Tangent bundles of quasi-constant holomorphic sectional curvatures

C.L. Bejan and V. Oproiu

Dedicated to the memory of Radu Rosca (1908-2005)

Abstract. Among all the natural almost Kählerian structures on the tangent bundle TM, we select those with the property that any holomorphic plane making a certain angle with Liouville vector field have the same curvature. Mainly, we prove that this happens only for those structures with constant holomorphic sectional curvature.

Mathematics Subject Classification: 53C55, 53C15.
Key words: tangent bundle, natural lifts, Kählerian structures, quasi-constant holomorphic sectional curvature.

1 Introduction

The tangent bundle TM of a Riemannian manifold (M, g) has many nice geometric properties, and furnishes important examples arising in various geometric classifications.

It is well known (see [14], [17]) that the splitting of the tangent bundle to TM into the vertical and horizontal distributions, defined by the Levi Civita connection of g on M, and the corresponding Sasaki metric lead to an almost Kähler structure on TM. The results from [7] (see also [6], [11]), giving a general expression of the natural 1-st order lifts of the Riemannian metric g to TM, allow us to consider some interesting problems concerning the diagonal natural 1-st order almost Hermitian lifts of g to TM. The second author has studied some properties of a special natural 1-st order lift G of g and a natural almost complex structure J on TM (see [9], [10], [12], and see also [11], [13]).

In section 2 we provide a general construction of a family of natural almost Hermitian structures on the tangent bundle TM of a Riemannian manifold (M, g). Among all these structures (which are of diagonal type) an interesting goal is to select, in sections 3 and 4, those which are Kählerian. As for a Kähler manifold carrying a unit vector field ξ, in section 5 we recall from [1] the notion of quasi-constant holomorphic sectional curvature, meaning that the curvature of any holomorphic plane depends on both the point and its angle with ξ. We apply this to the set of non-zero tangent vectors, among which the existence of the Liouville vector field arises the question of

Balkan Journal of Geometry and Its Applications, Vol.11, No.1, 2006, pp. 11-22.

whether the above Kählerian structures are or are not of quasi-constant holomorphic sectional curvatures. In section 6 we prove that this happens if and only if TM has constant holomorphic sectional curvature.

Several computations have been done by using the RICCI package under Mathematica for doing tensor calculations in differential geometry.

All geometric objects are assumed to be smooth. We use the computations in local coordinates in a fixed local chart though many results admit an invariant form via the vertical and horizontal lifts. The summation convention is used throughout over the indices h, i, j, k, l running $\{1, \dots, n\}$.

2 Natural almost complex structures of diagonal type on TM

Let (M, g) be a smooth n-dimensional Riemannian manifold and denote its tangent bundle by $\tau : TM \longrightarrow M$. To fix notation, the manifold structure of TM is obtained from the manifold structure of M whose local charts $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \dots, x^n, y^1, \dots, y^n)$ are induced from the local charts $(U, \varphi) = (U, x^1, \dots, x^n)$ on M, where the local coordinates $x^i, y^i,\ i = 1, \dots, n$, are defined as follows. The first n local coordinates of a tangent vector $y \in \tau^{-1}(U)$ are the local coordinates in the local chart (U, φ) of its base point, i.e. $x^i = x^i \circ \tau$, by an abuse of notation. The last n local coordinates $y^i,\ i = 1, \dots n$, of $y \in \tau^{-1}(U)$ are the vector space coordinates of y with respect to the natural basis in the local chart (U, φ). A useful concept in the differential geometry of TM is that of M-tensor field (of type (p, q)) which is defined by sets of n^{p+q} components (functions of x and y) with p upper indices and q lower indices, assigned to induced local charts $(\tau^{-1}(U), \Phi)$ on TM, such that the local coordinate change rule is that of the local coordinate components of a (p, q)-tensor field on the base manifold M (see [8] for further details); e.g., the components $y^i,\ i = 1, \dots, n$, corresponding to the last n local coordinates of a tangent vector y, assigned to the induced local chart $(\tau^{-1}(U), \Phi)$ define an M-tensor field of type $(1, 0)$. Assume that $u : [0, \infty) \longrightarrow \mathbf{R}$ is a smooth function and let $\|y\|^2 = g_{\tau(y)}(y, y)$ be the square of the norm of the tangent vector y. If δ^i_j (the Kronecker symbols) are the local coordinate components of the identity $(1, 1)$-tensor field I on M, then the components $u(\|y\|^2)\delta^i_j$ define an M-tensor field of type $(1, 1)$ on TM. The components $u(\|y\|^2)g_{ij}$ define an M-tensor field of type $(0, 2)$ on TM, where g is the metric tensor field on M. The components $g_{0i} = y^k g_{ki}$ define an M-tensor field of type $(0, 1)$ on TM.

The Levi Civita connection $\dot{\nabla}$ of g on M gives the direct sum decomposition

$$TTM = VTM \oplus HTM \tag{2.1}$$

of the tangent bundle to TM into the vertical distribution $VTM = \mathrm{Ker}\ \tau_*$ and the horizontal distribution HTM. The set of vector fields $(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n})$ on $\tau^{-1}(U)$ defines a local frame field for VTM and for HTM we have the local frame field $(\frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n})$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{0i}\frac{\partial}{\partial y^h}, \quad \Gamma^h_{0i} = y^k \Gamma^h_{ki}$$

and $\Gamma^h_{ki}(x)$ are the Christoffel symbols of g.

The set $(\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}, \frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n})$ defines a local frame on TM, adapted to the direct sum decomposition (1). Remark that

$$\frac{\partial}{\partial y^i} = (\frac{\partial}{\partial x^i})^V, \quad \frac{\delta}{\delta x^i} = (\frac{\partial}{\partial x^i})^H,$$

where X^V and X^H denote the vertical and horizontal lift of the vector field X on M which help us to obtain invariant expressions later on. However, in local coordinates, the formulae are more direct, and more natural, in a certain sense.

We begin by considering the energy density of the tangent vector y

$$t = \frac{1}{2}\|y\|^2 = \frac{1}{2}g_{\tau(y)}(y, y) = \frac{1}{2}g_{ik}(x)y^i y^k, \quad y \in \tau^{-1}(U). \tag{2.2}$$

Obviously, we have $t \in [0, \infty)$ for all $y \in TM$. By direct computation we obtain

Lemma 1. *If $n > 1$ and u, v are smooth functions on TM such that either*

$$ug_{ij} + vg_{0i}g_{0j} = 0,$$

or

$$u\delta^i_j + vg_{0j}y^i = 0,$$

on the domain of any induced local chart on TM, then $u = v = 0$.

Denote by $C = y^i\frac{\partial}{\partial y^i}$ the Liouville vector field on TM and by $\widetilde{C} = y^i\frac{\delta}{\delta x^i}$ the similar horizontal vector field on TM. Let $a_1, a_2, b_1, b_2 : [0, \infty) \to \mathbf{R}$ be some smooth functions. A natural 1-st order almost complex structure J of diagonal type on TM is given by (see [7])

$$\begin{cases} J\frac{\delta}{\delta x^i} = a_1(t)\frac{\partial}{\partial y^i} + b_1(t)g_{0i}C, \\ \\ J\frac{\partial}{\partial y^i} = -a_2(t)\frac{\delta}{\delta x^i} - b_2(t)g_{0i}\widetilde{C}. \end{cases} \tag{2.3}$$

Proposition 2 [10]. *The operator J defines an almost complex structure on TM if and only if*

$$a_1 a_2 = 1, \quad (a_1 + 2tb_1)(a_2 + 2tb_2) = 1. \tag{2.4}$$

Remark (i) As all coefficients $a_1, a_2, a_1 + 2tb_1, a_2 + 2tb_2$ from (4) are non-zero and of the same sign, we may assume them positive for any $t \geq 0$.

(ii) By (4), two of the coefficients $a_1,\ a_2,\ a_3,\ b_1,\ b_2$ are functions of the other two; e.g. we have:

$$a_2 = \frac{1}{a_1}, \quad b_2 = \frac{-a_2 b_1}{a_1 + 2tb_1} = \frac{-b_1}{a_1(a_1 + 2tb_1)}. \tag{2.5}$$

To express the integrability condition of J we use the vanishing of its Nijenhuis tensor field N_J, defined by

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y],$$

for all vector fields X and Y on TM.

Theorem 3. [10] *Let (M, g) be an $n(> 2)$-dimensional connected Riemannian manifold. The almost complex structure J defined by (3) on TM is integrable if and only if (M, g) has constant sectional curvature c and the coefficient b_1 is given by:*

(2.6) $$b_1 = \frac{a_1 a_1' - c}{a_1 - 2ta_1'}$$

(compare with the corresponding expressions from [9] and [15]).

3 Natural diagonal almost Kählerian structures on TM

Consider a diagonal 1-st order, natural F-metric G on TM (see [7], see also , [6], [11]), given by

(3.1) $$G(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = c_1 g_{ij} + d_1 g_{0i} g_{0j}, G(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = c_2 g_{ij} + d_2 g_{0i} g_{0j},$$

$$G(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}) = G(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}) = 0,$$

where c_1, c_2, d_1, d_2 are smooth functions depending on the energy density $t \in [0, \infty)$. The conditions for G to be positive definite are assured if

(3.2) $$c_1 > 0,\ c_2 > 0,\ c_1 + 2td_1 > 0,\ c_2 + 2td_2 > 0.$$

We establish here the conditions under which the metric G is almost Hermitian with respect to the almost complex structure J, considered in the previous section, i.e.

$$G(JX, JY) = G(X, Y),$$

for all vector fields X, Y on TM.

Considering the coefficients of g_{ij} in the conditions

(3.3) $$\begin{cases} G(J\frac{\delta}{\delta x^i}, J\frac{\delta}{\delta x^j}) = G(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}), \\ G(J\frac{\partial}{\partial y^i}, J\frac{\partial}{\partial y^j}) = G(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}), \end{cases}$$

we obtain the following expressions

(3.4) $$c_1 = \lambda a_1, \quad c_2 = \lambda a_2,$$

where $\lambda = \lambda(t)$ is a positive smooth function of $t \in [0, \infty)$. (Recall the assumptions $a_1, a_2 > 0$).

Next, considering the coefficients of $g_{0i} g_{0j}$ in the relations (9) and using (10), we obtain the following expressions

(3.5) $$\begin{cases} c_1 + 2td_1 = (\lambda + 2t\mu)(a_1 + 2tb_1), \\ c_2 + 2td_2 = (\lambda + 2t\mu)(a_2 + 2tb_2), \end{cases}$$

where $\lambda + 2t\mu = \lambda(t) + 2t\mu(t)$ is a positive smooth function of $t \in [0, \infty)$. The conditions (8) are automatically fulfilled, due to the properties (4) of the coefficients a_1, a_2, b_1, b_2. From (14), d_1 and d_2 have the following explicit expressions

(3.6) $$d_1 = \lambda b_1 + \mu(a_1 + 2tb_1), \quad d_2 = \lambda b_2 + \mu(a_2 + 2tb_2).$$

Remark If $\lambda = 1$ and $\mu = 0$, we obtain the almost Kählerian structure constructed in [10].

Consider now the two-form Ω defined by the almost Hermitian structure (G, J) on TM
$$\Omega(X, Y) = G(X, JY),$$
for all vector fields X, Y on TM.

The expression of the 2-form Ω in a local adapted frame $(\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}, \frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n})$ on TM, is given by
$$\Omega(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = 0,\ \Omega(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = 0,\ \Omega(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}) = \lambda g_{ij} + \mu g_{0i} g_{0j}$$
or, equivalently

(3.7) $$\Omega = (\lambda g_{ij} + \mu g_{0i} g_{0j}) \dot{\nabla} y^i \wedge dx^j,$$

where $\dot{\nabla} y^i = dy^i + \Gamma^i_{0h} dx^h$ is the absolute differential of y^i.

From the following formula
$$d\Omega = \frac{1}{2}(\lambda' - \mu)(g_{ij} g_{0k} - g_{0i} g_{jk}) \dot{\nabla} y^k \wedge \dot{\nabla} y^i \wedge dx^j,$$
obtained by a straightforward computation and following the same idea as in [11], we obtain

Theorem 4. *The almost Hermitian structure* (TM, G, J) *is almost Kählerian if and only if*
$$\mu = \lambda'.$$

Thus the family of almost Kählerian structures of diagonal type on TM depends on three essential coefficients a_1, b_1, λ. Combining the results from Theorems 3 and 4, it follows that the coefficient b_1 can be expressed as a function of a_1 and its first derivative, so that a natural Kählerian structure (G, J) of diagonal type on TM is defined by two essential coefficients a_1, λ, which have to satisfy some additional conditions $a_1 > 0, a_1 + 2tb_1 > 0, \lambda > 0$. Examples of such structures can be found in [15] (see also [9], [10], [12]).

4 The Levi Civita connection and its curvature tensor field on TM

Assume that (TM, G, J) is Kählerian, hence the base manifold M has constant sectional curvature and the parameters a_2, b_1, b_2 are given by (5), (6), while the coefficients c_1, c_2, d_1, d_2 are given by (10), (12), where $\mu = \lambda'$. Denote by $\delta_i = \frac{\delta}{\delta x^i},\ \partial_i = \frac{\partial}{\partial y^i},\ i = 1, \ldots, n$. The local expression of the Levi Civita connection of G is given in an adapted local frame $(\partial_1, ..., \partial_n, \delta_1, ..., \delta_n)$ by

(4.1) $$\begin{cases} \nabla_{\partial_i} \partial_j = Q^h_{ij} \partial_h, \quad \nabla_{\delta_i} \partial_j = \Gamma^h_{ij} \partial_h + P^h_{ji} \delta_h, \\ \nabla_{\partial_i} \delta_j = P^h_{ij} \delta_h, \quad \nabla_{\delta_i} \delta_j = \Gamma^h_{ij} \delta_h + S^h_{ij} \partial_h, \end{cases}$$

where Γ^h_{ij} are the Christoffel symbols of the connection $\dot\nabla$ and the $M-$tensor fields P^h_{ij}, Q^h_{ij}, S^h_{ij} are given by

$$P^h_{ij} = \frac{c_1'}{2c_1} g_{0i}\delta^h_j + \frac{d_1 - cc_2}{2c_1} g_{0j}\delta^h_i + \frac{d_1 + cc_2}{2(c_1 + 2td_1)} g_{ij} y^h -$$

$$-\frac{c_1' d_1 + d_1^2 - c_1 d_1' - cc_2 d_1}{2c_1(c_1 + 2td_1)} g_{0i} g_{0j} y^h,$$

$$Q^h_{ij} = \frac{c_2'}{2c_2}(g_{0i}\delta^h_j + g_{0j}\delta^h_i) + \frac{2d_2 - c_2'}{2(c_2 + 2td_2)} g_{ij} y^h + \frac{c_2 d_2' - 2d_2 c_2'}{2c_2(c_2 + 2td_2)} g_{0i} g_{0j} y^h,$$

$$S^h_{ij} = \frac{cc_2 - d_1}{2c_2} g_{0i}\delta^h_j - \frac{cc_2 + d_1}{2c_2} g_{0j}\delta^h_i - \frac{c_1'}{2(c_2 + 2td_2)} g_{ij} y^h +$$

$$+\frac{2d_1 d_2 - c_2 d_1'}{2c_2(c_2 + 2td_2)} g_{0i} g_{0j} y^h.$$

Denote by $\dot R^h_{kij} = \dot R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^k}$ the components of the curvature tensor field $\dot R$ of $\dot\nabla$, and by $\dot R^h_{0ij} = y^k \dot R^h_{kij}$ The curvature tensor field of ∇ is denoted by R. Its components in the local adapted frame $(\partial_1, ..., \partial_n, \delta_1, ..., \delta_n)$ are given by

$$R(\delta_i, \delta_j)\delta_k = XXX^h_{ijk}\delta_h,\ R(\delta_i, \delta_j)\partial_k = XXY^h_{ijk}\partial_h,$$

$$R(\partial_i, \partial_j)\partial_k = YYY^h_{ijk}\partial_h,\ R(\partial_i, \partial_j)\delta_k = YYX^h_{ijk}\delta_h, \tag{4.2}$$

$$R(\partial_i, \delta_j)\delta_k = YXX^h_{ijk}\partial_h, R(\partial_i, \delta_j)\partial_k = YXY^h_{ijk}\delta_h,$$

where the components $XXX^h_{ijk}, \ldots$ are given by

$$XXX^h_{ijk} = \dot R^h_{kij} + P^h_{lk}\dot R^l_{0ij} + P^h_{li} S^l_{jk} - P^h_{lj} S^l_{ik},$$

$$XXY^h_{ijk} = \dot R^h_{kij} + S^h_{il} P^l_{kj} - S^h_{jl} P^l_{ki} + Q^h_{lk}\dot R^l_{0ij},$$

$$YYX^h_{ijk} = \frac{\partial}{\partial y^i} P^h_{jk} - \frac{\partial}{\partial y^j} P^h_{ik} + P^h_{il} P^l_{jk} - P^h_{jl} P^l_{ik},$$

$$YYY^h_{ijk} = \frac{\partial}{\partial y^i} Q^h_{jk} - \frac{\partial}{\partial y^j} Q^h_{ik} + Q^h_{il} Q^l_{jk} - Q^h_{jl} Q^l_{ik}$$

$$YYX^h_{ijk} = \frac{\partial}{\partial y^i} S^h_{jk} + Q^h_{il} S^l_{jk} - S^h_{jl} P^l_{ik},$$

$$YXY^h_{ijk} = \frac{\partial}{\partial y^i} P^h_{kj} + P^h_{il} P^l_{kj} - P^h_{lj} Q^l_{ik}.$$

5 Kähler manifolds of quasi constant holomorphic sectional curvatures

Let (M, g, J) be a Kähler manifold endowed with a unit vector field ξ, where g is the Riemannian metric and J is the complex structure. The manifold (M, g, J, ξ) is said to be of quasi-constant holomorphic curvatures (see [1], [5]) if for any holomorphic section *span* $\{X, JX\}$ generated by the unit tangent vector $X \in T_pM,\ p \in M$ with $\varphi = \angle\ (span\{X, JX\}, \xi)$, the Riemannian sectional curvature $R(X, JX, JX, X)$ may only depend on the point $p \in M$ and the angle φ, i.e.

$$R(X, JX, JX, X) = f(p, \varphi), \quad p \in M,\ \varphi \in [0, \pi/2].$$

This notion is the Kählerian correspondent to the notion of a Riemannian manifold of quasi-constant sectional curvatures (see [3], [4]). One shows (see [1], [5]) that a Kählerian manifold (M, g, J, ξ) is of quasi-constant holomorphic sectional curvature if and only if the curvature tensor field R of ∇ satisfies the identity

$$R = \kappa_0 R_0 + \kappa_1 R_1 + \kappa_2 R_2,$$

where $\kappa_0, \kappa_1, \kappa_2$ are smooth functions on M and R_0, R_1, R_2 are certain tensor fields of curvature type on M which will be described below. The first tensor field R_0 is given by the expression which defines the Kählerian manifolds of constant holomorphic curvature, i.e.

$$\begin{aligned} R_0(X,Y)Z = \tfrac{1}{4}\{&g(Y,Z)X - g(X,Z)Y + \\ &+ g(JY,Z)JX - g(JX,Z)JY + 2g(X,JY)JZ\}. \end{aligned} \tag{5.1}$$

The next tensor field R_1 depends on the unitary vector field ξ and on the corresponding 1-form η defined by $\eta(X) = g(X, \xi)$. In order to define R_1 we introduce the following auxiliary $(1,3)$-tensor field

$$\begin{aligned} P(X,Y,Z) = \tfrac{1}{8}\{&\eta(Y)\eta(Z)X + \eta(X)\eta(JZ)JY + \\ &+ \eta(X)\eta(JY)JZ + g(Y,Z)\eta(X)\xi + g(X,JZ)\eta(Y)J\xi + \\ &+ \tfrac{1}{2}g(X,JY)\eta(JZ)\xi + \tfrac{1}{2}g(X,JY)\eta(Z)J\xi\}. \end{aligned} \tag{5.2}$$

Then the tensor field R_1 is defined by

$$\begin{aligned} R_1(X,Y)Z &= P(X,Y,Z) - P(Y,X,Z) + \\ &+ P(JX,JY,Z) - P(JY,JX,Z). \end{aligned} \tag{5.3}$$

The last tensor field R_2 is given by

$$\begin{aligned} R_2(X,Y)Z = &\{\eta(X)\eta(JY) - \eta(JX)\eta(Y)\} \\ &\{\eta(JZ)\xi + \eta(X)J\xi\}. \end{aligned} \tag{5.4}$$

One can check easily that the tensor fields R_0, R_1, R_2 have the symmetry and skew-symmetry properties as well as the invariance properties with respect to J, specific to the curvature tensor field on a Kählerian manifold. Moreover, they verify the first Bianchi identity.

6 Tangent bundle as a Kähler manifold of quasi constant holomorphic sectional curvatures

In the case of the tangent bundle TM of a Riemannian manifold of constant sectional curvature we have the Kählerian structure (G,J) considered above and the Liouville vector field $C=y^i\partial_i$. This vector field is non-zero on the subset $T_0M\subset TM$ of all non-zero tangent vectors. Then we could consider the unitary vector field $\frac{1}{\|C\|}C$ and study the property of T_0M to be of quasi-constant holomorphic curvatures. However, we should prefer to work with the vector field C as ξ, since the scalar factors can be incorporated in κ_1,κ_2.

The following notations will simplify our calculus

$$R_0(\delta_i,\delta_j)\delta_k=(XXX_0)^h_{ijk}\delta_h,\ R_0(\delta_i,\delta_j)\partial_k=(XXY_0)^h_{ijk}\partial_h,$$

$$R_0(\partial_i,\partial_j)\delta_k=(YYX_0)^h_{ijk}\delta_h,\ R_0(\partial_i,\partial_j)\partial_k=(YYY_0)^h_{ijk}\partial_h,$$

$$R_0(\partial_i,\delta_j)\delta_k=(YXX_0)^h_{ijk}\partial_h,\ R_0(\partial_i,\delta_j)\partial_k=(YXY_0)^h_{ijk}\delta_h,$$

where

$$(XXX_0)^h_{ijk}=\frac{1}{4}(\delta^h_i(c_1g_{jk}+d_1g_{0j}g_{0k})-\delta^h_j(c_1g_{ik}+d_1g_{0i}g_{0k})),$$

$$(XXY_0)^h_{ijk}=\frac{1}{4}(c_2g_{kl}+d_2g_{0k}g_{0l})\{(a_1\delta^h_i+b_1g_{0i}y^h)(a_1\delta^l_j+b_1g_{0j}y^l)-$$

$$-(a_1\delta^h_j+b_1g_{0j}y^h)(a_1\delta^l_i+b_1g_{0i}y^l)\}$$

$$(YYX_0)^h_{ijk}=\frac{1}{4}(c_1g_{kl}+d_1g_{0k}g_{0l})\{(a_2\delta^h_i+b_2g_{0i}y^h)(a_2\delta^l_j+b_2g_{0j}y^l)-$$

$$-(a_2\delta^h_j+b_2g_{0j}y^h)(a_2\delta^l_i+b_2g_{0i}y^l)\},$$

$$(YYY_0)^h_{ijk}=1/4\{(\delta^h_i(c_2g_{jk}+d_2g_{0j}g_{0k})-\delta^h_j(c_2g_{ik}+d_2g_{0i}g_{0k})\},$$

$$(YXX_0)^h_{ijk}=1/4\{\delta^h_i(c_1g_{jk}+d_1g_{0j}g_{0k})+$$

$$+(a_1\delta^h_j+b_1g_{0j}y^h)(c_1g_{kl}+d_1g_{0k}g_{0l})(a_2\delta^l_i+b_2g_{0i}y^l)+$$

$$+2(a_1\delta^l_j+b_1g_{0j}y^l)(c_2g_{il}+d_2g_{0i}g_{0l})(a_1\delta^h_k+b_1g_{0k}y^h)\},$$

$$(YXY_0)^h_{ijk}=\frac{1}{4}\{-\delta^h_j(c_2g_{ik}+d_2g_{0i}g_{0k})-$$

$$-(a_2\delta^h_i+b_2g_{0i}y^h)(c_2g_{kl}+d_2g_{0k}g_{0l})(a_1\delta^l_j+b_1g_{0j}y^l)-$$

$$-2(a_2\delta^h_k+b_2g_{0k}y^h)(c_2g_{il}+d_2g_{0i}g_{0l})(a_1\delta^l_j+b_1g_{0j}y^l)\}.$$

The components of the tensor field R_1 are obtained in a similar way

$$(XXX_1)^h_{ijk}=\frac{1}{8}(\lambda+2\lambda' t)^2(\delta^h_ig_{0j}g_{0k}-\delta^h_jg_{0i}g_{0k})+$$

$$+\frac{a_1\lambda(a_1-2a_1't)(\lambda+2\lambda' t)}{8(a_1^2-2ct)}(g_{0i}g_{jk}y^h-g_{0j}g_{ik}y^h)$$

$$(XXY_1)^h_{ijk} = \frac{a_1(a_1-2a_1't)(\lambda+2\lambda't)^2}{8(a_1^2-2ct)}(\delta^h_i g_{0j}g_{0k} - \delta^h_j g_{0i}g_{0k})+$$

$$+\frac{1}{8}\lambda(\lambda+2\lambda't)(g_{0i}g_{jk}y^h - g_{0j}g_{ik}y^h),$$

$$(YYX_1)^h_{ijk} = \frac{(a_1-2a_1't)(\lambda+2\lambda't)^2}{8a_1(a_1^2-2ct)}(\delta^h_i g_{0j}g_{0k} - \delta^h_j g_{0i}g_{0k})+$$

$$+\frac{\lambda(a_1-2a_1't)^2(\lambda+2\lambda't)}{8(a_1^2-2ct)^2}(g_{0i}g_{jk}y^h - g_{0j}g_{ik}y^h),$$

$$(YYY_1)^h_{ijk} = \frac{(a_1-2a_1't)^2(\lambda+2\lambda't)^2}{8(a_1^2-2ct)^2}(\delta^h_i g_{0j}g_{0k} - \delta^h_j g_{0i}g_{0k})+$$

$$+\frac{\lambda(a_1-2a_1't)(\lambda+2\lambda't)}{8a_1(a_1^2-2ct)}(g_{0i}g_{jk}y^h - g_{0j}g_{ik}y^h),$$

$$(YXX_1)^h_{ijk} = \frac{a_1(a_1-2a_1't)(\lambda+2t\lambda')^2}{8(a_1^2-2ct)}(2\delta^h_k g_{0i}g_{0j} + \delta^h_j g_{0i}g_{0k} + g_{jk}g_{0i}y^h)+$$

$$+\frac{(\lambda+2t\lambda')^2}{8}\delta^h_i g_{0j}g_{0k} + \frac{\lambda(\lambda+2\lambda't)}{8}(g_{ik}g_{0j}y^h + 2g_{ij}g_{0k}y^h)+$$

$$+\frac{(\lambda+2\lambda't)(2a_1a_1'\lambda - 2c\lambda + 2a_1^2\lambda' + 3a_1a_1'\lambda't - 7c\lambda't)}{4(a_1^2-2ct)}g_{0i}g_{0j}g_{0k}y^h,$$

$$(YXY_1)^h_{ijk} = -\frac{(a_1-2a_1't)(\lambda+2\lambda't)^2}{8a_1(a_1^2-2ct)}(2\delta^h_k g_{0i}g_{0j} + \frac{1}{(a_1^2-2ct)}\delta^h_i g_{0j}g_{0k} + \delta^h_i g_{0j}g_{0k})-$$

$$-\frac{\lambda(a_1-2a_1't)^2(\lambda+2\lambda't)}{8a_1(a_1^2-2ct)^2}(g_{jk}g_{0i}y^h + (a_1^2-2ct)g_{ik}g_{0j}y^h)-$$

$$+\frac{\lambda(a_1-2a_1't)^2(\lambda+2\lambda't)}{4(a_1^2-2ct)^2}g_{ij}g_{0k}y^h-$$

$$-\frac{(a_1-2a_1't)(\lambda+2\lambda't)(-2a_1a_1'\lambda + 2c\lambda + 2a_1^2\lambda' - 7a_1a_1'\lambda't + 3c\lambda't)}{4a_1(a_1^2-2ct)^2}g_{0i}g_{0j}g_{0k}y^h.$$

Finally, the components of the tensor field R_2 are obtained as follows

$$(XXX_2)^h_{ijk} = 0,\ (XXY_2)^h_{ijk} = 0,\ (YYX_2)^h_{ijk} = 0,\ (YYY_2)^h_{ijk} = 0,$$

$$(YXX_2)^h_{ijk} = \frac{(a_1-2a_1't)(\lambda^3+6\lambda^2\lambda't+12\lambda\lambda'^2t^2+8\lambda'^3t^3)}{a_1^2-2ct}g_{0i}g_{0j}g_{0k}y^h,$$

$$(YXY_2)^h_{ijk} = -\frac{(a_1-2a_1't)^3(\lambda^3+6\lambda^2\lambda't+12\lambda\lambda'^2t^2+8\lambda'^3t^3)}{(a_1^2-2ct)^3}g_{0i}g_{0j}g_{0k}y^h$$

Remark. We could consider a more general vector field $\xi = \alpha C + \beta\widetilde{C}$, where α, β are smooth functions on TM and the vector field $\widetilde{C} = y^i\delta_i$ is the horizontal vector field corresponding to the Liouville vector field C. A simple computation shows that the tensor fields R_1 and R_2 have not local expressions similar to that obtained in (15) for the tensor field R unless if $\beta = 0$. So our choice of Liouville vector field C for ξ is the only one possible.

To obtain the conditions under which the Kählerian manifold (T_0M,G,J,C) is of quasi-constant holomorphic sectional curvatures, we have to consider the following equations

$$XXX^h_{ijk}-\kappa_0(XXX_0)^h_{ijk}-\kappa_1(XXX_1)^h_{ijk}-\kappa_2(XXX_2)^h_{ijk}=0,$$

$$XXY^h_{ijk}-\kappa_0(XXY_0)^h_{ijk}-\kappa_1(XXY_1)^h_{ijk}-\kappa_2(XXY_2)^h_{ijk}=0,$$
$$YYX^h_{ijk}-\kappa_0(YYX_0)^h_{ijk}-\kappa_1(YYX_1)^h_{ijk}-\kappa_2(YYX_2)^h_{ijk}=0,$$
$$YYY^h_{ijk}-\kappa_0(YYY_0)^h_{ijk}-\kappa_1(YYY_1)^h_{ijk}-\kappa_2(YYY_2)^h_{ijk}=0,$$
$$YXX^h_{ijk}-\kappa_0(YXX_0)^h_{ijk}-\kappa_1(YXX_1)^h_{ijk}-\kappa_2(YXX_2)^h_{ijk}=0,$$
$$YXY^h_{ijk}-\kappa_0(YXY_0)^h_{ijk}-\kappa_1(YXY_1)^h_{ijk}-\kappa_2(YXY_2)^h_{ijk}=0.$$

From the first four equations we obtain the same value for the coefficient κ_0

$$\kappa_0=-\{4a_1^2c\lambda^2-2a_1^2a_1'^2\lambda^2t-8a_1a_1'c\lambda^2t-4a_1^3a_1'\lambda\lambda't+8a_1^2c\lambda\lambda't-$$
$$-2a_1^4\lambda'^2t+4a_1'^2c\lambda^2t^2-8a_1a_1'c\lambda\lambda't^2+4a_1^2c\lambda'^2t^2\}/\{-a_1^3\lambda^3+2a_1^2a_1'\lambda^3t-$$
$$-2a_1^3\lambda^2\lambda't+4a_1^2a_1'\lambda^2\lambda't^2\}.$$

From the last two equations we obtain another value of the coefficient κ_0

$$\kappa_0=-\{-2a_1^3a_1'\lambda^2-2a_1^4\lambda\lambda'+2a_1^2a_1'^2\lambda^2t+4a_1a_1'c\lambda^2t+4a_1^2c\lambda\lambda't-$$
$$-2a_1^4\lambda'^2t-4a_1'^2c\lambda^2t^2+4a_1^2c\lambda'^2t^2\}/\{-a_1^3\lambda^3+2a_1^2a_1'\lambda^3t-$$
$$-2a_1^3\lambda^2\lambda't+4a_1^2a_1'\lambda^2\lambda't^2\}.$$

Asking for the equality of the two values of κ_0, we find the following relation which must be fulfilled by a_1 and λ

$$\frac{\lambda'}{\lambda}=\frac{2a_1'ct-a_1^2a_1'-2a_1c}{a_1^3+2a_1ct}, \tag{6.1}$$

which can be written, by an integration, as

$$\lambda=A\frac{a_1}{a_1^2+2ct}, \tag{6.2}$$

for a certain positive constant A.

Next we get

$$\kappa_0=\frac{4c}{A},$$

and

$$\kappa_1=0,\ \kappa_2=0.$$

Hence, in our case, we have

$$R=\kappa_0R_0,$$

and we may state our main result

Theorem 5. *Let (T_0M, G, J) be the manifold of all non-zero tangent vectors to M, carrying the above Kählerian structure and the Liouville vector field C, where the Riemannian manifold (M, g) has constant sectional curvature. If (T_0M, G, J, C) is a manifold of quasi-constant holomorphic sectional curvature then TM has constant holomorphic sectional curvature.*

Remark. By theorem 5, the manifold T_0M satisfies a generalization of Schur type lemma, namely if we suppose that the sectional curvature of any holomorphic plane depends on the angle ϕ with the Liouville vector field and the point $p \in M$ only, then TM is of constant holomorphic sectional curvature.

Acknowledgement. This work was partially supported by the Grant 18/1463/2005, CNCSIS, Ministerul Educaţiei şi Cercetării, România.

References

[1] C.L. Bejan, M. Benyounes, *Kähler η-Einstein manifolds*, accepted for publication in Journal of Geometry.

[2] C.L. Bejan, *Some structures induced on the tangent bundle of an almost contact manifold*, An. St. Univ. "Al.I.Cuza" Iaşi, Mat. 30 (3) (1984), 69-78.

[3] V. Boju, M. Popescu, *Espaces à courbure quasi-constante*, J.Diff. Geom. 13 (1978), 373-383.

[4] G. Ganchev, V. Mihova, *Riemannian manifolds of quasi-constant sectional curvatures*, J.Reine Angew. Math. 522 (2000), 119-141.

[5] G. Ganchev, V. Mihova, *Kähler manifolds of quasi-constant holomorpic sectional curvatures*, arXiv:math.DG /0505671 v1 31 May 2005.

[6] I. Kolář, P. Michor, J. Slovak, *Natural Operations in Differential Geometry*, Springer Verlag, Berlin, 1993, vi, 434 pp.

[7] D. Krupka, J. Janyška, *Lectures on Differential Invariants,* Folia Fac. Sci. Nat. Univ. Purkinianae Brunensis, 1990.

[8] K.P. Mok, E.M. Patterson, Y.C. Wong, *Structure of symmetric tensors of type (0,2) and tensors of type (1,1) on the tangent bundle*, Trans. Amer. Math. Soc., 234 (1977), 253-278.

[9] V. Oproiu, *A Kaehler Einstein structure on the tangent bundle of a space form*, Int. J. Math. and Math. Sci. 25(3) (2001), 183-195.

[10] V. Oproiu, *Some new geometric structures on the tangent bundles*, Publ. Math. Debrecen, 55/3-4 (1999), 261-281.

[11] V. Oproiu, *A generalization of natural almost Hermitian structures on the tangent bundle*, Math. J. Toyama Univ. 22 (1999), 1-14.

[12] V. Oproiu, N. Papaghiuc, *A Kaehler structure on the nonzero tangent bundle of a space form,* Diff. Geom. Applic. 11 (1999), 1-12.

[13] V. Oproiu, D. Poroşniuc, *A class of Kaehler Einstein structures on the cotangent bundle of a space form*, Public. Math. Debrecen 66 (2005), 457-478.

[14] S. Sasaki, *On the Differential Geometry of the Tangent Bundle of Riemannian Manifolds*, Tôhoku Math. J., 10 (1958), 238-354.

[15] M. Tahara, L. Vanhecke, Y. Watanabe, *New structures on tangent bundles*, Note di Matematica (Lecce), 18(1998), 131-141.

[16] M. Tahara, S. Marchiafava, Y. Watanabe, *Quaternion Kähler structures on the tangent bundle of a complex space form*, Rend. Istit. Mat. Univ. Trieste, Suppl. 30 (1999), 163-175..

[17] K. Yano, S. Ishihara, *Tangent and Cotangent Bundles*, M. Dekker Inc., New York, 1973.

Authors' addresses:

Cornelia Livia Bejan
Seminarul Matematic, Universitatea "Al.I.Cuza", Iaşi,
RO-700506, Romania.
email: bejan@math.tuiasi.ro

Vasile Oproiu
Faculty of Mathematics, University "Al.I.Cuza", Iaşi
RO-700506, Romania.

Institute of Mathematics,
"O.Mayer", Romanian Academy, Iaşi Branch.
email: voproiu@uaic.ro