A note on Euclidean spheres

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Abstract. For an orientable compact and connected hypersurface in the Euclidean space R^{n+1} with scalar curvature S, mean curvature α and sectional curvatures bounded below by a constant $\delta > 0$, it is shown that the inequality

$$S \leq n(n-1)\alpha^2 - (n-1)\delta^{-1} \|\nabla\alpha\|^2$$

implies that the hypersurface is a sphere, where $\nabla\alpha$ is the gradient of α.

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1 Introduction

The class of positively curved compact hypersurfaces in the Euclidean space R^{n+1} is quite large and therefore it is an interesting question in Geometry to obtain conditions which characterize the spheres in this class. For any hypersurface in R^{n+1} its scalar curvature S is given by $S = n^2\alpha^2 - \|A\|^2$, where $\|A\|$ is the length of the shape operator A and α is the mean curvature. In light of the Schwarz inequality $\|A\|^2 \geq n\alpha^2$, the scalar curvature S satisfies $S \leq n(n-1)\alpha^2$ for any hypersurface of R^{n+1}, and in case of a hypersphere the equality holds. It is therefore suggestive that in the inequality $S \leq n(n-1)\alpha^2$ the right hand side be decreased by a factor so that it forces the hypersurface to be a sphere. In this paper for a compact and connected hypersurface with sectional curvatures bounded below by a constant $\delta > 0$, we show that this factor is $(n-1)\delta^{-1}\|\nabla\alpha\|^2$. Indeed we prove the following:

Theorem 1.1. *Let M be an orientable compact and connected hypersurface of the Euclidean space R^{n+1} whose sectional curvatures are bounded below by a constant $\delta > 0$ If the scalar curvature S and the mean curvature α of M satisfy*

$$S \leq n(n-1)\alpha^2 - (n-1)\delta^{-1}\|\nabla\alpha\|^2$$

then α is a constant and $M = S^n(\alpha^2)$.

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2 Preliminaries

Let M be an orientable hypersurface of the Euclidean space R^{n+1}. We denote the induced metric on M by g. Let $\overline{\nabla}$ be the Euclidean connection and ∇ be the Riemannian connection on M with respect to the induced metric g. Let N be the unit normal vector field and A be the shape operator. Then the Gauss and Weingarten formulas for the hypersurface are

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \overline{\nabla}_X N = -AX, \quad X, Y \in \mathfrak{X}(M) \tag{2.1}$$

where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M. We also have the following Codazzi equation

$$(\nabla A)(X, Y) = (\nabla A)(Y, X), \quad X, Y \in \mathfrak{X}(M) \tag{2.2}$$

where $(\nabla A)(X, Y) = \nabla_X AY - A\nabla_X Y$. The mean curvature α of the hypersurface is given by $n\alpha = \sum_i g(Ae_i, e_i)$, where $\{e_1, ..., e_n\}$ is a local orthonormal frame on M. The square of the length of the shape operator A is given by

$$\|A\|^2 = \sum_{ij} g(Ae_i, e_j)^2 = tr.A^2$$

The scalar curvature S of the hypersurface is given by

$$S = n^2\alpha^2 - \|A\|^2 \tag{2.3}$$

3 Some Lemmas

Let M be a hypersurface of R^{n+1}. We define a symmetric operator $B : \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $B = A - \alpha I$. Let $\nabla\alpha$ be the gradient of the mean curvature function α.

Lemma 3.1. *The operator B satisfies*

(i) $trB = 0$,

(ii) $g((\nabla B)(X, Y), Z) = g(Y, (\nabla B)(X, Z))$

(iii) $(\nabla B)(X, Y) = (\nabla B)(Y, X) + R_0(X, Y)\nabla\alpha$,

where $R_0(X, Y)Z = g(Y, Z)X - g(X, Z)Y$, $X, Y, Z \in \chi(M)$.

The proof is straightforward and follows from the definition of B and the equation (2.2).

Lemma 3.2. *Let $\{e_1, ..., e_n\}$ be a local orthonormal frame on the hypersurface M. Then*

$$\sum_i (\nabla B)(e_i, e_i) = (n - 1)\nabla\alpha$$

Proof. Since $tr.B = 0$, choosing a pointwise constant local orthonormal frame, for $X \in \mathfrak{X}(M)$ we have

$$\begin{aligned} 0 &= \sum_i Xg(Be_i, e_i) = \sum_i g((\nabla B)(X, e_i), e_i) \\ &= \sum_i [g((\nabla B)(e_i, X) + R_0(X, e_i)\nabla\alpha, e_i)] \\ &= -(n-1)g(\nabla\alpha, X) + \sum_i g((\nabla B)(e_i, e_i), X) \end{aligned}$$

and the Lemma is proved. □

We define the second covariant derivative $(\nabla^2 B)(X, Y, Z)$ as

$$(\nabla^2 B)(X, Y, Z) = \nabla_X(\nabla B)(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)$$

Then using Lemma 3.1, we immediately obtain the following

Lemma 3.3. $(\nabla^2 B)(X, Y, Z) = (\nabla^2 B)(X, Z, Y) + H_\alpha(X, Z)Y - H_\alpha(X, Y)Z$, $X, Y, Z \in \chi(M)$, *where* $H_\alpha(X, Y) = g(\nabla_X(\nabla\alpha), Y)$ *is the Hessian of* α.

Lemma 3.4. *Let* $\{e_1, ..., e_n\}$ *be a local orthonormal frame that diagonalizes* B. *If* $Be_i = \lambda_i e_i$, *then*

$$\sum_{i<j} (\lambda_i - \lambda_j)^2 = n\|A\|^2 - n^2\alpha^2$$

Proof. We have $\sum_i \lambda_i = 0$ by Lemma 3.1, and consequently we get

$$\begin{aligned} \sum_{ij} (\lambda_i - \lambda_j)^2 &= \sum_{ij} \lambda_i^2 + \sum_{ij} \lambda_j^2 - 2\sum_{ij} \lambda_i\lambda_j \\ &= 2n\|B\|^2 - 2\sum_i \left(\sum_j \lambda_j\right) \lambda_i \\ &= 2n\|B\|^2 \end{aligned}$$

Since $\sum_{ij} (\lambda_i - \lambda_j)^2 = 2\sum_{i<j} (\lambda_i - \lambda_j)^2$, we get $\sum_{i<j} (\lambda_i - \lambda_j)^2 = n\|B\|^2 = n\|A\|^2 - n^2\alpha^2$. □

Lemma 3.5. *Let* M *be an orientable compact hypersurface of the Euclidean space* R^{n+1}. *Then*

$$\int_M \left(\sum_i g(\nabla_{e_i}(\nabla\alpha), Be_i)\right) dV = -(n-1)\int_M \|\nabla\alpha\|^2 dV$$

where $\{e_1, ..., e_n\}$ *is a local orthonormal frame on* M.

Proof. Choosing a point wise covariant constant local orthonormal frame $\{e_1, ..., e_n\}$ on M, we compute

$$\begin{aligned} div\,(B(\nabla\alpha)) = \sum_i e_i g(\nabla\alpha, Be_i) &= \sum_i g(\nabla_{e_i}(\nabla\alpha), Be_i) + \sum_i g(\nabla\alpha, (\nabla B)(e_i, e_i)) \\ &= \sum_i g(\nabla_{e_i}(\nabla\alpha), Be_i) + (n-1)\|\nabla\alpha\|^2 \end{aligned}$$

Integrating this equation we get the Lemma. □

4 Proof of the Theorem 1.1

Let M be an orientable compact and connected hypersurface of the Euclidean space R^{n+1}. Define a function $f : M \to R$ by $f = \frac{1}{2}\|B\|^2$. Then by a straightforward computation we get the Laplacian Δf of the smooth function f as

$$\Delta f = \|\nabla B\|^2 + \sum_{ij} g\left((\nabla^2 B)(e_j, e_j, e_i), Be_i\right) \tag{4.1}$$

where $\{e_1, ..., e_n\}$ is local orthonormal frame on M.

Using Lemma 3.3 and (i) in Lemma 3.1, we arrive at

$$\begin{aligned} g(\left(\nabla^2 B\right)(e_j, e_j, e_i), Be_i) &= g(\left(\nabla^2 B\right)(e_j, e_i, e_j), Be_i) \\ &+ H_\alpha(e_j, e_i) g(e_j, Be_i) \end{aligned} \tag{4.2}$$

Now using the Ricci identity

$$\left(\nabla^2 B\right)(X, Y, Z) = \left(\nabla^2 B\right)(Y, X, Z) + R(X, Y)BZ - BR(X, Y)Z,\ X, Y, Z \in \chi(M)$$

where R is the curvature tensor field of M, in equation (4.2) we get

$$\begin{aligned} g(\left(\nabla^2 B\right)(e_j, e_j, e_i), Be_i) &= g(\left(\nabla^2 B\right)(e_i, e_j, e_j), Be_i) + g(R(e_j, e_i)Be_j, Be_i) \\ &- g(R(e_j, e_i)e_j, B^2 e_i) + H_\alpha(e_j, e_i) g(e_j, Be_i). \end{aligned}$$

Thus in light of this equation the equation (4.1) takes the form

$$\begin{aligned} \Delta f &= \|\nabla B\|^2 + \sum_{ij} g(\left(\nabla^2 B\right)(e_i, e_j, e_j), Be_i) + \sum_i H_\alpha(e_i, Be_i) \\ &+ \sum_{ij} \left[g(R(e_j, e_i)Be_j, Be_i) - g(R(e_j, e_i)e_j, B^2 e_i)\right] \end{aligned} \tag{4.3}$$

Using Lemma 3.2, we get

$$\sum_i \left(\nabla^2 B\right)(e_i, e_j, e_j) = (n-1)\nabla_{e_i}(\nabla\alpha). \tag{4.4}$$

Also we have

$$H_\alpha(e_i, Be_i) = g(\nabla_{e_i}(\nabla\alpha), Be_i) \tag{4.5}$$

We choose a local orthonormal frame $\{e_1, ..., e_n\}$ that diagonalizes B with $Be_i = \lambda_i e_i$ to compute

$$\begin{aligned}
\sum_{ij} & \left[g(R(e_j, e_i)Be_j, Be_i) - g(R(e_j, e_i)e_j, B^2 e_i)\right] \\
&= -\sum_{ij} \lambda_i\lambda_j K_{ij} + \sum_{ij} \lambda_i^2 K_{ij} \\
&= \frac{1}{2}\left[2\sum_{ij} \lambda_i^2 K_{ij}\right] - \sum_{ij} \lambda_i\lambda_j K_{ij} \\
&= \frac{1}{2}\left[\sum_{ij} \lambda_i^2 K_{ij} + \sum_{ij} \lambda_j^2 K_{ij} - 2\sum_{ij} \lambda_i\lambda_j K_{ij}\right] \\
&= \frac{1}{2}\sum_{ij} \left(\lambda_i - \lambda_j\right)^2 K_{ij} = \sum_{i<j} \left(\lambda_i - \lambda_j\right)^2 K_{ij}
\end{aligned}$$

where $K_{ij} = g(R(e_i, e_j)e_j, e_i)$ is the sectional curvature of the plane section spanned by $\{e_i, e_j\}$. Using this last equation together with (4.4) and (4.5) in (4.3), we arrive at

$$\Delta f = \|\nabla B\|^2 + n\sum_i g(\nabla_{e_i}(\nabla\alpha), Be_i) + \sum_{i<j} \left(\lambda_i - \lambda_j\right)^2 K_{ij}$$

Integrating this equation and using $K_{ij} > \delta$, together with Lemmas 3.4 and 3.5, we arrive at

$$\int_M \left\{\|\nabla B\|^2 - n(n-1)\|\nabla\alpha\|^2 + \delta\left(n\|A\|^2 - n^2\alpha^2\right)\right\} dV \leq 0 \tag{4.6}$$

The condition $S \leq n(n-1)\alpha^2 - (n-1)\delta^{-1}\|\nabla\alpha\|^2$ in the statement of the theorem together with equation (2.3) yields

$$n^2\alpha^2 - \|A\|^2 \leq n(n-1)\alpha^2 - (n-1)\delta^{-1}\|\nabla\alpha\|^2$$

that is,

$$n\alpha^2 - \lambda|A\|^2 \leq -(n-1)\delta^{-1}\|\nabla\alpha\|^2$$

which takes the form

$$-n(n-1)\|\nabla\alpha\|^2 + \delta\left(n\|A\|^2 - n^2\alpha^2\right) \geq 0.$$

Consequently, from the integral inequality (4.6) we conclude that $\nabla B = 0$, and since M is irreducible (being of positive curvature), we must have $B = \lambda I$ for some λ. However, $tr.B = 0$ gives $\lambda = 0$ and consequently that $B = 0$, that is $A = \alpha I$. Hence by equation (2.2) we get that α is a constant and M is a totally umbilical hypersurface and it is therefore the sphere $S^n(\alpha^2)$ of constant curvature α^2. □

Finally we note that exactly on the similar lines the following theorem can be proved for hypersurfaces of a real space form $\overline{M}(c)$ (A Riemannian manifold of constant sectional curvature)

Theorem 4.1. *Let M be an n-dimensional compact hypersurface of a real space form $\overline{M}(c)$ with sectional curvatures bounded below by a constant $\delta > 0$. If the scalar curvature S and the mean curvature α of M satisfy*

$$S \leq n(n-1)(c+\alpha^2) - (n-1)\delta^{-1}\|\nabla\alpha\|^2$$

then M is totally umbilical.

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