Geometry and Vision

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Abstract

The exposition emphasizes basic aspects of the interrelation between geometry and vision.

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There exists a profound interplay between the workings of the natural world and the laws and sensitivities of thought (cf. [1]). The most fundamental and firmly accepted parts of our general scientific knowledge of the world involve mathematical models (cf. [2]). And, as Chern begins his introduction to [3]: "While algebra and analysis provide the foundations of mathematics, geometry is at the core". Geometry is the field of mathematics whose main source of intuition is human visual perception. So, it seems appropriate that geometry would contribute somewhat to a better understanding of visual perception. Paraphrasing Feynman's saying that "Nature speaks to us in the language of mathematics", what follow may illustrate that "Nature likes to be looked at with geometer's eyes and brains". Vision is one of our most important senses giving contact with the world outside of us, even to the point of often being of essential importance for our survival. Thus, in humans, as in all creatures gifted with vision, by inheritance and through individual experience, in both of which in a wide sense cultural facts guiding the meaning of vision having been and being of great influence, through permanent eye and brain activities evolved and evolves an individual visual system. By human "early vision" is meant our mere actual registration of visual data, and it is the purpose of the present note to qualitatively describe a geometrical model for this process. For a more technical description of this model, see [4] [5]. For fundamentals on visual sensation and perception and on the psychology of seeing, see e.g. [6] [7] [8] [9] [10].

Primarily, any visual observation amounts to the recording of light-energy or "luminosity" L. For simplicity, consider a static image I in a plane P; (the following exposition readily can be adapted to more general images). As stated so well by Koenderink and van Doorn in their comments on the nature of observation [11], our only knowledge about the "real" image I in P is its observation, which formally is a surface in a 3-dimensional space S, called the "visual stimulus surface" M: z = L(x,y) corresponding to I, whereby x and y are Cartesian co-ordinates in P and the values

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z = L(x,y) are the luminosities recorded at the points (x,y). For simplicity, S will be considered as the 3-dimensional Euclidean space E; (the following exposition readily can be adapted to other geometries on the 3-dimensional space S which in a more subtle way describe the physical reality of vision, cf. [4] [5] [12]). So when looking at an image I in a plane P, our visual system observes a visual stimulus surface M in E based on which it makes us aware of a corresponding image I' which is our actual registration of this visual observation. It is only natural then to claim that, in early vision, our visual system registrates in I' the main shape-characteristics of the surface M in E. Although in the study of the geometry of surfaces M in E mostly the mean curvature H and the Gauss curvature K have been studied so far, a curvature like e.g. the Casorati curvature which is defined as the average of the squared principal curvatures of M in E (or, equivalently, the square of the norm of the second fundamental form or still the "curvedness" [13] [14] [15]) is a scalar-valued shape-characteristic for surfaces in 3-space that corresponds far better with our intuitive concept of surface-curvature. So, in accordance to the visual stimulus surface M in E given by an image I in a plane P, the proposed model yields an image I' in P which is determined by the Casorati curvatures of M in E.

Based on experience, we generally believe what we see, sometimes to the extreme as expressed in the saying that " I will only believe it when I have seen it!" Yet, human vision contains so-called "illusions" that have intrigued many people since antiquity, and concern essentially "misperceptions" of concrete distances, area's, directions, ... and perceptions of "non-existing" lines, contours and figures, as for instance the illusions of Muller-Lyer, Hering, Wundt, Oppell-Kundt, Zollner, Poggendorf, Judd, Ponzo, Titchener, Ebbinghaus, Ehrenstein, Orbison, Kanisza, Schumann, brightness, Mach, Hermann, the Bristol cafe-wall, the corridor, ..., (see e.g. [6] [7] [8] [9] [16] [17] [18]). In such visual illusions which in mathematical terms all concern very elementary planar images I (like just a few lines, triangles, squares, etc.), the deviations of our perceptions from the "real" images I can readily be measured. Quoting (somewhat freely) from Gregory's "Eye and Brain" [6] : "... Physicists and physiologists, psychologists and philosophers, have tried to explain such illusions for over a hundred years. ... The physical sciences take immense trouble to avoid errors. In the science of vision we seek out and study errors for understanding how we see and to suggest something of how the brain works. The weird and wonderful errors of illusions are not trivial. They are truly phenomenal phenomena, central to art and a major reason for the experimental methods of science". Applying the above model for early vision to the images I which yield the classical and continuously additionally introduced so-called illusions, it is such that the images I' (lines, triangles, squares, etc.) given by the extrema of the Casorati curvatures of the visual stimulus surface M in E corresponding to the images I, do indeed produce the human visual perceptions of these images I : "Wonder en is gheen Wonder" as read Simon Stevin's motto!

Likewise, for instance, the phenomena of visual sensitivity [19] and of the Julesz binocular so-called "stereopsis" [6] are equally naturally described by this model. And, of course, everyone is invited to study some of the manifold interesting phenomena in the broad science of vision in the light of the above geometrical model.

Finally, recall that Gregory's approach to the study of perception is based on regarding perception "as constructing hypotheses ... which may hit upon truth by producing symbolic structures matching physical reality" [20]. In view of the above, this approach could somewhat liberally be reformulated as follows [5] : visual perception is the construction, inspired by images registrated by early vision, of hypotheses which produce visual images that match physical reality in accordance with our previously acquired experience of this reality.

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