Ligthlike ruled surfaces in $\mathbb{R}^4_1$

Erol Kılıç and H. Bayram Karadağ

Abstract. In this paper, we introduce the lightlike ruled surfaces in semi-Euclidean space $\mathbb{R}^4_1$ and classify the lightlike ruled surfaces in $\mathbb{R}^4_1$. It is also investigated that their induced connection is a metric connection. Furthermore, we give the conditions of becoming striction line of base (directrix) curve.

Mathematics Subject Classification: 53A05, 53B30, 53C40, 53C50.
Key words: ruled surfaces, lightlike (null) surfaces, lightlike submanifolds, Lorentzian space.

1 Introduction

The ruled surfaces in Euclidean 3-space is one of the important topics of differential geometry. Because, the ruled surfaces have the most important positions in the study of rational design problems in spatial mechanisms and physical applications. Hence, the ruled surfaces have been studied by many authors, [1], [8],..., etc. In those studies, many properties of the ruled surfaces have been investigated.

After then, the ruled surfaces have been generalized to Euclidean n-space, [11], [6]. These ruled surfaces have been considered as submanifolds and various geometric properties have been studied. It is known that the theory of submanifolds of a Riemannian (or semi-Riemannian) manifold is one of the most important topics of differential geometry (see for example, Chen [2] and O'Neill [10]).

Recently, the differential geometry of the ruled surfaces by means of the Lorentzian metric has been studied by several authors [12], [9]. Particularly, in those studies the various properties of the ruled surfaces with non-degenerate reduced metric have been investigated. For example, the time-like ruled surfaces in Minkowski 3-space studied by Turgut and Hacısalihoğlu in [12]. The classification of the ruled surfaces in Minkowski 3-space given by Kim and et all in [9].

Furthermore, the lightlike submanifolds of a semi-Riemannian manifold have been studied by some authors in [3], [4]. Also, the totally umbilical half lightlike submanifolds of semi-Riemannian manifolds have been investigated by Duggal and Jin in [5].

Balkan Journal of Geometry and Its Applications, Vol.11, No.1, 2006, pp. 88-101.

Our goal in this study is to introduce the lightlike ruled surfaces in $\mathbb{R}^4_1$. We have used technics given by Duggal and Jin in [5]. We have divided the lightlike ruled surfaces in $\mathbb{R}^4_1$ into two classes. For each class of lightlike ruled surfaces, the totally geodesic, totally umbilical and as being a case the metric connection of induced metric have been investigated. Also, we have investigated the condition of becoming striction line of base (directrix) curve of each of the light-like ruled surfaces.

2 Preliminaries

Let $\mathbb{R}^4_1$ be the 4-dimensional semi-Euclidean space of a index 1 with the natural metric $\overline{g}$. So, for $x = (x^1, x^2, x^3, x^4) \in \mathbb{R}^4_1$,

$$\overline{g}(x, x) = -(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2. \tag{2.1}$$

A vector x of $\mathbb{R}^4_1$ is said to be spacelike if $\overline{g}(x, x) > 0$, (or $x = 0$), timelike if $\overline{g}(x, x) < 0$ and lightlike (or null) if $\overline{g}(x, x) = 0$. For any two vectors x, y of $\mathbb{R}^4_1$ is called orthogonal if $\overline{g}(x, y) = 0$. A timelike or lightlike vector in $\mathbb{R}^4_1$ is said to be causal.

Lemma 2.1. *[7] There are no causal vectors in $\mathbb{R}^4_1$ orthogonal to a time-like vector, and two null vector are orthogonal if and only if they are linearly dependent.*

We recall that the geometrical properties of the 1-lightlike 2-surfaces in $\mathbb{R}^4_1$. The 1-lightlike 2-surface in $\mathbb{R}^4_1$ is a half lightlike submanifold of $\mathbb{R}^4_1$, of codimension 2, [5]. Let M be a 1-lightlike surface in $\mathbb{R}^4_1$. Then the induced metric g is degenerate on M. Thus, there exits a lightlike vector field ξ_1 which is locally defined on a open subset $\mathcal{U}$ of M such that

$$g(\xi_1, \xi_1) = 0, \; g(\xi_1, X) = 0,$$

for any $X \in \Gamma(TM)$. Then, for each tangent space T_xM, $x \in M$, we consider

$$T_xM^\perp = \{u : g(u, v) = 0, \forall v \in T_xM\}$$

which is a degenerate 2-dimensional subspace of $T_x\mathbb{R}^4_1$. Since M is a lightlike surface, both T_xM and $T_xM^\perp$ are degenerate orthogonal subspaces but no longer complementary. Thus, for each $x \in M$, $T_xM \bigcap T_xM^\perp \neq \phi$ and there exists a lightlike distribution $Rad\, TM$ which is called radical distribution. Then, the radical distribution $Rad\, TM$ is a smooth distribution which is locally spanned by ξ_1. Hence, $Rad\, TM$ is a subbundle of TM and $TM^\perp$ with rank 1. Such a surface M in $\mathbb{R}^4_1$ is called a half-lightlike (1-lightlike) surface in $\mathbb{R}^4_1$ [5]. In this case, there exists a supplementary distribution to $Rad\, TM$ in TM. We choose such a non-degenerate distribution $S(TM)$ of M which is spanned by a unit spacelike vector field U. Thus we have

$$TM = Rad\, TM \perp S(TM). \tag{2.2}$$

We consider the orthogonal complementary distribution $S(TM)^\perp$ to $S(TM)$ in $T\mathbb{R}^4_1$. Then there exists a unit spacelike vector field ξ_2 belong to $\Gamma(S(TM)^\perp)$ such that $\overline{g}(\xi_2, X) = 0$, for all $X \in \Gamma(TM)$. Since $Rad\, TM$ is a 1-lightlike vector subbundle of $TM^\perp$, we may consider a supplementary distribution D to $Rad\, TM$ such that it is locally spanned by ξ_2. The distribution D is called a screen canonical affine normal bundle of M. Then we have

$$S(TM)^{\perp} = D \perp D^{\perp},$$

where $D^{\perp}$ is the orthogonal complementary distribution to D in $S(TM)^{\perp}$. Thus, there exists a unique locally defined vector field $N_1 \in \Gamma(D^{\perp})$ satisfying

$$\overline{g}(N_1, \xi_1) = 1,\ \overline{g}(N_1, N_1) = \overline{g}(N_1, \xi_2) = 0. \tag{2.3}$$

Hence, the canonical affine normal bundle $tr(TM)$ of M is given by

$$tr(TM) = D \perp ltr(TM),$$

where $ltr(TM)$ is a 1-dimensional vector bundle locally represented by N_1 with respect to the screen distribution $S(TM)$. Thus we have the following decomposition:

$$\begin{aligned} T\mathbb{R}^4_1 &= S(TM) \perp (Rad\ TM \oplus tr(TM)) \\ &= S(TM) \perp D \perp (Rad\ TM \oplus ltr(TM)). \end{aligned} \tag{2.4}$$

Denote by P the projection of TM on $S(TM)$ with respect to the decomposition (2.4) and obtain

$$X = PX + \eta(X) \tag{2.5}$$

for any $X \in \Gamma(TM)$, where η is a local differential 1-form on M given by

$$\eta(X) = \overline{g}(X, N_1). \tag{2.6}$$

From (2.2) and (2.3) we choose the field of frames $\{\xi_1, U\}$ and $\{\xi_1, U, \xi_2, N_1\}$ on M. Let $\overline{\nabla}$ be the standard Levi-Civita connection of $\mathbb{R}^4_1$ and ∇ be the induced connection on M. According to (2.2) and (2.3) we get

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + h(X, Y) \\ \overline{\nabla}_X N_1 &= -A_{N_1} X + \nabla^{\perp}_X N_1 \\ \overline{\nabla}_X \xi_2 &= -A_{\xi_2} X + \nabla^{\perp}_X \xi_2 \end{aligned} \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$, where $\nabla_X Y$, $A_{N_1} X$, $A_{\xi_2} X \in \Gamma(TM)$ and $h(X, Y)$, $\nabla^{\perp}_X N_1$, $\nabla^{\perp}_X \xi_2 \in \Gamma(tr(TM))$. It is well known that ∇ is a torsion-free linear connection on M, but not the metric connection. Here A_{N_1} and A_{ξ_2} are linear operators on $\Gamma(TM)$ which are called shape operators and h is a symmetric bilinear form on M which is called the second fundamental form of M. Moreover, we rewrite on the locally coordinate neighborhood $\mathcal{U}$ of M,

$$\overline{\nabla}_X Y = \nabla_X Y + D_1(X, Y) N_1 + D_2(X, Y) \xi_2 \tag{2.8}$$

$$\overline{\nabla}_X N_1 = -A_{N_1} X + \rho_1(X) N_1 + \rho_2(X) \xi_2 \tag{2.9}$$

$$\overline{\nabla}_X \xi_2 = -A_{\xi_2} X + \epsilon_1(X) N_1 + \epsilon_2(X) \xi_2 \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$, where $D_i(X, Y) = \overline{g}(h(X, Y), \xi_i)$, $\rho_i(X) = \overline{g}(\nabla^{\perp}_X N_1, \xi_i)$ and $\epsilon_i(X) = \overline{g}(\nabla^{\perp}_X \xi_2, \xi_i)$, $i = 1, 2$, on $\mathcal{U}$. From (2.8) and (2.9), we have

$$D_1(X,\xi_1)=0,\quad \overline{g}(A_{N_1}X,N_1)=0. \tag{2.11}$$

Since $\overline{\nabla}$ is a metric connection, from (2.6) and (2.8), we obtain

$$(\nabla_X g)(Y,Z)=D_1(X,Y)\eta(Z)+D_1(X,Z)\eta(Y) \tag{2.12}$$

for any $X,Y,Z\in\Gamma(TM)$.

We consider the decomposition (2.2), then we have

$$\nabla_X PY = \nabla^*_X PY + h^*(X,PY) \tag{2.13}$$

$$\nabla_X \xi_1 = -A^*_{\xi_1}X + \nabla^{*^{\perp}}_X \xi_1 \tag{2.14}$$

for any $X,Y\in\Gamma(TM)$, where $\nabla^*_X PY$, $A^*_{\xi_1}X\in\Gamma(S(TM))$ and $h^*(X,PY)$, $\nabla^{*^{\perp}}_X\xi_1\in\Gamma(Rad\ TM)$.

If h (resp. h^*) is vanish on M, then M (resp. $S(TM)$) is called totally geodesic. M (resp. $S(TM)$) is to be totally umbilical in $\mathbb{R}^4_1$ if there exists a smooth affine normal vector field $\mathcal{Z}\in\Gamma(tr(TM))$ (resp. a smooth vector field $\mathcal{W}\in\Gamma(Rad\ TM)$) on M, called the affine normal curvature vector field of M, such that $h(X,Y)=\mathcal{Z}\ \overline{g}(X,Y)$ (resp. $h^*(X,PY)=\mathcal{W}\ g(X,PY)$), for all $X,Y\in\Gamma(TM)$.

The following lemma will be used later:

Lemma 2.2. *[4] Let M be a 1-lightlike submanifold of a semi-Riemannian manifold $(\overline{M},\overline{g})$. Suppose $\mathcal{U}$ is a coordinate neighborhood of M and $\xi_1\in\Gamma(Rad\ TM|_{\mathcal{U}})$ everywhere non-zero on $\mathcal{U}$. Then there exists a unique section N_1 of $S(TM^{\perp})^{\perp}$ which is given by*

$$N_1=\frac{1}{\overline{g}(V,\xi_1)}\{V-\frac{\overline{g}(V,V)}{2\overline{g}(V,\xi_1)}\xi_1\} \tag{2.15}$$

such that $\overline{g}(N_1,\xi_1)=1$, $\overline{g}(N_1,N_1)=0$, where $V\in\Gamma(S(TM^{\perp})^{\perp}|_{\mathcal{U}})$ such that V is non-null and $\overline{g}(V,\xi_1)\neq 0$.

For the dependence of all the induced geometric objects of M, we refer to [4] and [5].

Let

$$\begin{aligned}\alpha:\ & I && \to \mathbb{R}^4_1\\ & u && \to \alpha(u)=(\alpha_1(u),\alpha_2(u),\alpha_3(u),\alpha_4(u))\end{aligned} \tag{2.16}$$

be a differentiable curve, where I is open interval such that $0\in I$ and let ℓ be a straight line along α given by

$$\begin{aligned}\ell:\ & \mathbb{R} && \to \mathbb{R}^4_1\\ & v && \to \ell(v)=\alpha(u)+ve(u),\end{aligned} \tag{2.17}$$

where $e(u)$ is the director vector of ℓ at the point $\alpha(u)$ such that $e(u)$ and the tangent vector of α are linearly independent at every point of the curve α. We assume that $\overline{g}(\alpha^{'},e)=0$, $\overline{g}(\alpha^{'},\alpha^{'})=\varepsilon_1$ and $\overline{g}(e,e)=\varepsilon$, where $\varepsilon_1=\pm1$ or $\varepsilon_1=0$, $\varepsilon=\pm1$ or $\varepsilon=0$

and $\alpha' = \frac{d\alpha}{du}$, i.e. α' and e are orthogonal and given by arc-length or null-arc.

A ruled surface M in $\mathbb{R}^4_1$ is given by (the image of) a map $\phi : I \to \mathbb{R}^4_1$ of the form

$$\phi(u, v) = \alpha(u) + ve(u), \tag{2.18}$$

where we call α a base curve and the various positions of the generating line ℓ the rulings of the surface. If e is constant along α, then M is called cylindrical (or developable) ruled surface, otherwise it is called non-cylindrical (or skew) ruled surface. Then the tangent bundle TM of M is spanned by $\{\phi_u, \phi_v\}$, where

$$\phi_u = \frac{\partial \phi}{\partial u} = \alpha' + ve' \tag{2.19}$$

$$\phi_v = \frac{\partial \phi}{\partial v} = e \tag{2.20}$$

and $\overline{g}(\phi_u, \phi_v) = 0$. The distribution $A(u)$ spanned by $\{e, e'\}$ is called asymptotic bundle of M with respect to generating line ℓ. The vector subbundle $T(u)$ spanned by the set $\{\alpha', e, e'\}$ is said to be tangential bundle of M with respect to generating line ℓ. If $\overline{g}(\alpha', e') = 0$, then we call the curve α a striction curve. Denote by g the induced tensor field on M of $\overline{g}$. If g is non-degenerate on M, then M is a semi-Riemannian ruled surface, otherwise M is a lightlike (degenerate) ruled surface in $\mathbb{R}^4_1$.

In this study, we consider only the lightlike ruled surfaces in $\mathbb{R}^4_1$. For those ruled surfaces, we give some results, theorems and examples.

3 The classification of lightlike ruled surfaces in $\mathbb{R}^4_1$

We consider the ruled surface M in $\mathbb{R}^4_1$ given by (2.18). Then, from (2.19) and (2.20), the induced metric is given by

$$[g_{ij}] = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix},$$

where $g_{11} = \overline{g}(\phi_u, \phi_u) = \overline{g}(\alpha', \alpha') + 2v\overline{g}(\alpha', e') + v^2\overline{g}(e', e')$, $g_{12} = g_{21} = \overline{g}(\phi_u, \phi_v) = 0$ and $g_{22} = \overline{g}(\phi_v, \phi_v) = \varepsilon$. If $det[g_{ij}] = 0$, then M is a lightlike ruled surface. Thus, M is a lightlike ruled surface if and only if $g_{11} = 0$ or $g_{22} = \varepsilon = 0$.

Let M be a lightlike ruled surface in $\mathbb{R}^4_1$. The lightlike ruled surface M is said to be *type I* or *type II*, according to the cases $g_{11} = 0$ or $\varepsilon = 0$, respectively. We note that, in the case of *type I*, e is a spacelike vector field along α, i.e. $\varepsilon = 1$. If M is of *type II*, then ϕ_u is a spacelike vector field along α, i.e. $g_{11} > 0$. We recall that g_{11} and ε are not vanish at the same time along α. If $g_{11} = 0$ and $\varepsilon = 0$, then ϕ can not be an immersion with *rank* 2.

3.1 Ruled surfaces of *Type I*

Let M be a ruled of *type I* in $\mathbb{R}^4_1$. Then $g_{11} = 0$, that is

$$\overline{g}(e', e')v^2 + 2\overline{g}(\alpha', e')v + \overline{g}(\alpha', \alpha') = 0. \tag{3.1.1}$$

The equation (3.1.1) is a second degree equation with respect to v. If the discriminant $\triangle = b^2 - 4ac \geqslant 0$ of (3.1.1), there are solutions in $\mathbb{R}$ and $v_{1,2} = \frac{-b\mp\sqrt{\triangle}}{2a}$, where $a = \overline{g}(e', e')$, $b = 2\overline{g}(\alpha', e')$ and $c = \overline{g}(\alpha', \alpha')$. Then, we set $J = \mathbb{R} \setminus \{v_1, v_2\}$ if $\triangle \geqslant 0$, $J = \mathbb{R}$ if $\triangle < 0$. Thus the parametrization of M is given by

$$\phi(u, v) = \alpha(u) + v\ e(u),\ u \in I,\ v \in J. \tag{3.1.2}$$

Hence, the lightlike ruled surface M is of *type I* if and only if

$$\overline{g}(\alpha', \alpha') = \overline{g}(e', e') = \overline{g}(\alpha', e') = 0. \tag{3.1.3}$$

From (3.1.3), we obtain that α is a lightlike (null) curve and e' is either lightlike (null) vector field or $e' = 0$ (i.e. M is cylindrical) along α. Since α' and e are linear independent and $\overline{g}(\alpha', e) = 0$ along α, e is a unit spacelike vector field along α, that is $\varepsilon = 1$. Since $\overline{g}(\alpha', e') = 0$ in (3.1.3), there is a smooth function λ on I such that

$$e' = \lambda\alpha'. \tag{3.1.4}$$

If $e' = 0$, then λ is vanish on I. Thus, from (2.19) and (3.1.4), we get

$$\phi_u = (1 + \lambda v)\alpha'. \tag{3.1.5}$$

Then, the tangent bundle of M is spanned by $\{\alpha', e\}$ and α' is a lightlike (degenerate) vector field on M. If we set $\xi_1 = \alpha'$, then the radical distribution $Rad\ TM$ and the screen distribution $S(TM)$ are spanned by ξ_1 and the spacelike vector field e, respectively. Thus the ruled surface M is a half lightlike submanifold in $\mathbb{R}^4_1$, of codimension 2. Since $Rad\ TM$ is the 1-dimensional vector subbundle of $TM^\perp$, there exists a unit spacelike vector field ξ_2. Then we have the following orthogonal distribution

$$S(TM)^\perp = D \perp D^\perp,$$

where $D^\perp$ is the orthogonal complementary the distribution to D in $S(TM)^\perp$. Thus there exists a unique locally defined lightlike vector field $N_1 \in \Gamma(D^\perp)$ such that

$$\overline{g}(N_1, \xi_1) = 1,\ \overline{g}(N_1, N_1) = \overline{g}(N_1, \xi_2) = \overline{g}(N_1, e) = 0. \tag{3.1.6}$$

Furthermore, the lightlike transversal vector bundle $ltr(TM)$ is spanned by N_1. Hence, we have a local quasi orthonormal frame $\{\xi_1, e, N_1, \xi_2\}$ along α.

Let $\overline{\nabla}$ be the standard connection of $\mathbb{R}^4_1$ and ∇ be the induced connection on M, respectively. Then, from (2.7) and (3.1.5), we obtain

$$\overline{\nabla}_{\xi_1}\xi_1 = \nabla_{\xi_1}\xi_1 + h(\xi_1, \xi_1) = \frac{1}{1+\lambda v}\alpha'' = \frac{a}{1+\lambda v}\xi_1 + \frac{b}{1+\lambda v}\xi_2, \tag{3.1.7}$$

$$\overline{\nabla}_{\xi_1} e = \nabla_{\xi_1} e + h(\xi_1, e) = \frac{1}{1+\lambda v}e' = \frac{\lambda}{1+\lambda v}\xi_1, \tag{3.1.8}$$

$$\overline{\nabla}_{\xi_1} N_1 = -A_{N_1}\xi_1 + \nabla^\perp_{\xi_1} N_1 = -\frac{\lambda}{1+\lambda v}\ e - \frac{a}{1+\lambda v}\ N_1 + c\ \xi_2, \tag{3.1.9}$$

$$\overline{\nabla}_{\xi_1}\xi_2 = -A_{\xi_2}\xi_1 + \nabla^\perp_{\xi_1}\xi_2 = -c\ \xi_1 - \frac{b}{1+\lambda v}N_1, \tag{3.1.10}$$

where $a = \overline{g}(\alpha'', N_1)$, $b = \overline{g}(\alpha'', \xi_2)$ and $c = \overline{g}(\overline{\nabla}_{\xi_1} N_1, \xi_2)$ are smooth functions on I. On the other hand, we have

$$\overline{\nabla}_e \xi_1 = \overline{\nabla}_e e = \overline{\nabla}_e N_1 = \overline{\nabla}_e \xi_2 = 0. \tag{3.1.11}$$

From (2.7) and (3.1.7)-(3.1.11), we get

$$\alpha'' = a\ \xi_1 + b\ \xi_2, \tag{3.1.12}$$

$$h(\xi_1, \xi_1) = \frac{b}{1 + \lambda v}\xi_2, \quad h(\xi_1, e) = h(e, e) = 0, \tag{3.1.13}$$

$$A_{N_1}\xi_1 = \frac{\lambda}{1 + \lambda v}\ e, \quad A_{N_1} e = 0, \tag{3.1.14}$$

$$A_{\xi_2}\xi_1 = c\ \xi_1, \quad A_{\xi_2} e = 0, \tag{3.1.15}$$

and $A^*_{\xi_1}$ and h^* are vanish on M.

It is known that a lightlike surface in the Lorentzian space is either totally geodesic or totally umbilical [4].

Thus, from (3.1.12) and (3.1.13), we have the following theorem.

Theorem 3.1. *Let M be a ruled surface of type I in $\mathbb{R}^4_1$. Then the following assertions are equivalent:*
(1) M is totally geodesic.
(2) $b = 0$.
(3) α'' is a null vector.

Since the standard connection $\overline{\nabla}$ is a Levi-Civita connection, from (2.12), we obtain

$$\nabla_{\xi_1} g = 0, \quad \nabla_e g = 0. \tag{3.1.16}$$

Thus, from(3.1.16), we have

Theorem 3.2. *Let M be a ruled surface of type I in $\mathbb{R}^4_1$. Then the induced connection ∇ is always a metric connection.*

Now we consider the asymptotic bundle $A(u)$ of M. $A(u)$ either 2-dimensional or 1-dimensional. If $A(u)$ is 2-dimensional, then $A(u)$ coincides with the tangent bundle TM of M. If $A(u)$ is 1-dimensional, then $e' = 0$, i.e. M is a cylindrical ruled surface. Hence, we have

Corollary 3.1. *Let M be a ruled surface of type I in $\mathbb{R}^4_1$. If $dimA(u) = 1$ (resp. $dimA(u) = 2$), then M is a cylindrical ruled surface (resp. a skew ruled surface).*

From (3.1.4), we have the following corollary

Corollary 3.2. *Let M be a ruled surface of type I in $\mathbb{R}^4_1$. Then α is a striction curve.*

Example 3.1. *Consider the lightlike curve α in $\mathbb{R}^4_1$ given by*

$$\alpha(u) = (r + \sin u,\ 0,\ r - \sin u\ ,0),\ r \in \mathbb{R},\ u \neq k\pi + \frac{\pi}{2}, k \in \mathbb{Z}$$

and the unit spacelike vector field e along α is given by

$$e(u) = (\cos u,\ \frac{1}{\sqrt{2}},\ -\cos u,\ \frac{1}{\sqrt{2}}).$$

Then the ruled surface M is parametrized by

$$\phi(u, v) = \alpha(u) + v\ e(u), v \in \mathbb{R}.$$

Thus, we get

$$\phi_u = (1 + v\lambda)\alpha',\ \phi_v = e,\ \overline{g}(\phi_u, \phi_v) = 0,$$

where $\lambda = -\tan u$. Hence, the ruled surface M is a ruled surface of type I. Then, the radical distribution Rad TM and the screen distribution $S(TM)$ are spanned by $\xi_1 = \alpha'$ and e, respectively. Furthermore, the lightlike transversal bundle $ltr(TM)$ and the screen canonical bundle D are spanned by

$$N_1 = (-\frac{1 + 2\cos^2 u}{2\cos u}, -\sqrt{2}, \frac{-1 + 2\cos^2 u}{2\cos u},\ 0),\quad \xi_2 = (-\cos u, -\frac{1}{\sqrt{2}},\ \cos u,\ \frac{1}{\sqrt{2}}),$$

respectively. Thus, we have the quasi-orthonormal frame $\{\xi_1, e, N_1, \xi_2\}$ of $\mathbb{R}^4_1$. According to the quasi-orthonormal frame $\{\xi_1, e, N_1, \xi_2\}$, we obtain

$$\begin{aligned}
\overline{\nabla}_{\xi_1}\xi_1 &= \frac{a}{1 + \lambda v}\xi_1,\quad a = -\tan u,\quad b = 0,\\
\overline{\nabla}_{\xi_1} e &= \frac{\lambda}{1 + \lambda v} e,\ \lambda = -\tan u,\\
\overline{\nabla}_{\xi_1} N_1 &= -\frac{\lambda}{1 + \lambda v}\ e - \frac{a}{1 + \lambda v} N_1 + c\ \xi_2,\quad c = \frac{-\tan u}{1 - v\tan u},\\
\overline{\nabla}_{\xi_1}\xi_2 &= -c\xi_1.
\end{aligned}$$

This ruled surface is a totally geodesic ruled surface in $\mathbb{R}^4_1$.

Example 3.2. *In $\mathbb{R}^4_1$ consider the lightlike curve α and the unit spacelike vector e given by*

$$\alpha = \left(\frac{4}{3}u^3 + u,\ \sqrt{2}u^2,\ \frac{4}{3}u^3 - u,\ \sqrt{2}u^2\right),\ e = \left(0,\ \frac{-1}{\sqrt{2}},\ 0,\ \frac{-1}{\sqrt{2}}\right),$$

respectively. Thus, we have the parametrization of a ruled surface M given by

$$\phi(u, v) = \alpha(u) + v\ e,\quad u, v \in \mathbb{R}.$$

It is easy to check that M is a lightlike ruled surface in $\mathbb{R}^4_1$. If we choose $V = (-1, 0, 0, 0)$, from (2.15), we obtain

$$\begin{aligned} \xi_1 &= \alpha' = (4u^2+1,\ 2\sqrt{2}\ u,\ 4u^2-1,\ 2\sqrt{2}\ u), \\ e &= \left(0,\ \frac{-1}{\sqrt{2}},\ 0,\ \frac{-1}{\sqrt{2}}\right), \\ N_1 &= \frac{1}{2(4u^2+1)^2}(-(4u^2+1),\ 2\sqrt{2}\ u,\ 4u^2-1,\ 2\sqrt{2}\ u), \\ \xi_2 &= \frac{1}{\sqrt{2}(4u^2+1)}(0,\ 4u^2-1,\ -4\sqrt{2}\ u,\ 4u^2-1). \end{aligned}$$

Furthermore, we get

$$\lambda = 0,\ a = \frac{8u}{4u^2+1},\ b = -4,\ c = \frac{-2}{(4u^2+1)^2},$$

and

$$\begin{aligned} \overline{\nabla}_{\xi_1}\xi_1 &= a\xi_1 + b\xi_2, & \overline{\nabla}_{\xi_1}e &= 0, \\ \overline{\nabla}_{\xi_1}N_1 &= -aN_1 + c\ \xi_2, & \overline{\nabla}_{\xi_1}\xi_2 &= -c\ \xi_1 - bN_1. \end{aligned}$$

Hence, M is a cylindrical ruled surface of type I and it is totally umbilical.

3.2 Ruled Surfaces of *Type II*

Let M be a ruled surface of *type II* in $\mathbb{R}^4_1$ given by (2.18). Then, the curve α is a spacelike curve in $\mathbb{R}^4_1$ and e is a null vector field along α such that $\overline{g}(\alpha', e) = 0$. Since e is a null vector along α, $\overline{g}(e', e) = 0$. Thus, from Lemma 2.1, e' either e' is a spacelike vector, or e' is a null vector, or $e' = 0$. The tangent bundle TM of M is spanned by $\{\phi_u, \phi_v\}$. If we set $\xi_1 = \phi_v = e$, then ξ_1 is a degenerate vector field along α and the radical distribution $Rad\,TM$ of M is spanned by ξ_1. The screen distribution $S(TM)$ of M is spanned by a unit spacelike vector field U, where we can take $U = \frac{1}{\|\phi_u\|}\phi_u$. Thus, we have a quasi-orthonormal frame field $\mathcal{F} = \{\xi_1, U, N_1, \xi_2\}$ along M, where $ltr(TM)$ and D spanned by N_1 and ξ_2 which are a lightlike vector field and a unit spacelike vector field along M, respectively, such that

$$\overline{g}(\xi_1, U) = \overline{g}(\xi_1, \xi_2) = \overline{g}(\xi_2, N_1) = 0,\ \overline{g}(\xi_1, N_1) = 1. \tag{3.2.1}$$

Proposition 3.1. *Let M a ruled surface of type II in $\mathbb{R}^4_1$. Then the quasi-orthonormal field of frame field $\mathcal{F} = \{\xi_1, U, N_1, \xi_2\}$ of $\mathbb{R}^4_1$ along M is a quasi-orthonormal frame along the curve α.*

Proof: We prove the proposition with respect to the cases of e'.

i. We assume that e' is a spacelike vector field along the curve α. Then, from (3.2.1) we have $\overline{g}(\alpha', N_1) = \overline{g}(e', N_1) = 0$ and $\overline{g}(\alpha', \xi_2) = \overline{g}(e', \xi_2) = 0$, that is e' belong to $\Gamma(TM)$. Thus, we can choose $U = \alpha'$.

ii. We assume that e' is a null vector field along the curve α. Then, from Lemma 2.1, e and e' are linear depended. Thus, we can write $e' = \mu\ e$, where μ is a real parameter on I. Thus we choose $U = \phi_u - v\mu e = \alpha'$.

iii. If $e' = 0$, i.e. M is a cylindrical ruled surface, then $U = \alpha'$. Hence we have the assertion of the proposition. □

From now on, we assume that the frame $\mathcal{F}$ is along the curve α.

According to the frame $\mathcal{F}$, by the direct calculation, we have

$$\overline{\nabla}_{\xi_1}\xi_1 = \overline{\nabla}_{\xi_1}U = \overline{\nabla}_{\xi_1}N_1 = \overline{\nabla}_{\xi_1}\xi_2 = 0, \tag{3.2.2}$$

$$\overline{\nabla}_U\xi_1 = a_{11}\xi_1 + a_{12}U = e', \tag{3.2.3}$$

$$\overline{\nabla}_U U = a_{21}\xi_1 - a_{12}N_1 + a_{24}\xi_2, \tag{3.2.4}$$

$$\overline{\nabla}_U N_1 = -a_{21}U - a_{11}N_1 + a_{34}\xi_2, \tag{3.2.5}$$

$$\overline{\nabla}_U\xi_2 = -a_{34}\xi_1 - a_{24}U, \tag{3.2.6}$$

where $a_{11} = \overline{g}(\overline{\nabla}_U\xi_1, N_1)$, $a_{12} = \overline{g}(\overline{\nabla}_U\xi_1, U)$, $a_{21} = \overline{g}(\overline{\nabla}_U U, N_1)$, $a_{24} = \overline{g}(\overline{\nabla}_U U, \xi_2)$ and $a_{34} = \overline{g}(\overline{\nabla}_U N_1, \xi_2)$. From (3.2.2), we have

$$h(\xi_1, \xi_1) = h(\xi_1, U) = 0, \quad A_{N_1}\xi_1 = A_{\xi_2}\xi_1 = 0. \tag{3.2.7}$$

Thus, from (3.2.7), we have the following corollary:

Corollary 3.3. *Let M be a ruled surface of type II in $\mathbb{R}^4_1$ with the frame $\mathcal{F} = \{\xi_1, U, N_1, \xi_2\}$ along the curve α. Then, ξ_1 is an eigenvector field for A_{N_1} and A_{ξ_2} with respect to eigenfunctions $\lambda_1 = 0$ and $\lambda_2 = 0$, respectively.*

From (3.2.2)-(3.2.6), we have the following theorems:

Theorem 3.3. *Let M be a ruled surface of type II in $\mathbb{R}^4_1$. Then the following assertions are equivalent:*
(1) M is totally geodesic.
(2) e' is null vector field along the curve α and A_{ξ_2} is Rad TM-valued.

On the other hand, from (2.7) and (3.2.2)-(3.2.6), for the basis $\{\xi_1, U\}$ of TM, we have

$$h(\xi_1, \xi_1) = 0, \; h(\xi_1, u) = 0, \; h(U, U) = -a_{12}N + a_{24}\xi_2. \tag{3.2.8}$$

If we set $\mathcal{Z} = -a_{12}N + a_{24}\xi_2$, then we can write

$$h(X, Y) = g(X, Y)\mathcal{Z}$$

for any $X, Y \in \Gamma(TM)$. Thus the ruled surface of *type II* in $\mathbb{R}^4_1$ is also always a totally umbilical ruled surface.

Theorem 3.4. *Let M be a skew ruled surface of type II in $\mathbb{R}^4_1$. Then the induced connection ∇ is a metric connection if and only if e' is a null vector field along the curve α.*

Proof: Assume that ∇ is a metric connection on M. Then $\nabla_X g = 0$, for any $X \in \Gamma(TM)$. From (2.12) and (3.2.8), we obtain

$$(\nabla_U g)(\xi_1, U) = -a_{12}.$$

Thus, from (3.2.3), we get $e' = a_{11}\xi_1$, i.e. e' is a null vector field along α. Conversely, let e' be a null vector field along α. Then $a_{12} = 0$. From (2.12) and (3.2.8), we get

$$\nabla_{\xi_1} g = 0,\ (\nabla_U g)(\xi_1, \xi_1) = 0,\ (\nabla_U g)(\xi_1, U) = -a_{12} = 0,\ (\nabla_U g)(U, U) = 0.$$

Hence, we have $\nabla_X g = 0$, for any $X \in \Gamma(TM)$, that is ∇ is a metric connection. □

Theorem 3.5. *Let M a ruled surface of type II in $\mathbb{R}^4_1$. If M is a cylindrical ruled surface, then the induced connection ∇ is a metric connection.*

Proof: Let M be a cylindrical ruled surface. Then, from (3.2.3), we have $a_{11} = a_{12} = 0$. Thus, $A^*_{\xi_1}$ is vanish on M. So ∇ is a metric connection. □

Theorem 3.6. *Let M be a ruled surface of type II in $\mathbb{R}^4_1$. M is a cylindrical ruled surface if and only if the normal connection $\nabla^\perp$ is D-value.*

Let M be a ruled surface of *type II* in $\mathbb{R}^4_1$. We want to find parametrized curve $\gamma(u)$ such that $\overline{g}(\gamma'(u), e') = 0,\ u \in I$, and $\gamma(u)$ lies on the trace of ϕ; that is

$$\gamma(u) = \alpha(u) + v(u)e(u),$$

for some real-valued function $v = v(u)$. Assuming the existence of such a curve γ, we obtain

$$\gamma' = \alpha' + v'e + ve'.$$

Since $\overline{g}(e, e') = 0$, we have

$$v = -\frac{\overline{g}(\alpha', e')}{\overline{g}(e', e')}.$$

If e' is a spacelike vector field along the curve α, then from Proposition 3.5, $v \neq 0$. Hence α is not a striction line. If e' is a null vector field along α, then from Lemma 2.1 α is a striction line.

Now, we consider the asymptotic bundle $A(u)$ of M. If $dim\ A(u) = 2$ (resp.$dim\ A(u) = 1$), then e' is a spacelike (resp. e' is a null) vector field along the curve α.

Thus, we have the following theorem:

Theorem 3.7. *Let M be a ruled surface of type II in $\mathbb{R}^4_1$. Then the following statements are equivalent:*
i) $dim\ A(u) = 1$.
ii) e' is a null vector field along the curve α.
iii) α is a striction line.

Example 3.3. *We consider the ruled surface given by*

$$\phi(u, v) = \alpha(u) + ve(u),\ u \neq 0,$$

where $\alpha(u) = (0, 0, \cos u, \sin u)$ and $e(u) = (u, u, 0, 0)$. Then we have

$$\begin{aligned}\phi_u &= \alpha' + v\ e' = (0, 0, -\sin u, \cos u) + v(1, 1, 0, 0),\\ \phi_v &= e = (u, u, 0, 0).\end{aligned}$$

We can easily see that $\overline{g}(\phi_u, \phi_v) = \overline{g}(e, e) = 0$ *and* $\overline{g}(\phi_u, \phi_u) = 1$*. If we set* $\xi_1 = e$ *and* $U = \alpha'$*, then the radical distribution Rad* TM *and the screen distribution* $S(TM)$ *are spanned by* ξ_1 *and* U*, respectively. Moreover, we find that* $N_1 = \frac{-1}{2u}(1, -1, 0, 0)$ *and* $\xi_2 = (0, 0, \cos u, \sin u)$*, then the lightlike transversal vector bundle* $ltr(TM)$ *and the screen canonical affine normal bundle* D *are spanned by* N_1 *and* ξ_2*, respectively. Hence, the quasi-orthonormal frame field of* $\mathbb{R}^4_1$ *along the curve* α *is* $\mathcal{F} = \{\xi_1, U, N_1, \xi_2\}$*. According to this frame, we get*

$$\begin{aligned}\overline{\nabla}_U \xi_1 &= e' = a_{11}\xi_1,\\ \overline{\nabla}_U U &= a_{24}\xi_2,\\ \overline{\nabla}_U N_1 &= -a_{11}N_1,\\ \overline{\nabla}_U \xi_2 &= -a_{24}U_1,\end{aligned}$$

where $a_{11} = \frac{1}{u},\ a_{24} = -1,\ a_{12} = a_{21} = a_{34} = 0$*. Thus the ruled surface* M *is a totally umbilical surface.*

We recall that the ruled surface M *is intersect of the unit pseudo sphere* $S^3_1 = \{(x^1, x^2, x^3, x^4) : -(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1\}$ *with the hypersurface* $x^1 - x^2 = 0$ *in* $\mathbb{R}^4_1$*.*

Example 3.4. *Let* α *be a smooth curve in* $\mathbb{R}^4_1$ *given by* $\alpha(u) = (0, u, f(u), 0)$ *and* $e = (1, 0, 0, 1)$*, where* f *is an arbitrary smooth function. Then the ruled surface* M *is parametrized by*

$$\phi(u, v) = \alpha(u) + ve.$$

Thus we have

$$\phi_u = \alpha' = (0, 1, f'(u), 0),\ \ \phi_v = e.$$

Here α *is aspacelike curve and* e *is a lightlike vector field along* α*. The radical distribution Rad* TM *is spanned by* $\xi_1 = e$ *and the screen distribution* $S(TM)$ *is spanned by the unit spacelike vector field* $U = \frac{1}{\|\alpha'\|}\ \alpha'$*. Hence we get*

$$N_1 = \frac{1}{2}(-1, 0, 0, 1),\ \ \xi_2 = \frac{1}{\sqrt{1 + f'(u)^2}}(0, f'(u), -1, 0),$$

where $ltr(TM)$ *is spanned by* N_1 *which is a lightlike vector field and* D *is spanned by* ξ_2 *which is a unit spacelike vector field. Then* M *is a cylindrical ruled surface of type* II*. It is easy calculated that* $a_{11} = a_{12} = a_{21} = a_{34} = 0,\ a_{24} = \frac{-f''(u)}{(1+f'(u)^2)^{\frac{3}{2}}}$*. Thus we obtain*

$$\begin{aligned}&\overline{\nabla}_U \xi_1 = e' = 0, \quad \overline{\nabla}_U U = a_{24}\xi_2,\\ &\overline{\nabla}_U N_1 = 0, \qquad\quad \overline{\nabla}_U \xi_2 = -a_{24}U.\end{aligned}$$

Hence M *is a totally umbilical ruled surface and the induced connection* ∇ *is a metric connection.*

References

[1] M.P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, New Jersey, 1976.

[2] B.Y. Chen, *Geometry of Submanifolds*, Marcel Dekker, Inc. New York, 1973.

[3] K.L. Duggal and A. Bejancu, *Lightlike submanifolds of codimension two*, Math. J. Toyama Univ. 15 (1992), 59-82.

[4] K.L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Kluver Acad. Pub., Dordrecht, 1996.

[5] K.L. Duggal and D.H. Jin, *Half lightlike submanifolds of codimension 2*, Math. J. Toyama Univ. 22 (1999), 121-161.

[6] H. Frank and O. Giering, *Weralgemeinerte Regalflächen im Grassen I*, Arch. Math. 38 (1982), 106-115.

[7] L. Graves, *Codimension one isometric immersions between Lorentz spaces*, Trans. Amer. Math. Soc. 252 (1979), 367-392.

[8] S. Izumiya and N. Takeuchi, *Geometry of ruled surfaces*, App. Math. in the Golden Age (ed., J.C.Misra), Narosa Pub. House, New Delhi, 2003, 305-338.

[9] Y.H. Kim and D.W. Yoon, *Classification of ruled surfaces in Minkowski 3-space*, Journal of Geometry and Physics, 49 (2004), 89-100.

[10] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Acad. Press, 1983.

[11] C. Thas, *Properties of ruled surfaces in the Euclidean space E^n*, Bull. Inst. Math. Acad. Sinica, 6 (1978), 133-142.

[12] A. Turgut and H.H. Hacısalihoğlu, *Time-like ruled surfaces in Minkowski 3-space-II*, Tr. J. of Math. 22 (1998), 33-46.

Authors' address:

Erol Kılıç and H. Bayram Karadağ,
Inonu University, Faculty of Arts and Science,
Department of Mathematics, 44280 Malatya, Turkey.
email: ekilic@inonu.edu.tr and hbkaradag@inonu.edu.tr