Gheorghe Titeica and Affine Differential Geometry

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Abstract

In 1908 Titeica showed that for a surface in Euclidean 3-space the ratio of the Gaussian curvature to the fourth power of the distance from a fixed point to the tangent plane is invariant under an affine transformation fixing O. He defined an S-surface to be any surface for which this ratio is constant. These S-surfaces turn out to be what are now called proper affine spheres with center at O.

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Since 1748, in "**Introductio in analysis infinitorum**", Euler has considered transformations of the plane, more general than similarities. The images of plane curves by these transformations were curves related to the initial ones and Euler called them affine curves, from the Latin "affinitas". These transformations are combinations of translations and homotheties in two orthogonal directions given by the formulae:

$$\begin{cases} x' &= a_{11}x + a_{12}y + a_1 \\ y' &= a_{21}x + a_{22}y + a_2 \end{cases},$$

called later affine transformations. During the first three decades of the 19^{th} century, Möbius studied properties of geometrical figures obtained one from another by affine transformations.

But the notation of affine geometry arose after the Program from Erlagen of Felix Klein, appeared in 1872, which stated that to every group of geometrical transformations of a space corresponds its geometry, studying properties of figures, invariant under the action of these transformations.

From elementary point of view, the Euclidean geometry represents the study of the properties concerning the length, area, volume, measure of angle. These properties are invariants of the group of Euclidian motions. Considering a wider group, Felix Klein obtained a more general geometry.

The group of affine transformations has different subgroups such as the equiaffine subgroup, the center affine subgroup, etc. Hence simultaneously with the general affine geometry appeared geometries emerged from it, corresponding to these subgroups: equiaffine geometry, center affine geometry, etc.

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Even the Euclidian geometry, from this point of view, subordinates to affine geometry. This one is a special case of projective geometry and it is possible to extend the theory to more general geometries such as the pseudo-Euclidean geometry.

We should remark that the Programm from Erlagen was published only after 20 years, first in Italy (1890), then in France (1891) and finally in Germany (1893).

Although the original programme of Felix Klein was conceived for elementary geometries, later, at the beginning of the 20th century, this programme was applied to differential geometry studying problems corresponding to certain groups of transformations.

The first result of affine differential geometry was obtained by A. Transon in 1841, considering the affine normal to a curve in "**Recherches sur la courbure des lignes et des surfaces**", J. Math. Pures et Appl. 6(1841), 191-208.

Only after 70 years a systematical and intensive study of affine properties of curves and surfaces began. So, in 1906 Pick initiated the development of the theory of natural equations in the geometry of an arbitrary group of continuous transformations of the plane. Using methods from the theory of Lie groups he gave a procedure to obtain invariants, generalizing the notion of length of an arc of curve- the curvature of a curve in Euclidian geometry.

In 1907 Gheorghe Titeica discovered a class of surfaces of the Euclidian space $\mathbf{R}^3$, the S-surfaces, which are examples of the so called today affine spheres. From a historical point of view, Gheorghe Titeica is the first geometer who studied the affine spheres, using Euclidian invariants; a Titeica surface has the property that the ratio between the Gauss Euclidian curvature of the surface at a point P and the fourth power of the distance from a fixed point O to the tangent plane at P of the surface is constant, so it is invariant with respect to center affine transformations, O being invariable.

The paper from 1907 **Sur certaines surfaces reglées** was published in C. R. Acad. Paris, 145, 132-133. Next years Titeica published other articles on the same line. Following the ideas of Felix Klein, at the Vth International Congress of Mathematics, Edinburgh (Cambridge), 1912, Gheorghe Titeica rised the problem of the study of curves and surfaces with respect to different groups of transformations.

Ten years after Titeica's discovery many geometers began to study properties of curves and surfaces with respect to the equiaffine group of transformations (the unimodular group): Berwald, Blaschke, Franck, Gross, Koning, Liebmann, Pick, Radon, Reidemaster, Salkowski, Thomsen and Winternitz. The development was fast and Blaske and Reidemeister published the first monograph of affine differential geometry in 1923.

The study of affine surfaces was extended also by E. Cartan who published two notes in Comptes Rendus in 1924 and also by Slebodzinski in 1937 and 1939:

"**Sur quelques problèmes de la théorie des surfaces de l'espace affine**" Prace Mat. Fiz 46 (1939), and

"**Sur la realization d'une variété à connexion affine par une surface plongée dans un espace affine**, C. R. (1937).

The research of Blaschke and his school was continued by the Japanese school: Kubota, Süss, Su Buging, Nakajima, etc, many papers being published in Tohoku Math. Journal and Japanese Journal of Math.

The last 20 years represented a period of great progress and development for the differential geometry. Besides many papers, also monographs presenting important classical results, modern and actual discoveries and open problems were published.

References

[1] W. Blaschke, *Vorlesurgen über Differentialgeometrie*, I, II, Berlin, Springer, 1921, 1923.

[2] K. Nomizu, T. Sasaki, *Affine Differential Geometry*, Cambridge University Press, 1994.

[3] P.A. Schirokow, A.P. Schirokow, *Affine Differentialgeometrie* Leipzig, Teubner, 1962.

[4] G. Tzitzeica, *Sur une nouvelle classe de surfaces*, Rend. Circ. Mat. Palermo, 25 (1908), 180-187; 28 (1909), 210-216.

[5] Gh. Vrănceanu, *Tzitzeica, fondateur de la géométrie centroaffine*, Revue Roumaine Math. Pures Appl. 24, 1979, 983-988.

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