On some property of the tangency relation of sets

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Abstract. In this paper the problem of the homogeneity of the tangency relation $T_l(a,b,k,p)$ of sets of the classes $A^*_{p,k}$ having the Darboux property in the generalized metric spaces (E,l) is considered. In Introduction of this paper we shall give the definition of the homogeneity of the tangency relation $T_l(a,b,k,p)$ in some class of the functions. Some sufficient conditions for the homogeneity of this tangency relation will be given in Section 2 of the present paper.

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1 Introduction

Let E be an arbitrary non-empty set and let l be a non-negative real function defined on the Cartesian product $E_0 \times E_0$ of the family E_0 of all non-empty subsets of the set E.

Let l_0 be a function defined by the formula

$$l_0(x,y) = l(\{x\},\{y\}) \quad \text{for} \quad x,y \in E. \tag{1.1}$$

If we put some conditions on the function l, then the function l_0 defined by (1.1) will be a metric of the set E. For this reason the pair (E,l) can be treated as a certain generalization of a metric space and we shall call it (see [9]) the generalized metric space. Using (1.1) we may define in the space (E,l), similarly as in a metric space, the following notions: the sphere $S_l(p,r)$ and the ball $K_l(p,r)$ with the centre at the point p and the radius r.

Let $S_l(p,r)_u$ denote here the so-called u-neighbourhood of the sphere $S_l(p,r)$ in the space (E,l) defined by the formula:

$$S_l(p,r)_u = \begin{cases} \bigcup\limits_{q \in S_l(p,r)} K_l(q,u) & \text{for} \quad u > 0 \\ S_l(p,r) & \text{for} \quad u = 0. \end{cases} \tag{1.2}$$

Let k be any but fixed positive real number and let a, b be arbitrary non-negative real functions defined in a certain right-hand side neighbourhood of 0 such that

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$$a(r) \xrightarrow[r\to 0^+]{} 0 \quad \text{and} \quad b(r) \xrightarrow[r\to 0^+]{} 0. \tag{1.3}$$

We say that the pair (A,B) of the sets $A,B \in E_0$ is (a,b)-clustered at the point p of the space (E,l), if 0 is the cluster point of the set of all real numbers $r>0$ such that the sets $A\cap S_l(p,r)_{a(r)}$ and $B\cap S_l(p,r)_{b(r)}$ are non-empty.

Let (see [9])

$$\begin{aligned} T_l(a,b,k,p) = \{(A,B):\ & A,B\in E_0,\ \text{the pair } (A,B) \text{ is } (a,b)\text{-clustered} \\ & \text{at the point } p \text{ of the space } (E,l) \text{ and} \\ & \tfrac{1}{r^k} l(A\cap S_l(p,r)_{a(r)}, B\cap S_l(p,r)_{b(r)}) \xrightarrow[r\to 0^+]{} 0\}. \end{aligned} \tag{1.4}$$

If $(A,B)\in T_l(a,b,k,p)$, then we say that the set $A\in E_0$ is (a,b)-tangent of order k to the set $B\in E_0$ at the point p of the space (E,l).

We call $T_l(a,b,k,p)$ defined by (1.4) the (a,b)-tangency relation of order k at the point $p\in E$ or briefly: the tangency relation of sets in the generalized metric space (E,l).

We say that the set $A\in E_0$ has the Darboux property at the point p of the generalized metric space (E,l), what we write: $A\in D_p(E,l)$ (see [3]), if there exists a number $\tau>0$ such that the set $A\cap S_l(p,r)\neq\varnothing$ for $r\in(0,\tau)$.

Let ρ be an arbitrary metric of the set E. We shall denote by $d_\rho A$ the diameter of the set $A\in E_0$, and by $\rho(A,B)$ the distance of sets $A,B\in E_0$ in the metric space (E,ρ).

Let f be any subadditive increasing real function defined in a certain right-hand side neighbourhood of 0, such that $f(0)=0$. By $\mathcal{F}_{f,\rho}$ we denote the class of all functions l fulfilling the conditions:

1^0 $\quad l: E_0\times E_0 \longrightarrow \langle 0,\infty)$,

2^0 $\quad f(\rho(A,B)) \le l(A,B) \le f(d_\rho(A\cup B))$ for $A,B\in E_0$.

Because

$$f(\rho(x,y)) = f(\rho(\{x\},\{y\})) \le l(\{x\},\{y\}) \le f(d_\rho(\{x\}\cup\{y\})) = f(\rho(x,y)),$$

then from here and from (1.1) it follows that

$$l_0(x,y) = l(\{x\},\{y\}) = f(\rho(x,y)) \quad \text{for} \quad l\in\mathcal{F}_{f,\rho} \quad \text{and} \quad x,y\in E. \tag{1.5}$$

It is easy to check that the function l_0 defined by (1.5) is the metric of the set E.

We say that the tangency relation $T_l(a,b,k,p)$ defined by (1.4) is additive in the class of functions $\mathcal{F}_{f,\rho}$, if

$$(A,B)\in T_{l_1+l_2}(a,b,k,p) \;\Leftrightarrow\; (A,B)\in (T_{l_1}(a,b,k,p)\cup T_{l_2}(a,b,k,p)) \tag{1.6}$$

for $A,B\in E_0$ and $l_1,l_2\in\mathcal{F}_{f,\rho}$.

In the paper [8] there were considered the problem of the additivity of the tangency relation $T_l(a,b,k,p)$ in the classes of sets $A^*_{p,k}$ having the Darboux property at the point p of the generalized metric space (E,l), where $l\in\mathcal{F}_{f,\rho}$.

If in Corollary 1 of Theorem 1 of the paper [8] we assume that the functions $l_1,l_2,...,l_m\in\mathcal{F}_{f,\rho}$ are equal to the function $l\in\mathcal{F}_{f,\rho}$, then

(1.7) $$(A,B) \in T_{ml}(a,b,k,p) \quad \text{if and only if} \quad (A,B) \in T_l(a,b,k,p)$$

for $A, B \in A^*_{p,k} \cap D_p(E,l)$, $m \in N$, and for the functions a, b fulfilling the condition

$$\frac{a(r)}{r^k} \xrightarrow[r\to 0^+]{} 0 \quad \text{and} \quad \frac{b(r)}{r^k} \xrightarrow[r\to 0^+]{} 0.$$

In connection with the above, arises the question: is the equivalence (1.7) true for an arbitrary $m \in R_+$? The answer to this question is positive, what will be proved in the present paper.

The tangency relation $T_l(a,b,k,p)$ we shall call homogeneous of order 0 in the class of the functions $\mathcal{F}_{f,\rho}$, if $(A,B) \in T_{ml}(a,b,k,p)$ if and only if $(A,B) \in T_l(a,b,k,p)$ for $m > 0$, $\mathcal{F}_{f,\rho}$ and $A, B \in E_0$.

In this paper the problem of the homogeneity of the tangency relation $T_l(a,b,k,p)$ in the class of the functions $\mathcal{F}_{f,\rho}$ for sets of the classes $A^*_{p,k}$ having the Darboux property in the generalized metric space (E,l) is considered. Some sufficient conditions for the homogeneity of order 0 of this tangency relation of sets of the classes $A^*_{p,k}$ will be given in Section 2 of this paper.

2 The homogeneity of the tangency relation of sets of the classes $A^*_{p,k}$

Let ρ be a metric of the set E and let A be an arbitrary set of the family E_0. Let A' denote the set of all cluster points of the set $A \in E_0$ and

(2.1) $$\rho(x,A) = \inf\{\rho(x,y) : \ y \in A\} \quad \text{for} \quad x \in E.$$

Let us put (see [3])

(2.2) $$A^*_{p,k} = \{A \in E_0 : \ p \in A' \ \text{ and there exists a number } \lambda > 0 \text{ such that}$$
$$\limsup_{[A,p;k]\ni(x,y)\to(p,p)} \frac{\rho(x,y) - \lambda\, \rho(x,A)}{\rho^k(p,x)} \le 0\},$$

where

(2.3) $$[A,p;k] = \{(x,y) : \ x \in E, \ y \in A \ \text{ and } \ \rho(x,A) < \rho^k(p,x) = \rho^k(p,y)\}.$$

Lemma 21.. *If the non-decreasing function a fulfils the condition*

(2.4) $$\frac{a(r)}{r^k} \xrightarrow[r\to 0^+]{} 0,$$

then for an arbitrary set $A \in A^*_{p,k}$ having the Darboux property at the point p of the metric space (E,ρ) and $m > 0$

(2.5) $$\frac{1}{r^k} d_\rho(A \cap S_\rho(p, r/m)_{a(r)/m}) \xrightarrow[r\to 0^+]{} 0.$$

Proof. In the proof of this lemma we shall consider two cases:

(i) $0 < m < 1$,

(ii) $m \geq 1$.

Let us suppose that $0 < m < 1$. From here, from the assumption (2.4) and from Lemma 1 of the paper [3]

$$\frac{1}{r^k} d_\rho(A \cap S_\rho(p,r)_{a(r)/m}) \xrightarrow[r\to 0^+]{} 0,$$

whence it follows that

$$\frac{1}{r^k} d_\rho(A \cap S_\rho(p,r/m)_{a(r/m)/m}) \xrightarrow[r\to 0^+]{} 0. \tag{2.6}$$

From the fact that a is the non-decreasing function and from the condition (i) it follows that $a(r) \leq a(r/m)$ for $r > 0$. Hence and from the definition of the set $S_l(p,r)_u$ we get the inequality

$$0 \leq d_\rho(A \cap S_\rho(p,r/m)_{a(r)/m}) \leq d_\rho(A \cap S_\rho(p,r/m)_{a(r/m)/m}).$$

From here and from (2.6) it follows the condition (2.5) of this lemma for $m \in (0,1)$.

Now we assume that $m \geq 1$. From (2.4) it follows that

$$\frac{a(mt)}{t^k} \xrightarrow[t\to 0^+]{} 0.$$

Hence and from Lemma 1 of the paper [3] we obtain

$$\frac{1}{t^k} d_\rho(A \cap S_\rho(p,t)_{a(mt)}) \xrightarrow[t\to 0^+]{} 0. \tag{2.7}$$

Setting $r = mt$, from (2.7) we get

$$\frac{1}{r^k} d_\rho(A \cap S_\rho(p,r/m)_{a(r)}) \xrightarrow[r\to 0^+]{} 0. \tag{2.8}$$

Because $A \cap S_\rho(p,r/m)_{a(r)/m} \subseteq A \cap S_\rho(p,r/m)_{a(r)}$ for $m \geq 1$, then

$$0 \leq d_\rho(A \cap S_\rho(p,r/m)_{a(r)/m}) \leq d_\rho(A \cap S_\rho(p,r/m)_{a(r)}).$$

From here and from (2.8) we get the condition (2.5) of this lemma for $m \in \langle 1, \infty)$. Therefore, the thesis of Lemma 2.1 is true for an arbitrary $m > 0$.

Because every function $l \in \mathcal{F}_{f,\rho}$ generates on the set $A \in E_0$ the metric (see (1.5)), then from Lemma 2.1 it follows that

$$\frac{1}{r^k} d_l(A \cap S_l(p,r/m)_{a(r)/m}) \xrightarrow[r\to 0^+]{} 0, \tag{2.9}$$

if $l \in \mathcal{F}_{f,\rho}$, $A \in A^*_{p,k} \cap D_p(E,l)$, and the function a fulfils the condition (2.4).

Let us put by the definition:

$$(ml)(A,B) = ml(A,B) \quad \text{for} \quad m > 0,\ l \in \mathcal{F}_{f,\rho} \text{ and } A, B \in E_0. \tag{2.10}$$

Lemma 22.. *If* $l \in \mathcal{F}_{f,\rho}$, *then*

$$S_{ml}(p,r)_u = S_l(p,r/m)_{u/m} \quad \textit{for} \quad m > 0. \tag{2.11}$$

`Proof.` Using (2.10) we have

$$\begin{gathered} S_{ml}(p,r) = \{x \in E : \ (ml)(\{p\},\{x\}) = r\} \\ = \{x \in E : \ ml(\{p\},\{x\}) = r\} = \{x \in E : \ l(\{p\},\{x\}) = r/m\} \\ = S_l(p,r/m), \end{gathered}$$

i.e.

$$S_{ml}(p,r) = S_l(p,r/m) \quad \text{for} \quad l \in \mathcal{F}_{f,\rho} \quad \text{and} \quad m > 0. \tag{2.12}$$

Analogously

$$K_{ml}(p,r) = K_l(p,r/m) \quad \text{for} \quad l \in \mathcal{F}_{f,\rho} \quad \text{and} \quad m > 0. \tag{2.13}$$

From (2.12), (2.13) and from the definition (1.2) of the set $S_l(p,r)_u$ we get the thesis of this lemma.

Theorem 21.. *If the non-decreasing functions* a, b *fulfil the condition*

$$\frac{a(r)}{r^k} \xrightarrow[r\to 0^+]{} 0 \quad \textit{and} \quad \frac{b(r)}{r^k} \xrightarrow[r\to 0^+]{} 0, \tag{2.14}$$

then the tangency relation $T_l(a,b,k,p)$ *is homogeneous of order* 0 *in the class of the functions* $\mathcal{F}_{f,\rho}$ *for the sets of the classes* $A^*_{p,k} \cap D_p(E,l)$.

`Proof.` Let us assume that $(A,B) \in T_{ml}(a,b,k,p)$ for $A, B \in A^*_{p,k} \cap D_p(E,l)$. From here it follows

$$\frac{1}{r^k}(ml)(A \cap S_{ml}(p,r)_{a(r)}, B \cap S_{ml}(p,r)_{b(r)}) \xrightarrow[r\to 0^+]{} 0.$$

Hence, from (2.10) and from Lemma 2.2 we obtain

$$\frac{1}{r^k} l(A \cap S_l(p,r/m)_{a(r)/m}, B \cap S_l(p,r/m)_{b(r)/m}) \xrightarrow[r\to 0^+]{} 0. \tag{2.15}$$

From (2.15) and from the fact that $l \in \mathcal{F}_{f,\rho}$ it results

$$\frac{1}{r^k} f(\rho(A \cap S_l(p,r/m)_{a(r)/m}, B \cap S_l(p,r/m)_{b(r)/m})) \xrightarrow[r\to 0^+]{} 0.$$

Hence and from Theorem 2.2 of the paper [7] on the compatibility of the tangency relations of sets of the classes $A^*_{p,k} \cap D_p(E,l)$ we get

$$\frac{1}{r^k} f(\rho(A \cap S_l(p,r/m)_{a(r)}, B \cap S_l(p,r/m)_{b(r)})) \xrightarrow[r\to 0^+]{} 0. \tag{2.16}$$

If $0 < m < 1$, then from the definition of the set $S_l(p,r)_u$ and from the assumption that a and b are non-decreasing functions it follows the inequality

$$0 \le \rho(A \cap S_l(p, r/m)_{a(r/m)}, B \cap S_l(p, r/m)_{b(r/m)}) \\ \le \rho(A \cap S_l(p, r/m)_{a(r)}, B \cap S_l(p, r/m)_{b(r)}).$$

Hence, from (2.16) and from the properties of the function that $l \in \mathcal{F}_{f,\rho}$ we obtain

$$\frac{1}{r^k} f(\rho(A \cap S_l(p, r/m)_{a(r/m)}, B \cap S_l(p, r/m)_{b(r/m)})) \xrightarrow[r \to 0^+]{} 0.$$

From here and from Theorem 2.1 of the paper [7] on the compatibility of the tangency relations of sets of the classes $A^*_{p,k} \cap D_p(E, l)$ we have

$$\frac{1}{r^k} f(d_\rho((A \cap S_l(p, r/m)_{a(r/m)}) \cup (B \cap S_l(p, r/m)_{b(r/m)}))) \xrightarrow[r \to 0^+]{} 0.$$

Hence and from the fact that $l \in \mathcal{F}_{f,\rho}$

$$\frac{1}{r^k} l(A \cap S_l(p, r/m)_{a(r/m)}, B \cap S_l(p, r/m)_{b(r/m)}) \xrightarrow[r \to 0^+]{} 0,$$

whence it follows

$$\frac{1}{t^k} l(A \cap S_l(p, t)_{a(t)}, B \cap S_l(p, t)_{b(t)}) \xrightarrow[t \to 0^+]{} 0. \tag{2.17}$$

From (2.16) and from Theorem 2.1 of the paper [7] it results

$$\frac{1}{r^k} f(d_\rho((A \cap S_l(p, r/m)_{a(r)}) \cup (B \cap S_l(p, r/m)_{b(r)}))) \xrightarrow[r \to 0^+]{} 0. \tag{2.18}$$

If $m \ge 1$, then from (2.18) and from the assumption on the functions a, b we get

$$\frac{1}{r^k} f(d_\rho((A \cap S_l(p, r/m)_{a(r/m)}) \cup (B \cap S_l(p, r/m)_{b(r/m)}))) \xrightarrow[r \to 0^+]{} 0. \tag{2.19}$$

Hence and from the fact that $l \in \mathcal{F}_{f,\rho}$ it follows

$$\frac{1}{r^k} l(A \cap S_l(p, r/m)_{a(r/m)}, B \cap S_l(p, r/m)_{b(r/m)}) \xrightarrow[r \to 0^+]{} 0,$$

which yields the condition (2.17).

From the fact that $A, B \in D_p(E, l)$ for that $l \in \mathcal{F}_{f,\rho}$ it follows that there exists a real number $\tau > 0$ such that the sets $A \cap S_l(p, r)$ and $B \cap S_l(p, r)$ are non-empty for $r \in (0, \tau)$. This denotes that the pair of sets (A, B) is (a, b)-clustered at the point p of the space (E, l). Hence and from (2.17) it follows that $(A, B) \in T_l(a, b, k, p)$ for $A, B \in A^*_{p,k} \cap D_p(E, l)$.

Now we assume that $(A, B) \in T_l(a, b, k, p)$ for $A, B \in A^*_{p,k} \cap D_p(E, l)$. From here it follows that

$$\frac{1}{r^k} l(A \cap S_l(p, r/m)_{a(r/m)}, B \cap S_l(p, r/m)_{b(r/m)}) \xrightarrow[r \to 0^+]{} 0.$$

Hence and from the fact that $l \in \mathcal{F}_{f,\rho}$ we obtain

$$\frac{1}{r^k} f(\rho(A \cap S_l(p, r/m)_{a(r/m)}, B \cap S_l(p, r/m)_{b(r/m)})) \xrightarrow[r\to 0^+]{} 0. \tag{2.20}$$

From here and from Theorem 2.1 of the paper [7] we have

$$\frac{1}{r^k} f(d_\rho((A \cap S_l(p, r/m)_{a(r/m)}) \cup (B \cap S_l(p, r/m)_{b(r/m)}))) \xrightarrow[r\to 0^+]{} 0. \tag{2.21}$$

If $0 < m < 1$, then from the fact that a, b are non-decreasing functions it follows

$$\begin{aligned} 0 &\le d_\rho((A \cap S_l(p, r/m)_{a(r)}) \cup (B \cap S_l(p, r/m)_{b(r)})) \\ &\le d_\rho((A \cap S_l(p, r/m)_{a(r/m)}) \cup (B \cap S_l(p, r/m)_{b(r/m)})). \end{aligned}$$

From here and from (2.21) we get

$$\frac{1}{r^k} f(d_\rho((A \cap S_l(p, r/m)_{a(r)}) \cup (B \cap S_l(p, r/m)_{b(r)}))) \xrightarrow[r\to 0^+]{} 0. \tag{2.22}$$

Hence and from Theorem 2.2 of the paper [7] we have

$$\frac{1}{r^k} f(d_\rho((A \cap S_l(p, r/m)_{a(r)/m}) \cup (B \cap S_l(p, r/m)_{b(r)/m}))) \xrightarrow[r\to 0^+]{} 0,$$

whence it follows

$$\frac{1}{r^k} l(A \cap S_l(p, r/m)_{a(r)/m}, B \cap S_l(p, r/m)_{b(r)/m}) \xrightarrow[r\to 0^+]{} 0,$$

i.e.

$$\frac{1}{r^k} (ml)(A \cap S_{ml}(p, r)_{a(r)}, B \cap S_{ml}(p, r)_{b(r)}) \xrightarrow[r\to 0^+]{} 0. \tag{2.23}$$

If $m \ge 1$, then from the fact that a, b are the non-decreasing functions we get the inequality

$$\begin{aligned} 0 &\le \rho(A \cap S_l(p, r/m)_{a(r)}, B \cap S_l(p, r/m)_{b(r)}) \\ &\le \rho(A \cap S_l(p, r/m)_{a(r/m)}, B \cap S_l(p, r/m)_{b(r/m)}). \end{aligned}$$

Hence and from (2.20) we have

$$\frac{1}{r^k} f(\rho(A \cap S_l(p, r/m)_{a(r)}, B \cap S_l(p, r/m)_{b(r)})) \xrightarrow[r\to 0^+]{} 0. \tag{2.24}$$

From (2.24) and from Theorems 2.1 and 2.2 (see also Corollary 2.1) of the paper [7] we obtain

$$\frac{1}{r^k} f(d_\rho((A \cap S_l(p, r/m)_{a(r)/m}) \cup (B \cap S_l(p, r/m)_{b(r)/m}))) \xrightarrow[r\to 0^+]{} 0.$$

From here and from the fact that $l \in \mathcal{F}_{f,\rho}$ we get

$$\frac{1}{r^k} l(A \cap S_l(p, r/m)_{a(r)/m}), (B \cap S_l(p, r/m)_{b(r)/m}) \xrightarrow[r\to 0^+]{} 0,$$

whence it follows the condition (2.23).

From the assumption $A, B \in D_p(E, l)$ for $l \in \mathcal{F}_{f,\rho}$ it follows that there exists a real number $\tau > 0$ such that

$$A \cap S_l(p,r)_{a(r)} \neq \varnothing \quad \text{and} \quad B \cap S_l(p,r)_{b(r)} \neq \varnothing \quad \text{for} \quad r \in (0,\tau). \tag{2.25}$$

If we set $\tau' = m\tau$, then $r/m \in (0,\tau)$ when $r \in (0,\tau')$. Hence, from (2.25) and from the equality $S_{ml}(p,r) = S_l(p,r/m)$ for $m > 0$ and $l \in \mathcal{F}_{f,\rho}$ it follows that the sets $A \cap S_{ml}(p,r)$, $B \cap S_{ml}(p,r)$ are non-empty for $r \in (0,\tau)$. From here it results that $A, B \in D_p(E, ml)$, what means that the pair of sets (A, B) is (a, b)-clustered at the point p of the space (E, ml). Hence and from the condition (2.23) it follows that $(A, B) \in T_{ml}(a, b, k, p)$ for $A, B \in A^*_{p,k} \cap D_p(E, l)$. This ends the proof of the theorem.

Let $A, B \in E_0$ and $l_1, l_2, \ldots, l_n$ be arbitrary functions belonging to the class $l \in \mathcal{F}_{f,\rho}$. Let by the definition (see [8])

$$(A,B) \in \bigcup_{i=1}^{n} T_{l_i}(a,b,k,p) \Longleftrightarrow (A,B) \in T_{l_j}(a,b,k,p) \quad \text{for an} \quad j \in \{1,2,\ldots,n\}.$$

From here, from Theorem 2.1 and from Theorem 1 on the additivity of the tangency relation $T_l(a, b, k, p)$ of the paper [8] we get

Corollary 21.. *If the non-decreasing functions a, b fulfil the condition* (2.14) *and $l, l_1, l_2, \ldots, l_n \in \mathcal{F}_{f,\rho}$, then $(A, B) \in T_{m_1 l_1 + \cdots + m_n l_n}(a, b, k, p)$ if and only if $(A, B) \in T_{l_j}(a, b, k, p)$ for an $j \in \{1, 2, \ldots, n\}$, and for arbitrary $A, B \in A^*_{p,k} \cap D_p(E, l)$ and $m_1, \ldots, m_n > 0$.*

Let A_p be the class of the rectifiable arcs with the Archimedean property at the point p of the metric space (E, ρ).

We say that the rectifiable arc A has the Archimedean property at the point p of the space (E, ρ) if

$$\lim_{A \ni x \to p} \frac{\ell(\widetilde{px})}{\rho(p,x)} = 1, \tag{2.26}$$

where $\ell(\widetilde{px})$ denotes the lenght of the arc $\widetilde{px}$.

Because the class A_p is contained in the class of sets $A^*_{p,1} \cap D_p(E, l)$, then from here and from Theorem 1 of this paper follows

Corollary 22.. *If the non-decreasing functions a, b fulfil the condition*

$$\frac{a(r)}{r} \xrightarrow[r \to 0^+]{} 0 \quad \textit{and} \quad \frac{b(r)}{r} \xrightarrow[r \to 0^+]{} 0, \tag{2.27}$$

then the tangency relation $T_l(a, b, k, p)$ is homogeneous of order 0 in the class of the functions $\mathcal{F}_{f,\rho}$ for arcs of the class A_p.

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