Geometry - an irreductible component of Culture

Nicolae Soare

Abstract

The present paper include some aspects and facets of geometry.

Mathematics Subject Classification: 00-02, 01-02.
Key words: history of Mathematics.

Geometry is a science whose definition can not be condensed in just one sentence, as it is accustomed when it comes to a mathematical topic. Its apparition and development lasted more than 4000 years and keep on, making up a well delimited domain in the knowledge accumulated by humanity during centuries.

Geometry is a part of the humanity culture, having different valences: the multiplicity of useful applications, but purely theoretical topics as well, which bring it close to art; practical interests, but also interests in knowing, together with activities upheld by the essentially human attraction to doubtful issues and, in lots of situations, by the perception of beauty.

Next we will briefly present its content and development stages.

1. Geometry as a logical system. Euclid. Euclid's "Elements of Geometry" (ca. 300 B.C.) is one of the great achievements of the human mind. It makes geometry a deductive science and geometrical phenomena as logical conclusions of a system of axioms and postulates. The content is not restricted to geometry as we now understand the term.

Its main geometrical results are:

a) Pythagoras' Theorem;

b) Angle-sum of a triangle being 180°.

The result b) is derived using the fifth (or the last) Euclid postulate, which says: "If a straight line falling on two straight lines makes the angles which are internal and on the same side, less than two right angles, then the two straight lines meet on the side on which the angles are less than two right angles.

Euclid realized that the parallel postulate was not as transparent as his other axioms and postulates. Efforts were made to prove it as a consequence. Their failure

Balkan Journal of Geometry and Its Applications, Vol.10, No.1, 2005, pp. 24-27.

led to the discovery of non-Euclidean geometry by C. F. Gauss, John Bolyai and N. I. Lobachevschi in the early 19th century.

The Euclid's "Elements" treated rectilinear figures and the circle. The last three of its thirteen Books were devoted to solid geometry.

2. Coordinatization of space. Descartes. The introduction of coordinates by Descartes (1596-1650) was a revolution in geometry. Descartes' work was published in 1637 as an appendix to his famous book on philosophy, entitled "Discours de la méthode". At about the same time Fermat (1601-1665) also found the concept of coordinates and used them to treat successfully geometric problems by algebraic methods. But Fermat's work was published only posthumously.

One immediate consequence was the study of curves defined by arbitrary equations $f(x, y) = 0$, thus enlarging the scope of the figures.

Fermat went on to introduce some of the fundamental concepts of the calculus, such as the tangent line and the maxima and minima. From two dimensions one goes to n dimensions, and to an infinite number of dimensions. In these spaces one studies loci defined by arbitrary systems of equations. Thus a great vista was opened, and geometry and algebra became inseparable.

A mystery is the role of differentiation. The analytic method is most effective when the functions involved are smooth.

Geometrical coordinates lightened the way to applications to physics. An example was the discovery of the law of gravitation by Newton, taking in consideration Kepler's laws, discovery that was possible only after an analytic theory of conics had been established.

3. Space based on the group concept. Felix Klein's Erlangen Program. Works on geometry led to the development of projective geometry, among whose founders were: J. V. Poncelet (1788-1867), A. F. Möbius (1790-1868), M. Chasles (1793-1880) and J. Steiner (1796-1863). Projective geometry studies the geometrical properties arising from the linear subspaces of a space and the transformations generated by projections and sections. Other geometries resulted, as affine geometry and conformal geometry.

In 1872 Felix Klein formulated his Erlangen Program, which defines geometry as the study of the properties of a space, that are invariant under a group of transformations.

The most important application of the Erlangen Program was the treatment of non-Euclidean geometry by the so-called Cayley-Klein projective metric. Sophus Lie founded a theory of transformation groups, which became a fundamental tool of all geometry.

4. Localization of geometry. Gauss and Riemann. In his monograph on surface theory published in 1827, Gauss (1777-1855) developed the geometry on a surface based on its fundamental form. This was generalized by B. Riemann (1826-1866) to n dimensions in his Habilitationschrift in 1854. Riemannian geometry is the geometry based on the quadratic differential form ds^2. Given ds^2, one can define the arc length of a curve, the angle between two intersecting curves, the volume of a domain and other geometrical concepts.

The main characteristic of this geometry is that it is local: it is valid in a neighborhood of the n-space. Because of this feature it fits well with field theory in physics.

Einstein's general theory of relativity interprets the physical universe as a four-dimensional Lorentzian space (with a ds^2 of signature +++-) satisfying the field equations.

It was observed that most properties of Riemannian geometry derive from its Levi-Civita parallelism, an infinitesimal transport of the tangent spaces.

In other words, Riemannian geometry studies the tangent bundle of a Riemannian space with the Levi-Civita connection.

5. Globalization. Topology. Riemannian geometry and its generalizations in differential geometry are local in character. It is very interesting to observe that we do need a whole space to piece the neighborhoods together. This is achieved by topology. The notion of a differentiable manifold is one of the most sophisticated concepts in mathematics. This idea was clear to Riemann, but the first formulation of a topological manifold was made by D. Hilbert in 1902. Hermann Weyl identified the Riemann surfaces with one-dimensional complex manifolds and used it as the central them of his famous book "Die Idee der Riemannschen Flӑhe" (Leipzig, 1913).

Hassler Whitney saw the merit of establishing an imbedding theorem on differentiable manifolds (1936), thus beginning the serious study of differential topology. That derivatives play a role in topology came as a shock when J. Milnor discovered the exotic differentiable structures on the seven-dimensional sphere (1956).

By studying the Yang-Mills equations on a four-dimensional manifold, S. Donaldson found in 1983 a remarkable theorem, which led to the existence of an infinite number of differentiable structures $\mathbf{R}^4$.

With the foundation of differentiable manifolds laid, geometrical structures, such as the Riemannian structure, the complex structure, the conformal structure, the f-structure can now be defined on them and studied using the exterior differential calculus and the tensor analysis.

A fundamental notion is "curvature", in its different forms, the simplest one being the circle in plane Euclidean geometry. It could also be the force of a physical system or the strength of a gravitational or electro-magnetic field. In mathematical terms it measures the non-commutativity of covariant differentiation.

It is remarkable that suitable algebraic combinations of curvature give topological invariants, as for example the Gauss-Bonnet Theorem, where the Euler-Poincaré characteristic appears.

6. Connections in a fiber bundle. Elie Cartan. A notion which includes both Klein's homogeneous spaces and Riemann's local geometry is Cartan's generalized spaces (espaces généralisés). In modern terms it is called a connection in a fiber bundle, which means a straightforward generalization of the Levi-Civita parallelism, which is a connection in the tangent bundle of a Riemannian manifold.

In general, we have a fiber bundle $\pi : E \longrightarrow M$, whose fibers $\pi^{-1}(x), x \in M$ are homogeneous spaces acted on by a Lie group G. A connection is an infinitesimal transport of the fibers compatible with the group action by G.

So far the most far-reaching applications of geometry are to physics, from which it is indeed inseparable. We can mention here an application to biology, namely, to the structures of DNA molecules. This is known to be a "double helix", which geometrically means a pair of closed curves. Their geometrical invariants will clearly be of significance in biology.

Contemporary geometry is thus a far echo from Euclid and new discoveries lead to more general definitions of geometry. It is satisfying to note that so far almost all the sophisticated notions introduced in geometry have been found useful. This proves that geometry in an irreductible component of the culture.

References

[1] R. Taton, *The General History of Science*, Vol. I-IV, (in Romanian), Ed. Şt. Enciclopedică, Bucharest, 1970, 1971, 1972, 1976.

Nicolae Soare
University of Bucharest, Faculty of Mathematics and Informatics
14 Academiei St., RO-010014, Bucharest, Romania
e-mail address: nsoare@geometry.math.unibuc.ro