambda + t\lambda')}{At(\lambda + 2t\lambda')}. \tag{3.11}$$

The parameter λ *must fulfill the conditons*

$$\lambda > 0,\ \frac{2c - A^2 t\lambda^2}{\lambda + 2t\lambda'} > 0\ \forall t > 0,\ A > 0. \tag{3.12}$$

Proof. From the condition $N = 0$, one obtains

$$\{At(\lambda + 2t\lambda')(b_1 + A\lambda)(\delta_i^h g_{jk} - \delta_j^h g_{ik}) - R_{kij}^h\}p_h = 0.$$

Differentiating with respect to p_l and taking $t \to 0$, it follows that the curvature tensor field of $\dot\nabla$ has the expression

$$R_{kij}^l = (\lim_{t\to 0} At(\lambda + 2t\lambda')(b_1 + A\lambda))(\delta_i^l g_{jk} - \delta_j^l g_{ik}).$$

Thus the sectional curvature $c = \lim_{t\to 0} At(\lambda + 2t\lambda')(b_1 + A\lambda)$ of (M, g) depends only on q^i. Using by the Schur theorem (in the case where M is connected and $dimM \geq 3$) it follows that (M, g) has the constant sectional curvature $c = \lim_{t\to 0} At(\lambda + 2t\lambda')(b_1 + A\lambda)$. Then we obtain the expression (3.11) of b_1.

Conversely, if (M, g) has constant curvature c and b_1 is given by (3.11), it follows in a straightforward way that $N = 0$.

Using by the relations (3.8), (3.9), (3.11) we obtain the conditions (3.12).

The class of natural complex structures J of diagonal type on T_0^*M depends on one essential parameter λ. The components of J are given by

$$\begin{cases} J_{ij}^{(1)} = At\lambda g_{ij} + \frac{c - A^2 t\lambda(\lambda + t\lambda')}{At(\lambda + 2t\lambda')} p_i p_j, \\ J_{(2)}^{ij} = \frac{1}{At\lambda} g^{ij} - \frac{c - A^2 t\lambda(\lambda + t\lambda')}{At^2\lambda(2c - A^2 t\lambda^2)} g^{0i} g^{0j}. \end{cases} \tag{3.13}$$

4 A class of natural Hermitian structures on T_0^*M

Consider the following symmetric $M-$tensor fields on T_0^*M, defined by the components

$$G_{ij}^{(1)} = c_1 g_{ij} + d_1 p_i p_j, \quad G_{(2)}^{ij} = c_2 g^{ij} + d_2 g^{0i} g^{0j}, \tag{4.14}$$

where c_1, c_2, d_1, d_2 are smooth functions depending on the energy density $t \in (0, \infty)$.

Obviously, $G^{(1)}$ is of type $(0, 2)$ and $G_{(2)}$ is of type $(2, 0)$. We shall assume that the matrices defined by $G^{(1)}$ and $G_{(2)}$ are positive definite. This happens if and only if

$$c_1 > 0,\ c_2 > 0,\ c_1 + 2td_1 > 0,\ c_2 + 2td_2 > 0\ \ \forall t > 0. \tag{4.15}$$

Then the following class of Riemannian metrics may be considered on T_0^*M

$$G = G_{ij}^{(1)} dq^i dq^j + G_{(2)}^{ij} Dp_i Dp_j, \tag{4.16}$$

where $Dp_i = dp_i - \Gamma_{ij}^0 dq^j$ is the absolute (covariant) differential of p_i with respect to the Levi Civita connection $\dot\nabla$ of g. Equivalently, we have

$$G(\frac{\delta}{\delta q^i},\frac{\delta}{\delta q^j})=G^{(1)}_{ij},\quad G(\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_j})=G^{ij}_{(2)},\quad G(\frac{\partial}{\partial p_i},\frac{\delta}{\delta q^j})=G(\frac{\delta}{\delta q^j},\frac{\partial}{\partial p_i})=0.$$

Remark that HT^*_0M, VT^*_0M are orthogonal to each other with respect to G, but the Riemannian metrics induced from G on HT^*_0M, VT^*_0M are not the same, so the considered metric G on T^*_0M is not a metric of Sasaki type. The $2n\times 2n$-matrix associated to G, with respect to the adapted local frame $(\frac{\delta}{\delta q^1},\dots,\frac{\delta}{\delta q^n},\frac{\partial}{\partial p_1},\dots,\frac{\partial}{\partial p_n})$ has two $n\times n$-blocks on the first diagonal

$$G=\begin{pmatrix} G^{(1)}_{ij} & 0\\ 0 & G^{ij}_{(2)}\end{pmatrix}.$$

The class of Riemannian metrics G is called a *class of natural lifts of diagonal type* of g. Remark also that the system of 1-forms $(Dp_1,...,Dp_n,dq^1,...,dq^n)$ defines a local frame on $T^*T^*_0M$, dual to the local frame $(\frac{\partial}{\partial p_1},\dots,\frac{\partial}{\partial p_n},\frac{\delta}{\delta q^1},\dots,\frac{\delta}{\delta q^n})$ on TT^*_0M, over $\pi^{-1}(U)$ adapted to the direct sum decomposition (1).

We shall consider another two M-tensor fields $H_{(1)}$, $H^{(2)}$ on T^*_0M, defined by the components

$$H^{jk}_{(1)}=\frac{1}{c_1}g^{jk}-\frac{d_1}{c_1(c_1+2td_1)}g^{0j}g^{0k},$$

$$H^{(2)}_{jk}=\frac{1}{c_2}g_{jk}-\frac{d_2}{c_2(c_2+2td_2)}p_jp_k.$$

The components $H^{jk}_{(1)}$ define an M-tensor field of type $(2,0)$ and the components $H^{(2)}_{jk}$ define an M-tensor field of type $(0,2)$. Moreover, the matrices associated to $H_{(1)}$, $H^{(2)}$ are the inverses of the matrices associated to $G^{(1)}$ and $G_{(2)}$, respectively. Hence we have

$$G^{(1)}_{ij}H^{jk}_{(1)}=\delta^k_i,\quad G^{ij}_{(2)}H^{(2)}_{jk}=\delta^i_k.$$

Now, we shall be interested in the conditions under which the class of the metrics G is Hermitian with respect to the class of the complex structures J, considered in the previous section, i.e.

$$G(JX,JY)=G(X,Y),$$

for all vector fields X,Y on T^*_0M.

Considering the coefficients of g_{ij},g^{ij} in the conditions

$$\text{(4.17)}\qquad \begin{cases} G(J\frac{\delta}{\delta q^i},J\frac{\delta}{\delta q^j})=G(\frac{\delta}{\delta q^i},\frac{\delta}{\delta q^j}),\\[2mm] G(J\frac{\partial}{\partial p_i},J\frac{\partial}{\partial p_j})=G(\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_j}),\end{cases}$$

we can express the parameters c_1,c_2 with the help of the parameters a_1,a_2 and a proportionality factor which must be $\lambda=\lambda(t)$ (see [9]). Then

$$\text{(4.18)}\qquad c_1=\lambda a_1=At\lambda^2,\qquad c_2=\lambda a_2=\frac{1}{At},$$

where the coefficients a_1,a_2 are given by (5) and (7).

Next, considering the coefficients of p_ip_j, $g^{0i}g^{0j}$ in the relations (17), we can express the parameters c_1+2td_1, c_2+2td_2 with help of the parameters a_1+2tb_1, a_2+2tb_2 and a proportionality factor $\lambda+2t\mu$

$$(4.19)\qquad \begin{cases} c_1+2td_1=(\lambda+2t\mu)(a_1+2tb_1), \\ c_2+2td_2=(\lambda+2t\mu)(a_2+2tb_2). \end{cases}$$

Remark that $\lambda(t)+2t\mu(t)>0\ \forall t>0$. It is much more convenient to consider the proportionality factor in such a form in the expression of the parameters c_1+2td_1, c_2+2td_2. Using by the relations (5), (7), (11),(18) we can obtain easily from (19) the explicit expressions of the coefficients d_1, d_2

$$(4.20)\qquad \begin{cases} d_1=\frac{\lambda[c-A^2t\lambda(\lambda+t\lambda')]+\mu t(2c-A^2t\lambda^2)}{At(\lambda+2t\lambda')}, \\ d_2=\frac{-c+A^2t\lambda(\lambda+t\lambda')+\mu A^2t^2(\lambda+2t\lambda')}{At^2(2c-A^2t\lambda^2)}. \end{cases}$$

Hence we may state:

Theorem 5. *Let J be the class of natural, complex structure of diagonal type on T_0^*M, given by (3) and (13). Let G be the class of the natural Riemannian metrics of diagonal type on T_0^*M, given by (14), (18), (20).*

Then we obtain a class of Hermitian structures (G,J) on T_0^*M, depending on two essential parameters λ and μ, which must fulfill the conditions

$$(4.21)\qquad \lambda>0,\quad \frac{2c-A^2t\lambda^2}{\lambda+2t\lambda'}>0,\quad \lambda+2t\mu>0\ \ \forall t>0,\quad A>0.$$

5 A class of Kähler structures on T_0^*M

Consider now the two-form ϕ defined by the class of Hermitian structures (G,J) on T_0^*M

$$\phi(X,Y)=G(X,JY),$$

for all vector fields X,Y on T_0^*M.

Using by the expression of ϕ and computing the values $\phi(\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_j}), \phi(\frac{\delta}{\delta q^i},\frac{\delta}{\delta q^j})$, $\phi(\frac{\partial}{\partial p_i},\frac{\delta}{\delta q^j})$, we obtain.

Proposition 6. *The expression of the 2-form ϕ in a local adapted frame $(\frac{\partial}{\partial p_1},\dots,\frac{\partial}{\partial p_n},\frac{\delta}{\delta q^1},\dots,\frac{\delta}{\delta q^n})$ on T_0^*M, is given by*

$$\phi(\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_j})=0,\ \ \phi(\frac{\delta}{\delta q^i},\frac{\delta}{\delta q^j})=0,\ \ \phi(\frac{\partial}{\partial p_i},\frac{\delta}{\delta q^j})=\lambda\delta^i_j+\mu g^{0i}p_j,$$

or, equivalently

$$(5.22)\qquad \phi=(\lambda\delta^i_j+\mu g^{0i}p_j)Dp_i\wedge dq^j.$$

Theorem 7. *The class of Hermitian structures (G, J) on T_0^*M is Kähler if and only if*

$$\mu = \lambda'.$$

Proof. The expressions of $d\lambda,\ d\mu,\ dg^{0i}$ and dDp_i are obtained in a straightforward way, by using the property $\dot\nabla_k g_{ij} = 0$ (hence $\dot\nabla_k g^{ij} = 0$)

$$d\lambda = \lambda' g^{0i} Dp_i,\ d\mu = \mu' g^{0i} Dp_i,\ dg^{0i} = g^{ik} Dp_k - g^{0h}\Gamma^i_{hk} dq^k,$$

$$dDp_i = -\frac{1}{2} R^0_{ikl} dq^k \wedge dq^l + \Gamma^l_{ik} dq^k \wedge Dp_l.$$

Then we have

$$d\phi = (d\lambda \delta^i_j + d\mu g^{0i} p_j + \mu dg^{0i} p_j + \mu g^{0i} dp_j) \wedge Dp_i \wedge dq^j +$$

$$+(\lambda \delta^i_j + \mu g^{0i} p_j) dDp_i \wedge dq^j.$$

By replacing the expressions of $d\lambda, d\mu, dg^{0i}$ and $d\dot\nabla y^h$, then using, again, the property $\dot\nabla_k g_{ij} = 0$, doing some algebraic computations with the exterior products, then using the well known symmetry properties of g_{ij}, Γ^h_{ij}, and of the Riemann-Christoffel tensor field, as well as the Bianchi identities, it follows that

$$d\phi = \frac{1}{2}(\lambda' - \mu) g^{0h} Dp_h \wedge Dp_i \wedge dq^i.$$

Therefore we have $d\phi = 0$ if and only if $\mu = \lambda'$.

Remark. The class of natural Kähler structures of diagonal type defined by (G, J) on T_0^*M depends on one essential parameter λ.

The paramater λ must fulfill the conditions

$$\lambda > 0, \quad 2c - A^2 t\lambda^2 > 0, \quad \lambda + 2t\lambda' > 0 \ \ \forall t > 0, \quad A > 0. \tag{5.23}$$

It follows that $c > 0$.

The components of the class of Kähler metrics G on T_0^*M are given by

$$\begin{cases} G^{(1)}_{ij} = At\lambda^2 g_{ij} + \frac{c - A^2 t\lambda^2}{At} p_i p_j, \\ G^{ij}_{(2)} = \frac{1}{At} g^{ij} - \frac{c - A^2 t[\lambda^2 + 2t\lambda'(\lambda + t\lambda')]}{At^2(2c - A^2 t\lambda^2)} g^{0i} g^{0j}. \end{cases} \tag{5.24}$$

We obtain, too

$$\begin{cases} H^{jk}_{(1)} = \frac{1}{At\lambda^2} g^{jk} - \frac{c - A^2 t\lambda^2}{At^2\lambda^2(2c - A^2 t\lambda^2)} g^{0j} g^{0k}, \\ H^{(2)}_{jk} = Atg_{jk} + \frac{c - A^2 t[\lambda^2 + 2t\lambda'(\lambda + t\lambda')])}{At(\lambda + 2t\lambda')^2} p_j p_k. \end{cases} \tag{5.25}$$

6 A class of locally symmetric Kähler Einstein structures on T_0^*M

The Levi Civita connection ∇ of the Riemannian manifold (T_0^*M, G) is determined by the conditions

$$\nabla G = 0, \qquad T = 0,$$

where T is its torsion tensor field. The explicit expression of this connection is obtained from the formula

$$2G(\nabla_X Y, Z) = X(G(Y,Z)) + Y(G(X,Z)) - Z(G(X,Y)) +$$

$$+G([X,Y],Z) - G([X,Z],Y) - G([Y,Z],X); \qquad \forall\ X,Y,Z \in \Gamma(T_0^*M).$$

The final result can be stated as follows.

Theorem 8. *The Levi Civita connection ∇ of G has the following expression in the local adapted frame $(\frac{\delta}{\delta q^1}, \dots, \frac{\delta}{\delta q^n}, \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n})$:*

$$\tag{6.26} \begin{cases} \nabla_{\frac{\partial}{\partial p_i}} \frac{\partial}{\partial p_j} = Q^{ij}_h \frac{\partial}{\partial p_h}, & \nabla_{\frac{\delta}{\delta q^i}} \frac{\partial}{\partial p_j} = -\Gamma^j_{ih} \frac{\partial}{\partial p_h} + P^{hj}_i \frac{\delta}{\delta q^h}, \\ \nabla_{\frac{\partial}{\partial p_i}} \frac{\delta}{\delta q^j} = P^{hi}_j \frac{\delta}{\delta q^h}, & \nabla_{\frac{\delta}{\delta q^i}} \frac{\delta}{\delta q^j} = \Gamma^h_{ij} \frac{\delta}{\delta q^h} + S_{hij} \frac{\partial}{\partial p_h}, \end{cases}$$

where $Q^{ij}_h, P^{hi}_j, S_{hij}$ are M-tensor fields on T_0^*M, defined by

$$\tag{6.27} \begin{cases} Q^{ij}_h = \frac{1}{2} H^{(2)}_{hk} (\frac{\partial}{\partial p_i} G^{jk}_{(2)} + \frac{\partial}{\partial p_j} G^{ik}_{(2)} - \frac{\partial}{\partial p_k} G^{ij}_{(2)}), \\ P^{hi}_j = \frac{1}{2} H^{hk}_{(1)} (\frac{\partial}{\partial p_i} G^{(1)}_{jk} - G^{il}_{(2)} R^0_{ljk}), \\ S_{hij} = -\frac{1}{2} H^{(2)}_{hk} \frac{\partial}{\partial p_k} G^{(1)}_{ij} + \frac{1}{2} R^0_{hij}. \end{cases}$$

Assuming that the base manifold (M, g) has positive constant sectional curvature c and replacing the expressions of the involved M-tensor fields, one obtains

$$\tag{6.28} \begin{cases} Q^{ij}_h = \frac{1}{2t} g^{ij} p_h - \frac{1}{2t} (\delta^i_h g^{0j} + \delta^j_h g^{0i}) + \\ \qquad \frac{c\lambda + 8ct\lambda' - 2A^2t^2\lambda\lambda'(\lambda - t\lambda') + 2t^2\lambda''(2c - A^2t\lambda^2)}{2t^2(2c - A^2t\lambda^2)(\lambda + 2t\lambda')} g^{0i} g^{0j} p_h, \\ P^{hi}_j = -\frac{1}{2t} g^{hi} p_j + \frac{1}{2t} \delta^i_j g^{0h} + \frac{\lambda + 2t\lambda'}{2t\lambda} \delta^h_j g^{0i} - \frac{c(\lambda + 2t\lambda')}{2t^2\lambda(2c - A^2t\lambda^2)} g^{0h} g^{0i} p_j, \\ S_{hij} = -\frac{\lambda(2c - A^2t\lambda^2)}{2(\lambda + 2t\lambda')} g_{ij} p_h - \frac{(2c - A^2t\lambda^2)}{2} g_{hi} p_j + \frac{A^2t\lambda^2}{2} g_{hj} p_i + \\ \qquad \frac{3c\lambda + 2ct\lambda' - 2A^2t\lambda^2(\lambda + t\lambda')}{2t(\lambda + 2t\lambda')} p_h p_i p_j. \end{cases}$$

The curvature tensor field K of the connection ∇ is obtained from the well known formula

$$K(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \qquad \forall\ X,Y,Z \in \Gamma(T_0^*M).$$

The components of curvature tensor field K with respect to the adapted local frame $(\frac{\delta}{\delta q^1},\dots,\frac{\delta}{\delta q^n},\frac{\partial}{\partial p_1},\dots,\frac{\partial}{\partial p_n})$ are obtained easily:

$$(6.29)\qquad \begin{cases} K(\frac{\delta}{\delta q^i},\frac{\delta}{\delta q^j})\frac{\delta}{\delta q^k} = QQQ^h_{ijk}\frac{\delta}{\delta q^h}, & K(\frac{\delta}{\delta q^i},\frac{\delta}{\delta q^j})\frac{\partial}{\partial p_k} = QQP^k_{ijh}\frac{\partial}{\partial p_h}, \\ K(\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_j})\frac{\delta}{\delta q^k} = PPQ^{ijh}_k\frac{\delta}{\delta q^h}, & K(\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_j})\frac{\partial}{\partial p_k} = PPP^{ijk}_h\frac{\partial}{\partial p_h}, \\ K(\frac{\partial}{\partial p_i},\frac{\delta}{\delta q^j})\frac{\delta}{\delta q^k} = PQQ^i_{jkh}\frac{\partial}{\partial p_h}, & K(\frac{\partial}{\partial p_i},\frac{\delta}{\delta q^j})\frac{\partial}{\partial p_k} = PQP^{ikh}_j\frac{\delta}{\delta q^h}, \end{cases}$$

where

$$(6.30)\qquad \begin{cases} QQQ^h_{ijk} = \lambda^2[\frac{A^2t}{2}(\delta^h_i g_{jk} - \delta^h_j g_{ik}) + \frac{A^2}{4}(g_{ik}p_j - g_{jk}p_i)g^{0h} - \\ \qquad \frac{A^2}{4}(\delta^h_i p_j - \delta^h_j p_i)p_k], \\ QQP^k_{ijh} = -QQQ^k_{ijh}, \\ PPQ^{ijh}_k = -\frac{1}{2t}(\delta^i_k g^{jh} - \delta^j_k g^{ih}) - \frac{1}{4t^2}(g^{ih}g^{0j} - g^{jh}g^{0i})p_k + \\ \qquad \frac{1}{4t^2}(\delta^i_k g^{0j} - \delta^j_k g^{0i})g^{0h}, \\ PPP^{ijk}_h = -PPQ^{ijk}_h, \\ PQQ^i_{jkh} = \frac{A^2t\lambda^2}{2}\delta^i_j g_{hk} + \frac{\lambda(2c-A^2t\lambda^2)}{4t(\lambda+2t\lambda')}\delta^i_k p_h p_j + \frac{\lambda[c-A^2\lambda t(\lambda+t\lambda')]}{2t(\lambda+2t\lambda')}\delta^i_j p_h p_k + \\ \qquad \frac{(2c-A^2t\lambda^2)}{4t}\delta^i_h p_j p_k + \frac{A^2\lambda^2}{4}g^{0i}g_{jk}p_h + \frac{A^2t\lambda\lambda'}{2}g^{0i}g_{hk}p_j + \\ \qquad \frac{A^2\lambda(\lambda+2t\lambda')}{4}g^{0i}g_{hj}p_k - \frac{\lambda[c+2A^2t^2\lambda'(\lambda+t\lambda')]}{2t^2(\lambda+2t\lambda')}g^{0i}p_h p_j p_k, \\ PQP^{ikh}_j = -\frac{1}{2t}\delta^i_j g^{hk} - \frac{1}{4t^2}g^{ik}g^{0h}p_j - \frac{\lambda'}{2t\lambda}g^{hk}g^{0i}p_j - \frac{\lambda+2t\lambda'}{4t^2\lambda}g^{hi}g^{0k}p_j - \\ \qquad \frac{A^2\lambda(\lambda+2t\lambda')}{4t(2c-A^2t\lambda^2)}\delta^k_j g^{0h}g^{0i} + \frac{c-A^2t\lambda(\lambda+t\lambda')}{2t^2(2c-A^2t\lambda^2)}\delta^i_j g^{0h}g^{0k} - \\ \qquad \frac{A^2(\lambda+2t\lambda')^2}{4t(2c-A^2t\lambda^2)}\delta^h_j g^{0i}g^{0k} + \frac{c(\lambda+2t\lambda')}{2t^3\lambda(2c-A^2t\lambda^2)}g^{0h}g^{0i}g^{0k}p_j. \end{cases}$$

are M-tensor fields on T^*_0M.

Remark. From the local coordinates expression of the curvature tensor field K, we obtain that the class of Kähler structures (G, J) on T^*_0M cannot have constant holomorphic sectional curvature.

The Ricci tensor field Ric of ∇ is defined by the formula:

$$Ric(Y,Z) = trace(X \longrightarrow K(X,Y)Z), \quad \forall\ X,Y,Z \in \Gamma(T^*_0M).$$

It follows

$$\begin{cases} Ric(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}) = \frac{An}{2} G^{(1)}_{ij}, \\ Ric(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}) = \frac{An}{2} G^{ij}_{(2)}, \\ Ric(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}) = Ric(\frac{\delta}{\delta q^j}, \frac{\partial}{\partial p_i}) = 0. \end{cases}$$

Thus

$$Ric = \frac{An}{2} G. \tag{6.31}$$

By straightforward computation, using the relations (28), (30) and the package Ricci, the following formulas are obtained:

(6.32)

$$\begin{cases} \frac{\delta}{\delta q^l} QQQ^h_{ijk} = -\Gamma^h_{ls} QQQ^s_{ijk} + \Gamma^s_{li} QQQ^h_{sjk} + \Gamma^s_{lj} QQQ^h_{isk} + \Gamma^s_{lk} QQQ^h_{ijs}, \\ \frac{\delta}{\delta q^l} PPQ^{ijh}_k = \ \Gamma^s_{lk} PPQ^{ijh}_s - \Gamma^i_{ls} PPQ^{sjh}_k - \Gamma^j_{ls} PPQ^{ish}_k - \Gamma^h_{ls} PPQ^{ijs}_k, \\ \frac{\delta}{\delta q^l} PQQ^i_{jkh} = -\Gamma^i_{ls} PQQ^s_{jkh} + \Gamma^s_{lj} PQQ^i_{skh} + \Gamma^s_{lk} PQQ^i_{jsh} + \Gamma^s_{lh} PQQ^i_{jks}, \\ \frac{\delta}{\delta q^l} PQP^{ikh}_j = \ \Gamma^s_{lj} PQP^{ikh}_s - \Gamma^i_{ls} PQP^{skh}_j - \Gamma^k_{ls} PQP^{ish}_j - \Gamma^h_{ls} PQP^{iks}_j, \\ \frac{\partial}{\partial p_l} QQQ^h_{ijk} = -P^{hl}_s QQQ^s_{ijk} + P^{sl}_i QQQ^h_{sjk} + P^{sl}_j QQQ^h_{isk} + P^{sl}_k QQQ^h_{ijs}, \\ \frac{\partial}{\partial p_l} PPQ^{ijh}_k = \ P^{sl}_k PPQ^{ijh}_s - P^{il}_s PPQ^{sjh}_k - P^{jl}_s PPQ^{ish}_k - P^{hl}_s PPQ^{ijs}_k, \\ \frac{\partial}{\partial p_l} PQQ^i_{jkh} = -P^{il}_s PQQ^s_{jkh} + P^{sl}_j PQQ^i_{skh} + P^{sl}_k PQQ^i_{jsh} + P^{sl}_h PQQ^i_{jks}, \\ \frac{\partial}{\partial p_l} PQP^{ikh}_j = \ P^{sl}_j PQP^{ikh}_s - P^{il}_s PQP^{skh}_j - P^{kl}_s PQP^{ish}_j - P^{hl}_s PQP^{iks}_j. \end{cases}$$

Due to the relations (26),(29), we have

$$(\nabla_{\frac{\delta}{\delta q^l}} K)(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j})\frac{\delta}{\delta q^k} = (\frac{\delta}{\delta q^l} QQQ^h_{ijk} + \Gamma^h_{ls} QQQ^s_{ijk} - \Gamma^s_{li} QQQ^h_{sjk} - \Gamma^s_{lj} QQQ^h_{isk} -$$

$$-\Gamma^s_{lk} QQQ^h_{ijs})\frac{\delta}{\delta q^h} + (S_{hls} QQQ^s_{ijk} + S_{slk} QQQ^s_{ijh} + S_{slj} PQQ^s_{ikh} - S_{sli} PQQ^s_{jkh})\frac{\partial}{\partial p_h}.$$

The coefficient of $\frac{\delta}{\delta q^h}$ is zero due to the relations (32). By straightforward computation, using the relations (28), (30) and the package Ricci, we obtain that the coefficient of $\frac{\partial}{\partial p_h}$ is zero. Thus

$$(\nabla_{\frac{\delta}{\delta q^l}} K)(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j})\frac{\delta}{\delta q^k} = 0.$$

Similarly,

$$(\nabla_{\frac{\partial}{\partial p_l}} K)(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j})\frac{\delta}{\delta q^k} = (\frac{\partial}{\partial p_l} QQQ^h_{ijk} + P^{hl}_s QQQ^s_{ijk} - P^{sl}_i QQQ^h_{sjk} - -P^{sl}_j QQQ^h_{isk} - P^{sl}_k QQQ^h_{ijs})\frac{\delta}{\delta q^h}.$$

The coefficient of $\frac{\delta}{\delta q^h}$ is zero due to the relations (32). Thus

$$(\nabla_{\frac{\partial}{\partial p_l}} K)(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j})\frac{\delta}{\delta q^k} = 0.$$

Similarly, we have computed the covariant derivatives of curvature tensor field K in the local adapted frame $(\frac{\delta}{\delta q^i}, \frac{\partial}{\partial p_i})$ with respect to the connection ∇ and we obtained in all the cases that the result is zero . Therefore

$$\nabla K = 0.$$

Hence we may state our main result.

Theorem 9. *Assume that the Riemannian manifold (M, g) has positive constant sectional curvature c. Let J be the class of natural, complex structure of diagonal type on T_0^*M, given by (3) and (13). Let G be the class of the natural Riemannian metrics of diagonal type on T_0^*M, given by (14) and (24).*

Then (G, J) is a class of locally symmetric Kähler Einstein structures on T_0^*M, depending on one essential parameter λ, which must fulfill the conditions (23):

$$\lambda > 0, \quad 2c - A^2 t\lambda^2 > 0, \quad \lambda + 2t\lambda' > 0 \ \ \forall t > 0, \quad A > 0.$$

Example. The function $\lambda = \frac{\sqrt{2c}}{A\sqrt{t+B}}$, $A, B \in \mathbf{R}_+$, fulfill the conditions (23).

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