Geometrical properties of a family of compactifications

Yotam I. Gingold and Harry Gingold

Abstract. We study the benefits of the explicit formulas of a parametrized family of bijections and the explicit formula of a certain metric. These formulas are induced by a family of compactifications of $\mathbb{C}$ that "account for all arguments of infinity". This family of bijections map the union of $\mathbb{C}$ and a continuum of ideal points onto a family of spherical bowls. This family of bijections are shown to give rise to a multitude of expressions that are "invariant with respect to independent rotations". These expressions help us generalize certain geometrical properties that are associated with the stereographic projection. Application of the metric to the approximation of unbounded functions is also demonstrated.

1 Introduction

The geometry of a plane and the geometry of a sphere play a unique role in mathematics and in mathematical physics. The mapping of the plane onto a sphere leads to the celebrated compactification known as the stereographic projection. However, the stereographic projection does not distinguish between positive infinity or negative infinity, or among other, different "values" of infinity. It is important both in mathematics and mathematical physics to possess computational tools that will distinguish between various "arguments of infinity." Compare e.g. with [8], and the texts [2, 7, 9]. In this article we study the tools that are the explicit formulas of the bijections and the metric induced by a family of compactifications that account for all arguments of infinity. We also provide some of their benefits. The fact that our bijections are infinitely smooth are of special value to the approximation of unbounded functions.

We first adopt some nomenclature and notations that will be used throughout this paper.

Notation 1.1. Denote by $Z = (x_1, x_2, x_3)$ a point in the Euclidean space $\mathbb{R}^3$, where x_j satisfy $-\infty < x_j < \infty$, $j = 1, 2, 3$. Denote by ID the continuum of ideal points $\mathrm{ID} := \{\infty(\cos\theta, \sin\theta) \mid 0 \leq \theta < 2\pi\}$. Call the set $\mathbb{C} \cup \mathrm{ID}$ the ultra extended complex plane. Denote by $z = (x, y) \in \mathbb{C}$ a point in the complex plane which is to be identified with the point $Q = (x, y, 0)$. Let $P = (0, 0, \gamma)$ be a fixed point on the x_3 coordinate, $0 < \gamma \leq 1$. We also put $r^2 = x^2 + y^2$, $R^2 = x_1^2 + x_2^2$ and $\omega = \gamma^2 + (1 - \gamma^2)r^2$. The word Bowl stands for the following set of points. Bowl $:= \left\{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1 \text{ and } -1 \leq x_3 \leq \gamma\right\}$.

The mapping soon to be developed, that matches each point $z \in \mathbb{C} \cup \mathrm{ID}$ with a point Z on the Bowl, will be denoted by $G(z)$. The derivation of the mapping from $\mathbb{C} \cup \mathrm{ID}$ to the Bowl is as follows. If P, Z, and Q lie on the same straight line, then the vectors $\overrightarrow{PZ}$ and $\overrightarrow{PQ}$ are collinear. This is if and only if $\overrightarrow{PZ} = t\overrightarrow{PQ}$ for some real number t. Videlicet, iff

$$(1.1.1) \qquad\qquad x_1 = tx, \qquad x_2 = ty, \qquad x_3 = (1 - t)\gamma.$$

Since we want our mapping to contain the stereographic projection as a particular case, we require t to be positive. Since x_1, x_2, and x_3 are points on the unit sphere, we have of course

$$(1.1.2) \qquad\qquad x_1^2 + x_2^2 + x_3^2 = 1.$$

We substitute the values of x_1, x_2, and x_3 from Equations (1.1.1) into Equation (1.1.2) to obtain $t^2 x^2 + t^2 y^2 + (1 - t)^2 \gamma^2 = 1$.

Solving for t, we obtain

$$t_{+,-} = \frac{\gamma^2 \pm \sqrt{\gamma^4 - (\gamma^2 - 1)(r^2 + \gamma^2)}}{\gamma^2 + r^2}.$$

Because we want $G(z)$ to map to the Bowl, so that $x_3 \leq \gamma$, we always choose

$$(1.1.3) \qquad\qquad t = \frac{\gamma^2 + \sqrt{\omega}}{\gamma^2 + r^2} = 1 - \frac{\sqrt{1 - R^2}}{\gamma}.$$

We now define a mapping from $\mathbb{C} \cup \mathrm{ID}$ into the Bowl as follows.

Definition 1.2. The mapping $G(z)$ from $\mathbb{C} \cup \mathrm{ID}$ into the Bowl is defined by

$(1.1.4)$

$$G(z) = \left\{ \begin{array}{ll} (x_1 = tx, x_2 = ty, x_3 = \gamma(1 - t)) & \text{if } z \in \mathbb{C} \\ (x_1 = \sqrt{1 - \gamma^2}\cos\theta, x_2 = \sqrt{1 - \gamma^2}\sin\theta, x_3 = \gamma) & \text{if } z = \infty(\cos\theta, \sin\theta) \end{array} \right\}.$$

The following theorem formalizes the previous discussion. Its simple proof is omitted.

Theorem 1.3. *G is a bijection from the ultra extended complex plane to the Bowl.*

Remark 1.4. *The definition of G given above is a natural one. Indeed, let $0 < \gamma < 1$. Given a sequence $z_n = r_n(\cos\theta_n, \sin\theta_n) = (x_n, y_n)$ where $r_n \to \infty$ and $(\cos\theta_n, \sin\theta_n) \to (\cos\theta, \sin\theta)$ as $n \to \infty$, the sequence is such that with $\omega_n = \gamma^2 + (1-\gamma^2)r_n^2$ and $t_n = \frac{\gamma^2 + \sqrt{\omega_n}}{\gamma^2 + r_n^2}$, we have $\sqrt{\omega_n} \sim \sqrt{1-\gamma^2}r_n, t_n \sim \frac{\sqrt{1-\gamma^2}}{r_n}$ as $n \to \infty$. Hence $x_{1n} \sim \sqrt{1-\gamma^2}\cos\theta_n, x_{2n} \sim \sqrt{1-\gamma^2}\sin\theta_n$.*

Remark 1.5. *For $\gamma = 1$, we obtain $t = \frac{2}{1+r^2}$, $x_1 = tx$, $x_2 = ty$, and $x_3 = 1-t$, the formulas of the stereographic projection. Compare e.g. with [1]. Then the sequence Z_n is such that $t_n \to 0, x_{1n} \to 0, x_{2n} \to 0, x_{3n} \to 1$, as $n \to \infty$.*

Remark 1.6. *The difference between each member of our family of compactifications, with $0 < \gamma^2 < 1$ and the stereographic projection, that corresponds to $\gamma^2 = 1$, is substantial indeed. First and foremost, Riemann's compactification of the complex plane is obtained by adding a single ideal point infinity. Our compactification augments $\mathbb{C}$ with a continuum of points ID. Riemann's compactification treats the point infinity like any other finite point. Our compactification recognizes among all the different arguments of infinity. Consequently, the image of the sequence $z_n = (-1)^n n$ converges on the Riemann sphere to the north pole. However, it will not converge on the Bowl in the metric to be introduced in this paper. Given a sequence z_n, e^{z_n} always converges on the Riemann sphere with $Re\{z_n\} \to \infty$. However, with $Re\{z_n\} \to \infty$, e^{z_n} does not converge in the ultra extended complex plane unless $e^{iIm\{z_n\}}$ converges.*

Remark 1.7. *The manner that the continuum of different arguments of infinity degenerate into one point is manifested in the formulas of the family of bijections when taking the limit $\gamma^2 \to 1^-$. It is noteworthy that the degeneration is accompanied by nonuniform convergence of $t(r^2)z$ as $\gamma^2 \to 1^-$.*

It is interesting to note the similarities and differences between the family of compactifications studied here and other compactifications related to the venerable sphere that are given in the literature. Poincare's compactification [8] uses the projection of $\mathbb{C}$ on two half spheres as an intermediary step that ultimately projects $\mathbb{C}$ onto a perpendicular plane to $\mathbb{C}$ that is tangent to the sphere. The bijection formulas that ensue are simple to work with in the realm of dynamical systems. However, the resulting transformations are not the mapping of the plane onto a compact surface but onto another non-compact plane. Poincare's half sphere is also utilized in the realization of non-Euclidean Geometry. See [5]. Equally interesting is Benedixon's transformations that are roughly speaking equivalent to an inversion of the plane. The relevance of these transformations to dynamical systems is documented in [2,7,9].

It is noteworthy that the derivations here uncover an abundance of expressions that are "invariant with respect to independent rotations." Essentially, these are mappings that are functions of the moduli of $z_1, \bar{z}_1, z_2, \bar{z}_2$. Quite a few of them turn out to be positive definite. Their useful occurrence motivates us to dedicate to them a formal definition.

Definition 1.8. We say that the mapping $S(z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n)$ of $2n$ points $z_j, \bar{z}_j \in \mathbb{C}$ is an expression invariant with respect to independent rotations, in short EIWRTIR, if

$$(1.1.5) \qquad S(z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n) = S(e^{i\theta_1}z_1, e^{-i\theta_1}\bar{z}_1, \ldots, e^{i\theta_n}z_n, e^{-i\theta_n}\bar{z}_n)$$

where $\theta_j \in \mathbb{R}$, $j = 1, 2, \ldots, n$.

For example, the mapping

$$S(z_1, \overline{z}_1, z_2, \overline{z}_2) = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (x_1 + x_2)^2 + (y_1 + y_2)^2$$

is an EIWRTIR. However, $D^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ is not an EIWRTIR. It is obvious that any $S(z_1, \overline{z}_1, \ldots, z_n, \overline{z}_n)$ that is an EIWRTIR is actually a function of the moduli $|z_1|, |z_2|, \ldots, |z_n|$. This is easily seen by choosing in Equation (1.1.5) $\theta_j = -\arg z_j$ for $j = 1, 2, \ldots, n$ and obtaining $S(z_1, \overline{z}_1, \ldots, z_n, \overline{z}_n) = S(|z_1|, |\overline{z}_1|, \ldots, |z_n|, |\overline{z}_n|)$. Of course, if S is a function of the moduli $|z_1|, |z_2|, \ldots, |z_n|$, then it is also an EIWRTIR.

The order of the contents in the remainder of this work is as follows. In section 2, we derive the explicit formulas of a metric. In this formula we encounter positive definite EIWRTIR's. Applications are discussed in section 3. Thanks to the explicit formulas of the bijections of this section, we are able to obtain a generalization of the theorem that asserts the similarity of two triangles in the setting of the stereographic projection. With the aid of the metric formula we are able to find certain manifestations of the power of a point with respect to a circle with infinite radius. They give rise to more EIWRTIR that are positive definite. Last but not least we show how to apply the induced metric to the approximation of unbounded functions.

2   An Induced Metric

In this section we derive a metric $\chi(z, \hat{z})$ for the ultra extended complex plane. To this end we need more notation.

Notation 2.1. In the sequel we denote by $\|G(z) - G(\hat{z})\|$ the Euclidean distance between two points. Denote by $\hat{Z} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ a point in the Euclidean space $\mathbb{R}^3$, where $\hat{x}_j$ satisfy $-\infty < \hat{x}_j < \infty$, $j = 1, 2, 3$. Denote by $\hat{z} = (\hat{x}, \hat{y})$ a point in the ultra extended complex plane which is identified with the point $\hat{Q} = (\hat{x}, \hat{y}, 0)$ such that $G(\hat{z}) = \hat{Z}$. We also put

$$\hat{r}^2 = \hat{x}^2 + \hat{y}^2, \ \hat{\omega} = \gamma^2 + (1 - \gamma^2)\hat{r}^2, \ \text{and} \ \hat{t} = \frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}.$$

Theorem 2.2. *The ultra extended complex plane is a complete metric space with respect to the chordal metric χ defined below as the Euclidean distance $\|G(z) - G(\hat{z})\|$.*

(2.2.1a) $\qquad \chi(z, \hat{z}) \equiv \|G(z) - G(\hat{z})\| = \sqrt{(x_1 - \hat{x}_1)^2 + (x_2 - \hat{x}_2)^2 + (x_3 - \hat{x}_3)^2}.$

Specifically, the square of the metric χ^2 is given by

(2.2.1b) $\qquad\qquad\qquad\qquad \chi^2(z, \hat{z}) = F(D^2 - \Delta)$

where F is a dilation factor

(2.2.1c) $\qquad\qquad\qquad\qquad F = \dfrac{(\gamma^2 + \sqrt{\omega})(\gamma^2 + \sqrt{\hat{\omega}})}{(\gamma^2 + r^2)(\gamma^2 + \hat{r}^2)},$

D^2 *is the square of the Euclidean distance between z and $\hat{z}$*

$$(2.2.1d) \qquad\qquad D^2 = (x - \hat{x})^2 + (y - \hat{y})^2,$$

and Δ is given by

(2.2.1e)

$$\Delta = \frac{(1 - \gamma^2)(r^2 - \hat{r}^2)^2}{(\sqrt{\omega} + \sqrt{\hat{\omega}})(\gamma^2 + \sqrt{\omega})(\gamma^2 + \sqrt{\hat{\omega}})} \left[\gamma^2 + \frac{(1 - \gamma^2)r^2\hat{r}^2 + \gamma^4(1 + \gamma^2) + \gamma^2(r^2 + \hat{r}^2)}{(\gamma^2 + r^2)\sqrt{\hat{\omega}} + (\gamma^2 + \hat{r}^2)\sqrt{\omega}} \right].$$

     F and Δ are EIWRTIR as functions of $|z|$, $|\bar{z}|$, $|\hat{z}|$, and $\left|\hat{\bar{z}}\right|$. F, Δ, and D^2 are positive definite functions of their variables. Specifically, $\Delta \geq 0$ for $\gamma^2 < 1$ and $\Delta = 0$ iff $\gamma^2 = 1$ or $r^2 = \hat{r}^2$.
     For $z = \infty(\cos\theta, \sin\theta)$, $\hat{z} = (\hat{x}, \hat{y})$,

$$(2.2.1f) \quad \chi^2(z, \hat{z}) = 2\gamma^2 \frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2} + 2(1 - \gamma^2) - 2\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\sqrt{1 - \gamma^2}(\hat{x}\cos\theta + \hat{y}\sin\theta)$$

and for $z = \infty(\cos\theta, \sin\theta)$, $\hat{z} = \infty(\cos\hat{\theta}, \sin\hat{\theta})$,

$$(2.2.1g) \qquad \chi^2(z, \hat{z}) = 4(1 - \gamma^2)\sin^2\left(\frac{\theta - \hat{\theta}}{2}\right) = \left(2\sqrt{1 - \gamma^2}\left|\sin\left(\frac{\theta - \hat{\theta}}{2}\right)\right|\right)^2.$$

Proof. We omit the trivial part that $\chi(z, \hat{z})$ is a distance function and proceed with the derivations of Equations (2.2.1a) to (2.2.1g).
     To derive χ, we turn to the definition of G, namely

$$(2.2.2) \qquad \|G(z) - G(\hat{z})\|^2 = (x_1 - \hat{x_1})^2 + (y_1 - \hat{y_1})^2 + (z_1 - \hat{z_1})^2.$$

     Expanding the squared terms in Equation (2.2.2) gives us

$$(2.2.3) \qquad \|Z - \hat{Z}\|^2 = x_1^2 + x_2^2 + x_3^2 + \hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 - 2x_1\hat{x}_1 - 2x_2\hat{x}_2 - 2x_3\hat{x}_3.$$

     Because x_1, x_2, x_3 and $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$ are coordinates on the unit sphere, we have

$$(2.2.4) \qquad\qquad x_1^2 + x_2^2 + x_3^2 = \hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 = 1.$$

     Substituting the expression given in Equation (2.2.4) into Equation (2.2.3) leaves us with

$$(2.2.5) \qquad\qquad \|Z - \hat{Z}\|^2 = 1 + 1 - 2\left[x_1\hat{x}_1 + x_2\hat{x}_2 + x_3\hat{x}_3\right].$$

     Expanding x_1, x_2, x_3, $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$ given by Notations 1.1 and 2.1 gives us

$$(2.2.6) \qquad \|Z - \hat{Z}\|^2 = 2 - 2\left[\begin{array}{l} \left(\frac{\gamma^2 + \sqrt{\omega}}{\gamma^2 + r^2}\right)\left(\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\right)(x\hat{x} + y\hat{y} + \gamma^2) \\ + \gamma^2 - \gamma^2\left(\frac{\gamma^2 + \sqrt{\omega}}{\gamma^2 + r^2}\right) - \gamma^2\left(\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\right) \end{array} \right].$$

     Expanding D^2 with the aid of $r^2 = x^2 + y^2$ and $\hat{r}^2 = \hat{x}^2 + \hat{y}^2$ yields the representation

$$(2.2.7) \qquad 2x\hat{x} + 2y\hat{y} = -D^2 + r^2 + \hat{r}^2.$$

Next we substitute the expression given in (2.2.7) into Equation (2.2.6) to obtain

(2.2.8)
$$\begin{aligned}
\|Z - \hat{Z}\|^2 &= \left(\frac{\gamma^2 + \sqrt{\omega}}{\gamma^2 + r^2}\right)\left(\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\right)(D^2 - r^2 - \hat{r}^2) + 2 - 2\gamma^2 \\
&\quad - 2\gamma^2\left(\frac{\gamma^2 + \sqrt{\omega}}{\gamma^2 + r^2}\right)\left(\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\right) + 2\gamma^2\left(\frac{\gamma^2 + \sqrt{\omega}}{\gamma^2 + r^2}\right) + 2\gamma^2\left(\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\right).
\end{aligned}$$

A lengthy but straightforward calculation leads us to the desired formulas (2.2.1b) to (2.2.1e).

Note that $(1 - \gamma^2)$ appears as a factor at the front of Δ. For $\gamma^2 = 1$, $\Delta = 0$ and $F = \frac{4}{(1+r^2)(1+\hat{r}^2)}$, leaving us with the Riemann Sphere chordal metric.

Next, when $z \in \text{ID}$ and $\hat{z} \in \mathbb{C}$, we shall derive the formula stated in Theorem 2.2 by observing the asymptotic behavior of χ as $r \to \infty$. First, observe that

(2.2.9a)
$$\omega = \gamma^2 + (1 - \gamma^2)r^2 = (1 - \gamma^2)r^2\left[1 + \frac{\gamma^2}{(1 - \gamma^2)r^2}\right] \sim (1 - \gamma^2)r^2 \text{ as } r \to \infty.$$

Furthermore, assume that as $r \to \infty$

$$(2.2.9b) \qquad \frac{x}{r} \sim \cos\theta, \qquad \frac{y}{r} \sim \sin\theta.$$

Substituting the asymptotic expression for ω given in Equation (2.2.9a) into Equation (2.2.6) and multiplying the first term inside the bracket by $\frac{r}{r}$, we see that

(2.2.10)
$$\|Z - \hat{Z}\|^2 \sim 2 - 2\left[\begin{array}{c}
\left(\frac{\gamma^2 + r\sqrt{1 - \gamma^2}}{\gamma^2 + r^2}\right)\left(\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\right)r\left(\frac{x}{r}\hat{x} + \frac{y}{r}\hat{y} + \frac{\gamma^2}{r}\right) \\
+ \gamma^2 - \gamma^2\left(\frac{\gamma^2 + r\sqrt{1 - \gamma^2}}{\gamma^2 + r^2}\right) - \gamma^2\left(\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\right)
\end{array}\right]$$

as $r \to \infty$. Substituting the asymptotic expressions for $\frac{x}{r}$ and $\frac{y}{r}$ given in Equation (2.2.9b) into Equation (2.2.10) and factoring powers of r yields

(2.2.11)
$$\begin{aligned}
\|Z - \hat{Z}\|^2 &\sim 2 - 2\left[\begin{array}{c}
\frac{r\sqrt{1 - \gamma^2}\left(1 + \frac{\gamma^2}{r\sqrt{1 - \gamma^2}}\right)}{r^2\left(1 + \frac{\gamma^2}{r^2}\right)}\left(\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\right)r\left(\hat{x}\cos\theta + \hat{y}\sin\theta + \frac{\gamma^2}{r}\right) \\
+ \gamma^2 - \gamma^2\frac{r\sqrt{1 - \gamma^2}\left(1 + \frac{\gamma^2}{r\sqrt{1 - \gamma^2}}\right)}{r^2\left(1 + \frac{\gamma^2}{r^2}\right)} - \gamma^2\left(\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\right)
\end{array}\right] \\
&\sim 2 - 2\left[\sqrt{1 - \gamma^2}\left(\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\right)(\hat{x}\cos\theta + \hat{y}\sin\theta) + \gamma^2 - \gamma^2\left(\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\right)\right] \\
&\sim 2\gamma^2\left(\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\right) + 2(1 - \gamma^2) - 2\sqrt{1 - \gamma^2}\left(\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\right)(\hat{x}\cos\theta + \hat{y}\sin\theta),
\end{aligned}$$

as $r \to \infty$. We have now arrived at the stated formula for χ when $z \in \text{ID}, \hat{z} \in \mathbb{C}$.

Note that for $\gamma^2 = 1$, $\chi^2(z, \hat{z}) = 2\frac{1+1}{1+\hat{r}^2} + 0 = \frac{4}{1+\hat{r}^2}$, leaving us with the Riemann Sphere chordal metric.

Finally, we shall derive the formula for χ when both $z, \hat{z} \in \mathrm{ID}$, as stated in Theorem 2.2. We shall do this by observing the asymptotic behavior of χ as both $r, \hat{r} \to \infty$. The asymptotic expression for $\hat{\omega}$ as $\hat{r} \to \infty$ is given by

$$(2.2.12\mathrm{a}) \qquad \hat{\omega} = \gamma^2 + (1-\gamma^2)\hat{r}^2 = (1-\gamma^2)\hat{r}^2 \left[1 + \frac{\gamma^2}{(1-\gamma^2)\hat{r}^2} \right] \sim (1-\gamma^2)\hat{r}^2.$$

Furthermore, assume that as $\hat{r} \to \infty$ we have

$$(2.2.12\mathrm{b}) \qquad\qquad \frac{\hat{x}}{\hat{r}} \sim \cos\hat{\theta}, \qquad \frac{\hat{y}}{\hat{r}} \sim \sin\hat{\theta}.$$

Then as $r, \hat{r} \to \infty$ we have

$$(2.2.13) \qquad \begin{aligned} \|Z - \hat{Z}\|^2 \quad\sim\quad & 2\gamma^2 \left(\frac{\gamma^2 + \hat{r}\sqrt{1-\gamma^2}}{\gamma^2 + \hat{r}^2} \right) + 2(1-\gamma^2) \\ & -2\sqrt{1-\gamma^2} \left(\frac{\gamma^2 + \hat{r}\sqrt{1-\gamma^2}}{\gamma^2 + \hat{r}^2} \right) \hat{r} \left(\frac{\hat{x}}{\hat{r}} \cos\theta + \frac{\hat{y}}{\hat{r}} \sin\theta \right). \end{aligned}$$

Substituting the asymptotic expressions from equations (2.2.12a) and (2.2.12b) into (2.2.13), after factoring powers of $\hat{r}$ in (2.2.13), yields as $r, \hat{r} \to \infty$

$$(2.2.14) \qquad \begin{aligned} \|Z - \hat{Z}\|^2 \quad\sim\quad & 2\gamma^2 \frac{\hat{r}\sqrt{1-\gamma^2}\left(1 + \frac{\gamma^2}{\hat{r}\sqrt{1-\gamma^2}}\right)}{\hat{r}^2\left(1 + \frac{\gamma^2}{\hat{r}^2}\right)} + 2(1-\gamma^2) \\ & -2\sqrt{1-\gamma^2} \frac{\hat{r}\sqrt{1-\gamma^2}\left(1 + \frac{\gamma^2}{\hat{r}\sqrt{1-\gamma^2}}\right)}{\hat{r}^2\left(1 + \frac{\gamma^2}{\hat{r}^2}\right)} \hat{r} \left(\cos\theta\cos\hat{\theta} + \sin\theta\sin\hat{\theta} \right) \\ \sim\quad & 2(1-\gamma^2) - 2(1-\gamma^2)\left(\cos\theta\cos\hat{\theta} + \sin\theta\sin\hat{\theta} \right) \\ \sim\quad & 2(1-\gamma^2)\left[1 - \left(\cos\theta\cos\hat{\theta} + \sin\theta\sin\hat{\theta} \right) \right] \\ \sim\quad & 2(1-\gamma^2)\left[1 - \cos(\theta - \hat{\theta}) \right] \sim 4(1-\gamma^2)\sin^2\left(\frac{\theta - \hat{\theta}}{2} \right). \end{aligned}$$

We have now arrived at the stated formula for χ when $z, \hat{z} \in \mathrm{ID}$. The task of proving the completeness of $\mathbb{C} \cup \mathrm{ID}$ in the metric $\chi(z, \hat{z})$ is omitted because of its simplicity. $\qquad\square$

Remark 2.3. *As we expect, for $\gamma^2 = 1$, $\chi\left(\infty(\cos\theta, \sin\theta), \infty(\cos\hat{\theta}, \sin\hat{\theta}) \right) = 0$. On the Riemann sphere, all arguments of infinity map to the north pole. The nonuniform convergence of our metric to the Riemann metric is manifested numerically by the term $\sqrt{1-\gamma^2}$ occurring in denominators in (2.2.9a), (2.2.11) and (2.2.14).*

3 Applications

As a first application to the bijections derived in section 1, we uncover a positive definite EIWRTIR. It helps us generalize, a well known theorem about the similarity of certain triangles in the setting of the stereographic projection. We keep in mind Notations 1.1 and 2.1. Then we have

Theorem 3.1. *The triangles $PZ\widehat{Z}$ and $P\widehat{Q}Q$ are similar if $\gamma^2 = 1$. If $\gamma^2 < 1$, then $PZ\widehat{Z}$ and $P\widehat{Q}Q$ are similar iff $r^2 = \widehat{r}^2$.*

Proof. We know that

$$(3.3.1) \qquad (PQ)^2 = x^2 + y^2 + \gamma^2 = r^2 + \gamma^2, \qquad (P\widehat{Q})^2 = \widehat{r}^2 + \gamma^2.$$

Expanding $(PZ)^2$, we see that

$$(3.3.2) \quad (PZ)^2 = x_1^2 + x_2^2 + (x_3 - \gamma)^2 = 1 - \gamma^2 + 2\gamma^2 t, \qquad (P\widehat{Z})^2 = 1 - \gamma^2 + 2\gamma^2 \widehat{t}.$$

Substituting the expressions for $(PQ)^2$, $(P\widehat{Q})^2$, $(PZ)^2$, and $(P\widehat{Z})^2$ given in Equations (3.3.1) and (3.3.2), respectively, into the expression I defined by

$$(3.3.3) \qquad\qquad I := (P\widehat{Z})^2(P\widehat{Q})^2 - (PZ)^2(PQ)^2,$$

we have

$$(3.3.4) \qquad I = [(1 - \gamma^2) + 2\gamma^2\widehat{t}][\widehat{r}^2 + \gamma^2] - [(1 - \gamma^2) + 2\gamma^2 t][r^2 + \gamma^2].$$

Factoring $\widehat{r}^2 - r^2$ and grouping terms sharing the coefficient $2\gamma^2$ in (3.3.4) gives us

$$(3.3.5) \qquad I = (1 - \gamma^2)(\widehat{r}^2 - r^2) + 2\gamma^2 \frac{(1 - \gamma^2)(\widehat{r}^2 - r^2)}{\sqrt{\gamma^2 + (1 - \gamma^2)\widehat{r}^2} + \sqrt{\gamma^2 + (1 - \gamma^2)r^2}}.$$

Finally, we obtain

$$(3.3.6) \qquad I = (P\widehat{Z})^2(P\widehat{Q})^2 - (PZ)^2(PQ)^2 = (1 - \gamma^2)(\widehat{r}^2 - r^2)\sigma,$$

where

$$(3.3.7) \qquad \sigma := \left\{ 1 + \frac{2\gamma^2}{\sqrt{\gamma^2 + (1 - \gamma^2)\widehat{r}^2} + \sqrt{\gamma^2 + (1 - \gamma^2)r^2}} \right\}.$$

There are three factors on the right hand side of Equation (3.3.7), $(1-\gamma^2)$, $(\widehat{r}^2 - r^2)$, and σ. Luckily, $\sigma \geq 1$. Hence, $PZ\widehat{Z}$ and $P\widehat{Q}Q$ are similar if $\gamma^2 = 1$ or, for $\gamma^2 < 1$, iff $r^2 = \widehat{r}^2$. It is no surprise that I is an EIWRTIR. However, it is a pleasant surprise that $\sigma - 1$ is a positive definite expression.

The power of a point Q with respect to a circle $|z - z_0| = \varsigma$ (with center z_0 and radius ς), is defined to be the product $QV \cdot QW$, where VW is any chord in the circle above, passing through a fixed point Q that is identified with z_0. This product is a property of the point Q and the circle $|z - z_0| = \varsigma$ and is not dependent on the chord VW. See e.g. [10, p. 232]. (This property is yet another manifestation of the similarity of certain triangles.) It would be natural to define the power of a point P with respect to the ideal circle $|z| = \infty$ as ∞. If V and W are ideal points, it would be natural to put $VQ = WQ = \infty$ and define $VQ \cdot WQ = \infty$. A question then arises. Is it possible to obtain via the family of bijections and the spherical Bowl metric of theorem 2.2 some well determined expressions that would be a manifestation of the power of a point with respect to a circle with infinite radius? The answer that we

provide is given in terms of two identities and some inequalities that involve positive definite EIWRTIR's.

Rather than start with a formal theorem we prefer at this stage to proceed with an informal discussion that will lead to the desired results. First we need a few notations. Fix a point $z = (x, y)$ that is situated inside the ideal circle $|z| = \infty$. Consider a triangle with vertices ZAB, where

$$(3.3.8) \quad Z = G(z), \quad A = G(\infty(\cos\theta, \sin\theta), \quad B = G(\infty(\cos(\theta + \pi), \sin(\theta + \pi)).$$

The edges of ZAB are

$$(3.3.9) \qquad \begin{aligned} AB = 2\sqrt{1-\gamma^2}, \qquad ZA = \chi(\infty(\cos\theta, \sin\theta, z)), \\ ZB = \chi(\infty(\cos(\theta + \pi), \sin(\theta + \pi)), z). \end{aligned}$$

Let us find the coordinates of A and B. The points A and B, respectively, are determined by the intersection of a certain line that passes through the point $P = (0, 0, \gamma)$ and intersects the circle $x_1^2 + x_2^2 = 1 - \gamma^2$. This line L2 is parallel to a line in the complex plane, which we denote by L1, that passes through a fixed point $Q = (x, y, 0)$, and has direction $(a, b, 0)$, where $a^2 + b^2 = 1$. It is easily verified that the parametric equations of the line L1 that possesses the direction $(a, b, 0)$ and passes through the point $Q = (x, y, 0)$ are

$$(3.3.10) \qquad x_1 = x + \eta a, \qquad x_2 = y + \eta b, \qquad x_3 = 0, \qquad -\infty < \eta < \infty.$$

The parametric equations of the line L2 that possesses the direction $(a, b, 0)$ and passes through the point $P = (0, 0, \gamma)$ is given by

$$(3.3.11) \qquad x_1 = \theta a, \qquad x_2 = \theta b, \qquad x_3 = \gamma, \qquad -\infty < \theta < \infty.$$

The line L2 intersects the sphere at the points

$$(3.3.12) \qquad A := (\sqrt{1-\gamma^2}a, \sqrt{1-\gamma^2}b, \gamma), \quad B := (-\sqrt{1-\gamma^2}a, -\sqrt{1-\gamma^2}b, \gamma).$$

Let $t = t(r^2)$, $r^2 = x^2 + y^2$. Then the vectors $\overrightarrow{ZA}$ and $\overrightarrow{ZB}$, their magnitudes, and their scalar product $\overrightarrow{ZA} \cdot \overrightarrow{ZB}$ are given respectively by

$$(3.3.13a) \qquad \overrightarrow{ZA} = \left\langle \sqrt{1-\gamma^2}a - tx, \ \sqrt{1-\gamma^2}b - ty, \ \gamma t \right\rangle,$$

$$(3.3.13b) \qquad \left\| \overrightarrow{ZA} \right\|^2 = 1 - \gamma^2 - 2\sqrt{1-\gamma^2}t(ax + by) + t^2 r^2 + t^2 \gamma^2,$$

$$(3.3.13c) \qquad \overrightarrow{ZB} = \left\langle -\sqrt{1-\gamma^2}a - tx, \ -\sqrt{1-\gamma^2}b - ty, \ \gamma t \right\rangle,$$

$$(3.3.13d) \qquad \left\| \overrightarrow{ZB} \right\|^2 = 1 - \gamma^2 + 2\sqrt{1-\gamma^2}t(ax + by) + t^2 r^2 + t^2 \gamma^2,$$

(3.3.13e)
$$\overrightarrow{ZA} \cdot \overrightarrow{ZB} = -\left[(1 - \gamma^2) - t^2(r^2 + \gamma^2)\right].$$

Notice that

(3.3.14)
$$|ax + by| \le (a^2 + b^2)^{\frac{1}{2}}(x^2 + y^2)^{\frac{1}{2}} \le r.$$

Denote by $\widehat{ZA}$ and $\widehat{ZB}$ the respective arcs of the circle generated by the intersection of the plane passing through the points P, Z, A, B and the sphere. Denote also by ZA^*, ZB^*, the arcs of the great circles passing through ZAO and ZBO, (with O being the origin). The arc $\widehat{ZA}$ corresponds to the chord ZA, and $\widehat{ZB}$ is the arc corresponding to the chord ZB. Use the relation (3.3.14) in (3.3.13b) and in (3.3.13d) to obtain

(3.3.15a)
$$\widehat{ZA}^2 \ge ZA^{*2} \ge \left\|\overrightarrow{ZA}\right\|^2 \ge (\sqrt{1 - \gamma^2} - tr)^2 + t^2\gamma^2,$$

(3.3.15b)
$$\widehat{ZB}^2 \ge ZB^{*2} \ge \left\|\overrightarrow{ZB}\right\|^2 \ge (\sqrt{1 - \gamma^2} - tr)^2 + t^2\gamma^2.$$

It is also easily verified that

(3.3.16)
$$\left\|\overrightarrow{ZA}\right\|^2 + \left\|\overrightarrow{ZA}\right\|^2 = 2\left[1 - \gamma^2 + t^2(r^2 + \gamma^2)\right].$$

In addition to the lower bounds in (3.3.15), we can obtain upper bounds on the product of $\left\|\overrightarrow{ZA}\right\|^2 \left\|\overrightarrow{ZB}\right\|^2$ in terms of EIWRTIR that are also positive definite. The relations (3.3.13b), (3.3.13d) and (3.3.14) also imply that

(3.3.17) $\quad \left\|\overrightarrow{ZA}\right\|^2 \le (\sqrt{1 - \gamma^2} + tr)^2 + t^2\gamma^2, \qquad \left\|\overrightarrow{ZB}\right\|^2 \le (\sqrt{1 - \gamma^2} + tr)^2 + t^2\gamma^2,$

(3.3.18) $\quad \left[(\sqrt{1 - \gamma^2} - tr)^2 + t^2\gamma^2\right]^2 \le \left\|\overrightarrow{ZA}\right\|^2 \left\|\overrightarrow{ZB}\right\|^2 \le \left[(\sqrt{1 - \gamma^2} + tr)^2 + t^2\gamma^2\right]^2.$

The relations (3.3.13e), (3.3.16), together with the inequalities (3.3.17), (3.3.18), are a manifestation of the power of the point z with respect to the circle with center at the origin and radius infinity. $\qquad \square$

Remark 3.2. *We notice that on the Riemann Sphere, in the case $\gamma^2 = 1$, the right hand side of (3.3.16) tends to zero as $r^2 \to \infty$. This is in glaring contradiction to the case $1 - \gamma^2 > 0$, where the right hand side of (3.3.16) is positive even when z is an ideal point. Then, $r^2 = \infty$, $t(r^2) = 0$, and the right hand side of (3.3.16) is $2(1 - \gamma^2)$.*

Next, we demonstrate the application of the formulas of the bijections and of the metric, to the problem of the approximation of unbounded functions. Consider the family of possibly unbounded real valued functions $f(s)$ on the interval $[0, 1]$ such that

$x_1(s) = t(f^2(s))f(s) \in C[0,1]$. Notice that unlike the theory of Pade approximants or the theory of Jacobi polynomials, we impose no restriction on the order of growth of the function $f(s)$. Let C_j^n be the binomial coefficients. Consider the Bernstein polynomials

$$(3.3.19) \qquad B(s,n) = \sum_{j=0}^{n} x_1\left(\frac{j}{n}\right) C_j^n s^j (1-s)^{n-j},$$

with

$$(3.3.20) \qquad x_1\left(\frac{j}{n}\right) = \frac{\left[\gamma^2 + \sqrt{\gamma^2 + (1-\gamma^2)f^2\left(\frac{j}{n}\right)}\right] f\left(\frac{j}{n}\right)}{\gamma^2 + f^2\left(\frac{j}{n}\right)}.$$

We shall prove the following theorem.

Theorem 3.3. *Given a function $f(s)$ such that $\frac{\left[\gamma^2 + \sqrt{\gamma^2 + (1-\gamma^2)f^2(s)}\right] f(s)}{\gamma^2 + f^2(s)} \in C[0,1]$,*

(i) the sequence of approximants $\frac{B(s,n)}{1 - \frac{\sqrt{1-B^2(s,n)}}{\gamma}}$ converges uniformly on $[0,1]$ to $f(s)$ in the Spherical Bowl metric given by Equation (2.2.1).

(ii) On every closed subset $I \in [0,1]$ where $f(s)$ is continuous, we have

$$(3.3.21) \qquad \lim_{n\to\infty} Sup_{s\in I} \left| f(s) - \frac{B(s,n)}{1 - \frac{\sqrt{1-B^2(s,n)}}{\gamma}} \right| = 0.$$

Proof. Notice that the properties of the bijection (1.1.4) together with the properties of the Bernstein Polynomials guarantee that

$$(3.3.22) \qquad \begin{aligned} -\sqrt{1-\gamma^2} &\leq \frac{\left[\gamma^2 + \sqrt{\gamma^2 + (1-\gamma^2)f^2(s)}\right] f(s)}{\gamma^2 + f^2(s)} \leq \sqrt{1-\gamma^2} \\ \Rightarrow -\sqrt{1-\gamma^2} &\leq \qquad\qquad B(s,n) \qquad\qquad \leq \sqrt{1-\gamma^2}. \end{aligned}$$

Consequently the preimage of $\frac{\left[\gamma^2 + \sqrt{\gamma^2 + (1-\gamma^2)f^2(s)}\right] f(s)}{\gamma^2 + f^2(s)}$ under our bijection is $f(s)$, and the image of $\frac{B(s,n)}{1 - \frac{\sqrt{1-B^2(s,n)}}{\gamma}}$ under our bijection is $B(s,n)$, where $-\sqrt{1-\gamma^2} \leq B(s,n) \leq \sqrt{1-\gamma^2}$. In other words, we are assured that the values of the image of $\frac{B(s,n)}{1 - \frac{\sqrt{1-B^2(s,n)}}{\gamma}}$ are on the spherical Bowl and never in its complement set with respect to the sphere. This concludes the proof of (i).

The proof of (ii) follows from the following observations. Consider a compact disk $CD := \{z \mid |z| \leq R < \infty\}$ together with its image $IM := \{Z \mid Z = G(z), z \in CD\}$ on the spherical Bowl. Then an arbitrarily small neighborhood in IM must be the image of a small neighborhood of CD and Equation (3.3.21) follows.   □

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Authors' addresses:

Yotam I. Gingold
Computer Science Department, New York University
719 Broadway, New York, NY 10003, USA
e-mail: gingold@cs.nyu.edu

Harry Gingold
Department of Mathematics, West Virginia University
Morgantown, WV 26505, USA
e-mail: gingold@math.wvu.edu