Nonlinear connections on dual Lie algebroids

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Dedicated to the memory of Radu Rosca (1908-2005)

Abstract. In this paper we start developing the so-called Klein's formalism on dual Lie algebroids. The nonlinear connection associated to a regular section is naturally obtained. Particularly, this connection is found for the Hamiltonian case.

Mathematics Subject Classification: 17B66, 53C05.
Key words: Lie algebroids, nonlinear connection, Klein's formalism.

1 Preliminaries on Lie algebroids

The notion of Lie algebroid is a generalization of the concepts of Lie algebra and integrable distribution. In [8] A. Weinstein gives a generalized theory of Lagrangian on Lie algebroids and obtains the Euler-Lagrange equations. The same equations were later obtained by E. Martinez [5] using the symplectic formalism and the notion of prolongation of Lie algebra over a mapping introduced by P.J. Higgins and K. Mackenzie [3]. In this paper Klein's formalism in the case of dual Lie algebroids is investigated. It should be mentioned that our approach is new even for the particular case of the cotangent bundle.

Let M be a differentiable, n-dimensional manifold and (TM, π_M, M) its tangent bundle. A Lie algebroid over the manifold M is the triple $(E, [\cdot,\cdot], \sigma)$ where $\pi : E \to M$ is a vector bundle of rank m over M, whose $C^\infty(M)$-module of sections $Sec(E)$ is equipped with a Lie algebra structure $[\cdot,\cdot]$ and $\sigma : E \to TM$ is a vector bundle homomorphism (called *the anchor*) which induces a Lie algebra homomorphism (also denoted σ) from $Sec(E)$ to $\chi(M)$, satisfying the compatibility conditions

$$[s_1, fs_2] = f[s_1, s_2] + (\sigma(s_1)f)s_2$$

for every $f \in C^\infty(M)$ and s_1, $s_2 \in Sec(E)$. From the above definition we easily get

$$[\sigma(s_1), \sigma(s_2)] = \sigma[s_1, s_2], \quad [s_1, [s_2, s_3]] + [s_2, [s_3, s_1]] + [s_3, [s_1, s_2]] = 0.$$

For $f \in C^\infty(M)$ the differential $df(x) \in E_x^*$ is defined by $\langle df(x), u\rangle = \sigma(u)f$, for every $u \in E_x$ and for differentiable k -form $\omega \in \bigwedge^k(E) = Sec((E^*)^k \to M)$, $k > 0$ its exterior derivative $d\omega \in \bigwedge^{p+1}(E)$ is defined as follows:

Balkan Journal of Geometry and Its Applications, Vol.11, No.1, 2006, pp. 73-80.

$$
\begin{aligned}
d\omega(s_1,...,s_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i+1}\sigma(s_i)\omega(s_1,...,\hat{s}_i,...,s_{k+1}) + \\
&\quad + \sum_{1\le i<j\le k+1}(-1)^{i+j}\omega([s_i,s_j],s_1,...,\hat{s}_i,...,\hat{s}_j,...s_{k+1}).
\end{aligned}
$$

Also, for $\xi \in Sec(E)$ on can define the *Lie derivative* with respect to ξ by $\mathcal{L}_\xi = i_\xi \circ d + d \circ i_\xi$, where i_ξ is the contraction with ξ. If we take the local coordinates (x^i) on an open $U \subset M$, a local basis $\{s_\alpha\}$ of sections of the bundle $\pi^{-1}(U) \to U$ generates local coordinates (x^i, y^α) on E. The local functions $\sigma^i_\alpha(x)$, $L^\gamma_{\alpha\beta}(x)$ on M given by

$$\sigma(s_\alpha) = \sigma^i_\alpha \frac{\partial}{\partial x^i}, \quad [s_\alpha, s_\beta] = L^\gamma_{\alpha\beta} s_\gamma, \quad i,j = \overline{1,n}, \quad \alpha,\beta = \overline{1,m},$$

capture the properties which define the Lie algebroid structure over M in so called *structure equations*:

$$\sigma^j_\alpha \frac{\partial \sigma^i_\beta}{\partial x^j} - \sigma^j_\beta \frac{\partial \sigma^i_\alpha}{\partial x^j} = \sigma^i_\gamma L^\gamma_{\alpha\beta}, \quad \sum_{(\alpha,\beta,\gamma)} \left(\sigma^i_\alpha \frac{\partial L^\delta_{\beta\gamma}}{\partial x^i} + L^\delta_{\alpha\eta} L^\eta_{\beta\gamma} \right) = 0.$$

Locally, if $f \in C^\infty(M)$ then $df = \frac{\partial f}{\partial x^i}\sigma^i_\alpha s^\alpha$ and if $\theta \in Sec(E^*)$, $\theta = \theta_\alpha s^\alpha$ then

$$d\theta = (\frac{\partial \theta_\beta}{\partial x^i}\sigma^i_\alpha - \frac{1}{2}\theta_\gamma L^\gamma_{\alpha\beta}) s^\alpha \wedge s^\beta, \tag{1.1}$$

where $\{s^\alpha\}$ is the dual basis of $\{s_\alpha\}$. Particularly, we have $dx^i = \sigma^i_\alpha s^\alpha$ and $ds^\alpha = -\frac{1}{2}L^\alpha_{\beta\gamma} s^\beta \wedge s^\gamma$.

2 Dual Lie algebroids

Let $\tau : E^* \to M$ be the dual of $\pi : E \to M$ and $(E, [\cdot,\cdot], \sigma)$ a Lie algebroid structure over M. One can construct a Lie algebroid structure over E^*, by taking the prolongation of $(E, [\cdot,\cdot], \sigma)$ over E^* (see [3],[4],[5]). This structure is given by the following objects:

- The associated vector bundle is $(\mathcal{T}E^*, \tau_1, E^*)$ where $\mathcal{T}E^* = \cup_{u^* \in E^*} \mathcal{T}_{u^*}E^*$ with

  $$\mathcal{T}_{u^*}E^* = \{(u_x, v_{u^*}) \in E_x \times T_{u^*}E^* | \sigma(u_x) = T_{u^*}\tau(v_{u^*}), \tau(u^*) = x \in M\}$$

  and the projection $\tau_1 : \mathcal{T}E^* \to E^*$, $\tau_1(u_x, v_{u^*}) = u^*$.

- The Lie algebra structure $[\cdot,\cdot]$ on $Sec(\tau_1)$ is defined in the following way: if $\rho_1, \rho_2 \in Sec(\tau_1)$ are such that $\rho_i(u^*) = (X_i(\tau(u^*)), U_i(u^*))$ where $X_i \in Sec(\pi), U_i \in \chi(E^*)$ and $\sigma(X_i(\tau(u^*)) = T_{u^*}\tau(U_i(u^*))$, $i = 1, 2$, then

  $$[\rho_1, \rho_2](u^*) = ([X_1, X_2](\tau(u^*)), [U_1, U_2](u^*))$$

- The anchor is the projection $\sigma^1 : \mathcal{T}E^* \to TE^*$, $\sigma^1(u, v) = v$.

Notice that if $\mathcal{T}\tau : \mathcal{T}E^* \to E, \mathcal{T}\tau(u,v) = u$ then $(V\mathcal{T}E^*, \tau_{1|V\mathcal{T}E^*}, E^*)$ with $V\mathcal{T}E^* := Ker\mathcal{T}\tau$ is a subbundle of $(\mathcal{T}E^*, \tau_1, E^*)$, called the *vertical subbundle.* If (x^i, μ_α) are local coordinates on E^* at $u^*$and $\{s_\alpha\}$ is a local basis of sections of $\pi : E \to M$ then a local basis of $Sec(\mathcal{T}E^*)$ is $\{\mathcal{X}_\alpha, \mathcal{P}^\alpha\}$ where

$$\mathcal{X}_\alpha(u^*) = \left(s_\alpha(\tau(u^*)), \sigma^i_\alpha \frac{\partial}{\partial x^i}|_{u^*}\right), \quad \mathcal{P}^\alpha(u^*) = \left(0, \frac{\partial}{\partial \mu_\alpha}|_{u^*}\right). \tag{2.1}$$

The Lie brackets on the elements of this basis are:

$$[\mathcal{X}_\alpha, \mathcal{X}_\beta] = L^\gamma_{\alpha\beta}\mathcal{X}_\gamma, \quad [\mathcal{X}_\alpha, \mathcal{P}^\alpha] = 0, \quad [\mathcal{P}^\alpha, \mathcal{P}^\beta] = 0 \tag{2.2}$$

and therefore

$$dx^i = \sigma^i_\alpha \mathcal{X}^\alpha, \quad d\mu_\alpha = \mathcal{P}_\alpha, \quad d\mathcal{X}^\gamma = -\frac{1}{2}L^\gamma_{\alpha\beta}\mathcal{X}^\alpha \wedge \mathcal{X}^\beta, \quad d\mathcal{P}_\alpha = 0$$

where $\{\mathcal{X}^\alpha, \mathcal{P}_\alpha\}$ is the dual basis of $\{\mathcal{X}_\alpha, \mathcal{P}^\alpha\}$. Also if $\rho = \rho^\alpha \mathcal{X}_\alpha + \rho_\alpha \mathcal{P}^\alpha$ is a section of $\mathcal{T}E^*$, then $\sigma^1(\rho) = \sigma^i_\alpha \rho^\alpha \frac{\partial}{\partial x^i} + \rho_\alpha \frac{\partial}{\partial \mu_\alpha}$. The canonical symplectic structure of a Lie algebroid $\mathcal{T}E^*$ is given by $\omega = -d\theta$ where $\theta = \mu_\alpha \mathcal{X}^\alpha$ is the Liouville form. In local coordinates we get

$$\omega = \mathcal{X}^\alpha \wedge \mathcal{P}_\alpha + \frac{1}{2}\mu_\alpha L^\alpha_{\beta\gamma}\mathcal{X}^\beta \wedge \mathcal{X}^\gamma. \tag{2.3}$$

We remark that $V\mathcal{T}E^*$ is Lagrangian for ω, i.e. $\omega(\rho_1, \rho_2) = 0$, for every vertical sections ρ_1, ρ_2.

3 Nonlinear connection on $\mathcal{T}E^*$

Definition 1. A nonlinear connection (or connection) on $\mathcal{T}E^*$ is an almost product structure $\mathcal{N}$ on $\tau_1 : \mathcal{T}E^* \to E^*$(i.e. a bundle morphism $\mathcal{N} : \mathcal{T}E^* \to \mathcal{T}E^*$, such that $\mathcal{N}^2 = id$) smooth on $\mathcal{T}E^* \backslash \{0\}$ such that $V\mathcal{T}E^* = Ker(id + \mathcal{N})$.

(i) If $\mathcal{N}$ is a connection on $\mathcal{T}E^*$ then $H\mathcal{T}E^* = Ker(id - \mathcal{N})$ is the horizontal subbundle associated to $\mathcal{N}$ and $\mathcal{T}E^* = V\mathcal{T}E^* \oplus H\mathcal{T}E^*$. Each $\rho \in Sec(\tau_1)$ can be written as $\rho = \rho^h + \rho^v$ where ρ^h, $\rho^v$are sections in the horizontal and respective vertical subbundles. If $\rho^h = 0$ then ρ is called *vertical* and if $\rho^v = 0$ then ρ is called *horizontal.* The section $\mathcal{C}$ given locally by $\mathcal{C} = \mu_\alpha \mathcal{P}^\alpha$ defines a global vertical section that is called *Liouville section.*

(ii) A connection $\mathcal{N}$ on E^* induces two projectors $h, v : \mathcal{T}E^* \to \mathcal{T}E^*$ such that $h(\rho) = \rho^h$ and $v(\rho) = \rho^v$ for every $\rho \in Sec(\tau_1)$. We have

$$h = \frac{1}{2}(id + \mathcal{N}), \quad v = \frac{1}{2}(id - \mathcal{N}), \tag{3.1}$$

$$\ker h = Imv = V\mathcal{T}E^*, \quad Imh = \ker v = H\mathcal{T}E^*. \tag{3.2}$$

(iii) Locally a connection can be expressed as

$$\mathcal{N}(\mathcal{X}_\alpha) = \mathcal{X}_\alpha + 2\mathcal{N}_{\alpha\beta}\mathcal{P}^\beta, \quad \mathcal{N}(\mathcal{P}^\alpha) = -\mathcal{P}^\alpha, \tag{3.3}$$

where $\mathcal{N}_{\alpha\beta}=\mathcal{N}_{\alpha\beta}(x,\mu)$ are the local coefficients of $\mathcal{N}$. The vector fields

$$\delta^*_\alpha = h(\mathcal{X}_\alpha)=\mathcal{X}_\alpha+\mathcal{N}_{\alpha\beta}\mathcal{P}^\beta \tag{3.4}$$

generate a basis of $H\mathcal{T}E^*$. The frame $\{\delta^*_\alpha,\mathcal{P}^\alpha\}$ is a local basis of $\mathcal{T}E^*$ called *adapted*. The dual adapted basis is $\{\mathcal{X}^\alpha,\delta\mathcal{P}_\alpha\}$ where $\delta\mathcal{P}_\alpha=\mathcal{P}_\alpha-\mathcal{N}_{\alpha\beta}\mathcal{X}^\beta$.

Definition 2. A connection $\mathcal{N}$ on $\mathcal{T}E^*$ is called symmetric if $H\mathcal{T}E^*$ is Lagrangian for ω.

Proposition 1. $\mathcal{N}$ is symmetric iff locally

$$\mathcal{N}_{\alpha\beta}-\mathcal{N}_{\beta\alpha}=\mu_\gamma L^\gamma_{\alpha\beta}. \tag{3.5}$$

Proposition 2. The Lie brackets of the adapted basis $\{\delta^*_\alpha,\mathcal{P}^\alpha\}$ are

$$[\delta^*_\alpha,\delta^*_\beta]=L^\gamma_{\alpha\beta}\delta^*_\gamma+\mathcal{R}_{\alpha\beta\gamma}\mathcal{P}^\gamma,\quad [\delta^*_\alpha,\mathcal{P}^\beta]=-\frac{\partial\mathcal{N}_{\alpha\gamma}}{\partial\mu_\beta}\mathcal{P}^\gamma,\quad [\mathcal{P}^\alpha,\mathcal{P}^\beta]=0, \tag{3.6}$$

where

$$\mathcal{R}_{\alpha\beta\gamma}=\sigma^i_\alpha\frac{\partial\mathcal{N}_{\beta\gamma}}{\partial x^i}-\sigma^i_\beta\frac{\partial\mathcal{N}_{\alpha\gamma}}{\partial x^i}+\mathcal{N}_{\alpha\delta}\frac{\partial\mathcal{N}_{\beta\gamma}}{\partial\mu_\delta}-\mathcal{N}_{\beta\delta}\frac{\partial\mathcal{N}_{\alpha\gamma}}{\partial\mu_\delta}+L^\varepsilon_{\alpha\beta}\mathcal{N}_{\varepsilon\gamma}. \tag{3.7}$$

Definition 3. The curvature of a connection $\mathcal{N}$ on $\mathcal{T}E^*$ is given by $\Omega=-\mathrm{N}_h$ where h is defined by (3.1), and $\mathrm{N}_h=-\frac{1}{2}[h,h]$ is the Nijenhuis tensor of h.

In the local coordinates

$$\Omega=-\frac{1}{2}\mathcal{R}_{\alpha\beta\gamma}\mathcal{X}^\alpha\wedge\mathcal{X}^\beta\otimes\mathcal{P}^\gamma$$

where $\mathcal{R}_{\alpha\beta\gamma}$ is given by (3.7) and is called the *curvature tensor* of $\mathcal{N}$.
The curvature is an obstruction to the integrability of $H\mathcal{T}E^*$. We have

Proposition 3. $H\mathcal{T}E^*$ is integrable if and only if the curvature vanishes.

Remark 1. Two connections $\mathcal{N}$ on $\mathcal{T}E^*$ and N on TE^* are called σ^1-related if $N\circ\sigma^1=\sigma^1\circ\mathcal{N}$. In this case $N(\sigma^1(\delta^*_\alpha))=\sigma^1(\delta^*_\alpha)$ from which we easily obtain

$$\sigma^1(\delta^*_\alpha)=\sigma^i_\alpha\delta_i,\quad \mathcal{N}_{\alpha\beta}=\sigma^i_\alpha N_{i\beta},$$

where $N_{i\beta}$ are the coefficients of N and $\delta_i=\frac{\partial}{\partial x^i}+N_{i\alpha}\frac{\partial}{\partial\mu_\alpha}$ is a local adapted frame of the horizontal subbundle HTE^*. Also for the curvature tensors of two σ^1-related connections we have:

$$\mathcal{R}_{\alpha\beta\gamma}=\sigma^i_\alpha\sigma^j_\beta R_{ij\gamma},$$

where $R_{ij\gamma}=\delta_i(N_{j\gamma})-\delta_j(N_{i\gamma})$.

4 Almost tangent structures and connections

Definition 4. An almost tangent structure $\mathcal{J}$ on $\mathcal{T}E^*$ is a bundle morphism $\mathcal{J}:\mathcal{T}E^*\to\mathcal{T}E^*$ of $\tau_1:\mathcal{T}E^*\to E^*$, of rank m, such that $\mathcal{J}^2=0$. An almost tangent structure $\mathcal{J}$ on $\mathcal{T}E^*$ is called adapted if $Im\mathcal{J}=Ker\mathcal{J}=V\mathcal{T}E^*$.

Locally, an adapted almost tangent structure is given by $\mathcal{J}=t_{\alpha\beta}\mathcal{X}^\alpha\otimes\mathcal{P}^\beta$ where the coefficients matrix $(t_{\alpha\beta}(x,\mu))$ has rank m.

Proposition 4. $\mathcal{J}$ is an integrable if and only if

$$\frac{\partial t^{\alpha\gamma}}{\partial\mu_\beta}=\frac{\partial t^{\beta\gamma}}{\partial\mu_\alpha} \tag{4.1}$$

where $t^{\alpha\gamma}t_{\gamma\beta}=\delta^\alpha_\beta$.

Proof. $\mathcal{J}$ is an integrable if and only if the Nijenhuis tensor $\mathrm{N}_{\mathcal{J}}(\rho,\upsilon)=[\mathcal{J}\rho,\ \ \mathcal{J}\upsilon]-\mathcal{J}[\mathcal{J}\rho,\upsilon]-\mathcal{J}[\rho,\mathcal{J}\upsilon]=0$. This is locally equivalent to

$$\mathrm{N}_{\mathcal{J}}(\mathcal{X}_\alpha,\mathcal{X}_\beta)=\left(t_{\alpha\gamma}\frac{\partial t_{\beta\varepsilon}}{\partial\mu_\gamma}-t_{\beta\gamma}\frac{\partial t_{\alpha\varepsilon}}{\partial\mu_\gamma}\right)\mathcal{P}^\varepsilon,\ \mathrm{N}_{\mathcal{J}}(\mathcal{X}_\alpha,\mathcal{P}^\beta)=\mathrm{N}_{\mathcal{J}}(\mathcal{P}^\alpha,\mathcal{P}^\beta)=0.$$

Therefore $\mathcal{J}$ is integrable iff $t_{\alpha\gamma}\frac{\partial t_{\beta\varepsilon}}{\partial\mu_\gamma}=t_{\beta\gamma}\frac{\partial t_{\alpha\varepsilon}}{\partial\mu_\gamma}$ that is equivalent to (4.1). □

Remark 2. (i) An adapted almost tangent structure $\mathcal{J}$ on $\mathcal{T}E^*$ is called symmetric if $\omega(\mathcal{J}\rho,\upsilon)=\omega(\mathcal{J}\upsilon,\rho)$. Locally, this requires the symmetry of the tensor $t_{\alpha\beta}$.
(ii) If g is a pseudo-Riemannian metric on the vertical bundle $V\mathcal{T}E^*$ then there exists an unique symmetric adapted almost tangent structure on $\mathcal{T}E^*$ such that

$$g(\mathcal{J}\rho,\mathcal{J}\upsilon)=-\omega(\mathcal{J}\rho,\upsilon). \tag{4.2}$$

In this case we say that $\mathcal{J}$ is induced by the metric g.
Locally if $g(x,\mu)=g^{\alpha\beta}\mathcal{P}_\alpha\otimes\mathcal{P}_\beta$ then $t^{\alpha\beta}=g^{\alpha\beta}$. In particular, any regular Hamiltonian $H:E^*\to\mathbb{R}$ on E^* induces a pseudo-Riemannian metric on $V\mathcal{T}E^*$ (the metric tensor is $g^{\alpha\beta}=\frac{\partial^2H}{\partial\mu_\alpha\partial\mu_\beta}$) therefore, it induces an unique symmetric adapted almost tangent structure (denoted $\mathcal{J}_H$) such that (4.2) is verified. Moreover, this is a tangent structure i.e., $\mathcal{J}_H$ is integrable.
(iii) Any symmetric adapted almost tangent structure $\mathcal{J}$ on $\mathcal{T}E^*$ induces a pseudo-Riemannian metric on the vertical bundle $V\mathcal{T}E^*$ as defined by (4.2).

Definition 5. The torsion of a connection $\mathcal{N}$ is the vector valued two form $T:=[\mathcal{J},h]$ where h is given by (3.1) and $[\mathcal{J},h]$ is the Frolicher-Nijenhuis bracket.

Remark 3. T is a semibasic vector-valued form. Its local expression is:

$$T=\frac{1}{2}\left(t_{\alpha\epsilon}\frac{\partial\mathcal{N}_{\beta\gamma}}{\partial\mu_\varepsilon}-t_{\beta\epsilon}\frac{\partial\mathcal{N}_{\alpha\gamma}}{\partial\mu_\varepsilon}+\delta^*_\alpha(t_{\beta\gamma})-\delta^*_\beta(t_{\alpha\gamma})-L^\epsilon_{\alpha\beta}t_{\epsilon\gamma}\right)\mathcal{X}^\alpha\wedge\mathcal{X}^\beta\otimes\mathcal{P}^\gamma.$$

Proposition 5. Let $\mathcal{N}$ be a bundle morphism of $\tau_1:\mathcal{T}E^*\to E^*$, smooth on $\mathcal{T}E^*\backslash\{0\}$. Then $\mathcal{N}$ is a connection on $\mathcal{T}E^*$ if and only if there exists an adapted almost tangent structure $\mathcal{J}$ on $\mathcal{T}E^*$ such that

$$\mathcal{J}\mathcal{N}=\mathcal{J},\quad \mathcal{N}\mathcal{J}=-\mathcal{J}.$$

Definition 6. Let $\mathcal{J}$ be an adapted almost tangent structure on $\mathcal{T}E^*$. A section ρ of $\mathcal{T}E^*$ is called $\mathcal{J}$-regular if

$$\mathcal{J}[\rho,\mathcal{J}\nu]=-\mathcal{J}\nu,$$

for every section ν of $\mathcal{T}E^*$.

Locally $\rho=\rho^\alpha\mathcal{X}_\alpha+\rho_\beta\mathcal{P}^\beta$ is $\mathcal{J}$-regular iff

$$t^{\alpha\beta} = \frac{\partial \rho^{\beta}}{\partial \mu_{\alpha}}. \tag{4.3}$$

Example 1. Let H be a regular Hamiltonian on E^*. One can associate to H a remarkable $\mathcal{J}_H$ -regular section $\rho \in Sec(\tau_1)$, locally given by

$$\rho = \frac{\partial H}{\partial \mu_{\alpha}} \mathcal{X}_{\alpha} + \rho_{\alpha} \mathcal{P}^{\alpha}, \tag{4.4}$$

which will be called a *semi-Hamiltonian* section. Moreover, the equation

$$i_{\rho_H} \omega = dH, \tag{4.5}$$

defines an unique $\mathcal{J}_H$ $-$regular section $\rho_H \in Sec(\tau_1)$ (see [4]) locally given by

$$\rho_H = \frac{\partial H}{\partial \mu_{\alpha}} \mathcal{X}_{\alpha} - (\sigma^i_{\alpha} \frac{\partial H}{\partial x^i} + \mu_{\gamma} L^{\gamma}_{\alpha\beta} \frac{\partial H}{\partial \mu_{\beta}}) \mathcal{P}^{\alpha}, \tag{4.6}$$

called the *Hamilton section.*

Theorem 1. Let $\mathcal{J}$ be an adapted almost tangent structure on $\mathcal{T}E^*$. If ρ is a $\mathcal{J}$-regular section of $\mathcal{T}E^*$ then

$$\mathcal{N} = -\mathcal{L}_{\rho} \mathcal{J}, \tag{4.7}$$

is a connection on $\mathcal{T}E^*$.

Proof. Since $\mathcal{N}(v) = -\mathcal{L}_{\rho}\mathcal{J}(v) = -[\rho, \mathcal{J}v] + \mathcal{J}[\rho, v]$ then $\mathcal{J}\mathcal{N}(v) = -\mathcal{J}[\rho, \mathcal{J}v] + \mathcal{J}^2[\rho, v] = \mathcal{J}\nu$ and $\mathcal{N}\mathcal{J}(v) = -[\rho, \mathcal{J}^2 v] + \mathcal{J}[\rho, \mathcal{J}v] = -\ \mathcal{J}\nu$. By using Proposition 5 we get the proof of the theorem. $\square$

Remark 4. The connection (4.7) is induced by $\mathcal{J}$ and ρ. Its local coefficients are given by

$$\mathcal{N}_{\alpha\beta} = \frac{1}{2} \left(t_{\alpha\gamma} \frac{\partial \rho_{\beta}}{\partial \mu_{\gamma}} - \sigma^i_{\alpha} t_{\gamma\beta} \frac{\partial \rho^{\gamma}}{\partial x^i} - \rho t_{\alpha\beta} + \rho^{\gamma} t_{\lambda\beta} L^{\lambda}_{\gamma\alpha} \right) \tag{4.8}$$

Proposition 6. The torsion of the connection (4.7) vanishes.

Proof. We have $T = [\mathcal{J}, h] = \frac{1}{2}\left([\mathcal{J}, id] + [\mathcal{J}, -[\rho, \mathcal{J}]]\right) = \frac{1}{2}[\mathcal{J}, [\mathcal{J}, \rho]]$. Using Jacobi identity we obtain that $T = 0$. $\square$

Proposition 7. The connection $N = -\mathcal{L}_{\rho_H} \mathcal{J}_H$ is symmetric.

Proof. Use (4.6) and (4.8) after some computations we get (3.5). $\square$

5 Homogeneous connections

Definition 7. An adapted almost tangent structure on $\mathcal{T}E^*$ is called homogeneous if $\mathcal{L}_{\mathcal{C}}\mathcal{J} = -\mathcal{J}$.

Notice that $\mathcal{J}$ is homogeneous if the local components $t_{\alpha\beta}(x, \mu)$ are 0-homogeneous with respect to μ.

Proposition 8. Let $\mathcal{J}$ be a homogeneous adapted tangent structure. A section $\rho \in Sec(\tau_1)$ is $\mathcal{J}$$-$regular if and only if $\mathcal{J}\rho = \mathcal{C}$.

Proof. If ρ is $\mathcal{J}$$-$regular then $t^{\alpha\beta} = \frac{\partial \rho^{\beta}}{\partial \mu_{\alpha}}$, hence ρ^{β} must be 1-homogeneous with respect to μ, therefore $\mu_\alpha t^{\alpha\beta} = \rho^\beta$, that is equivalent to $\mathcal{J}\rho = \mathcal{C}$. Vice versa, if $\mathcal{J}\rho = \mathcal{C}$ then $\rho^\alpha = \mu_\beta t^{\beta\alpha}$ and thus $\frac{\partial \rho^\alpha}{\partial \mu_\gamma} = t^{\gamma\alpha} + \mu_\beta \frac{\partial t^{\gamma\alpha}}{\partial \mu_\beta} = t^{\gamma\alpha}$. □

Remark 5. (i) Based on the above result, the local expression for a $\mathcal{J}-$ regular section with $\mathcal{J}$ a homogeneous adapted tangent structure is

$$\rho = \mu_\alpha t^{\alpha\beta} \mathcal{X}_\beta + \rho_\gamma \mathcal{P}^\gamma. \tag{5.1}$$

(ii) $\mathcal{J}_H$ is homogeneous iff $\mathcal{C}H = 2H$ (i.e. H is 2-homogeneous on the fibres) and therefore $H = \frac{1}{2} g^{\alpha\beta} \mu_\alpha \mu_\beta$. Accordingly $\rho_H = \mu_\alpha g^{\alpha\beta} \mathcal{X}_\beta + \rho_\gamma \mathcal{P}^\gamma$.
(iii) The coefficients of the connection (4.7) generated by ρ from (5.1) can be written in the following form:

$$\mathcal{N}_{\alpha\beta} = \frac{1}{2}\left(\mu_\epsilon t^{\epsilon\gamma} \left(\sigma^i_\alpha \frac{\partial t_{\gamma\beta}}{\partial x^i} - \sigma^i_\gamma \frac{\partial t_{\alpha\beta}}{\partial x^i} \right) + t_{\alpha\gamma} \frac{\partial \rho_\beta}{\partial \mu_\gamma} - \rho_\epsilon \frac{\partial t_{\alpha\beta}}{\partial \mu_\epsilon} + \mu_\epsilon t^{\epsilon\gamma} t_{\lambda\beta} L^\lambda_{\gamma\alpha} \right).$$

(iv) If $\rho \in Sec(\tau_1)$ is given by (5.1) and $\mathcal{N}$ is any connection on $\mathcal{T}E^*$ then $\xi = h(\rho)$ is a $\mathcal{J}$-regular section of τ_1 and is independent of ρ. We call this section associated to $\mathcal{N}$. The local expression of ξ is

$$\xi = \mu_\alpha t^{\alpha\beta} \mathcal{X}_\beta + \mu_\alpha t^{\alpha\beta} \mathcal{N}_{\beta\gamma} \mathcal{P}^\gamma.$$

For example the $\mathcal{J}$-regular section associated to $\mathcal{N} = -\mathcal{L}_\rho \mathcal{J}$ is $\xi = \frac{1}{2}(\rho + [\mathcal{C}, \rho])$ or locally

$$\xi = \mu_\alpha t^{\alpha\beta} \mathcal{X}_\beta + \frac{1}{2} \mu_\alpha \frac{\partial \rho_\gamma}{\partial \mu_\alpha} \mathcal{P}^\gamma.$$

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