A pseudo-Riemannian metric on the tangent bundle of a Riemannian manifold

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Abstract. On the tangent bundle of a Riemannian manifold (M, g) we consider a pseudo-Riemannian metric defined by a symmetric tensor field c on M and four real valued smooth functions defined on $[0, \infty)$. We study the conditions under which the above pseudo-Riemannian manifold has constant sectional curvature.

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1 Necessary facts about the tangent bundle TM

Let (M, g) be a smooth n-dimensional Riemannian manifold and let $\pi : TM \to M$ be its tangent bundle. Then TM has a structure of a $2n$-dimensional smooth manifold induced from the structure of smooth n-dimensional manifold of M as follows: every local chart $(U, \phi) = (U, x^1, ..., x^n)$ on M induced a local chart $(\pi^{-1}(U), \Phi) = (\pi^{-1}(U), x^1, ..., x^n, y^1, ..., y^n)$ on TM, where we made an abuse of notation,identifying x^i with $\pi^* x^i = x^i \circ \pi$ and y^i being the vector space coordinates of $y \in \pi^{-1}(U)$ with respect to the natural local frame $((\frac{\partial}{\partial x^1})_{\pi(y)}, ..., (\frac{\partial}{\partial x^n})_{\pi(y)})$ i.e. $y = y^i (\frac{\partial}{\partial x^i})_{\pi(y)}$

This special structure of TM allows us to introduce the notion of M-tensor fields on it (see [3]). An M-tensor field of type (p, q) on TM is defined by sets of n^{p+q} functions depending on x^i and y^i, assigned to induced local charts $(\pi^{-1}(U), \Phi)$ on TM, thus the change rule is that of the components of a tensor field of type (p, q) on M, when a change of local charts on the base manifold is performed. Remark that the components y^i define an M-tensor field of type $(1, 0)$ on TM. It is also obvious that a usual tensor field of type (p, q) on M may be thought as an M-tensor field of type (p, q) on TM. In the case of a covariant tensor field, the corresponding M-tensor field on the tangent bundle TM is nothing else but the pullback of the initial tensor field by the submersion $\pi : TM \to M$. Other useful M-tensor fields on TM may be obtained as follows. Let $a : [0, \infty) \to R$ be a smooth function and let $\| y \|^2 = g_{\pi(y)}(y, y)$ be the square of the norm of the tangent vector y. Then the components $a(\| y \|^2)\delta^i_j$ define a M-tensor field of type $(1, 1)$ on TM. Similarly, if $g_{ij}(x)$ are the local coordinate

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components of the metric tensor field g on M, then the components $a(\| y \|^2)g_{ij}$ define a symmetric M-tensor field of type $(0,2)$ on TM. The components $g_{0i} = y^j g_{ji}$ define an M-tensor field of type $(0,1)$ on TM.

Recall that the Levi-Civita connection $\dot{\nabla}$ of the Riemannian metric g defines the direct sum decomposition

$$TTM = VTM \oplus HTM$$

of the tangent bundle to TM into the vertical distribution $VTM = ker\ \pi_*$ and the horizontal distribution HTM. The vector fields $(\frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n})$ define a local frame field for VTM and for the horizontal distribution HTM we have the local frame field $(\frac{\delta}{\delta x^1}, ..., \frac{\delta}{\delta x^n})$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{i0}\frac{\partial}{\partial y^h}; \Gamma^h_{i0} = \Gamma^h_{ik}y^k$$

and Γ^h_{ij} are the Christoffel symbols defined by the Riemannian metric g. In [5] the author proves the following

Lemma 1. *If $n > 1$ and u, v are smooth function on TM such that*

$$ug_{ij} + vg_{0i}g_{0j} = 0, g_{0i} = y^j g_{ji}, \quad y \in \pi^{-1}(U)$$

on the domain of any induced local chart on TM, then $u = v = 0$.

In a similar way we can obtain

Lemma 2. *If $n > 1$ and u, v are smooth function on TM such that*

$$ug_{jk}\delta^h_i - ug_{ij}\delta^h_k + vg_{0i}g_{0j}\delta^h_k - vg_{0j}g_{0k}\delta^h_i = 0, g_{0i} = y^j g_{ji}, \quad y \in \pi^{-1}(U)$$

on the domain of any induced local chart on TM, then $u = v = 0$.

Remark. From the relation

$$ug_{jk}y_i y^h - ug_{ik}y_j y^h = 0, \quad y \in \pi^{-1}(U),$$

we obtain $u = 0$.

Since we work in a fixed local chart (U, ϕ) on M and in the corresponding induced local chart $(\pi^{-1}(U), \Phi)$ on TM, we shall use the following simpler notations

$$\frac{\partial}{\partial y^i} = \partial_i, \quad \frac{\delta}{\delta x^i} = \delta_i$$

We also denote by

$$t = \frac{1}{2} \| y \|^2 = \frac{1}{2} g_{\pi(y)}(y, y) = \frac{1}{2} g_{ij}(x) y^i y^j, y \in \pi^{-1}(U).$$

2 A pseudo-Riemannian metric on TM

Let c be a symmetric tensor field of type $(0,2)$ on M, and let $a_1, b_1, a_2, b_2 : [0,\infty) \to R$ be smooth functions. Consider the following symmetric tensor field of type $(0,2)$ on TM (see [6],[7],[4])

$$\begin{cases} G_y(X^V, Y^V) = 0, \\ G_y(X^H, Y^V) = a_1(t) g_{\pi(y)}(X,Y) + b_1(t) g_{\pi(y)}(y,X) g_{\pi(y)}(y,Y), \\ G_y(X^H, Y^H) = a_2(t) c_{\pi(y)}(X,Y) + b_2(t) g_{\pi(y)}(y,X) g_{\pi(y)}(y,Y). \end{cases} \tag{2.1}$$

The expression of G in local adapted frames is defined by the following M-tensor fields

$$G^1_{ij} = G(\delta_i, \partial_j) = a_1 g_{ij} + b_1 g_{0i} g_{0j},$$
$$G^2_{ij} = G(\delta_i, \delta_j) = a_2 c_{ij} + b_2 g_{0i} g_{0j}.$$

The associated matrix of G with respect to the adapted local frame is

$$\begin{pmatrix} 0 & G^1_{ij} \\ G^1_{ij} & G^2_{ij} \end{pmatrix}$$

The conditions for G to be nondegenerate are ensured if

$$a_1(a_1 + 2tb_1) \neq 0.$$

Under these conditions the matrix (G^1_{ij}) has the inverse with the entries

$$H_1^{ij} = \frac{1}{a_1} g^{ij} + \frac{b_1}{a_1 + 2tb_1} y^i y^j$$

We shall denote by

$$\partial_h G^1_{ij} = \frac{\partial G^1_{ij}}{\partial y^h}, \partial_h G^2_{ij} = \frac{\partial G^2_{ij}}{\partial y^h}, \quad \delta_h G^1_{ij} = \frac{\delta G^1_{ij}}{\delta x^h}, \delta_h G^2_{ij} = \frac{\delta G^2_{ij}}{\delta x^h}.$$

The following formulae can be easily checked and will be useful in our next computation:

$$\begin{cases} \dot\nabla_i G^1_{jk} = \delta_i G^1_{jk} - \Gamma^h_{ij} G^1_{hk} - \Gamma^h_{ik} G^1_{jh} = 0, \\ \dot\nabla_i G^2_{jk} = \delta_i G^2_{jk} - \Gamma^h_{ij} G^2_{hk} - \Gamma^h_{ik} G^2_{jh} = a_2 \dot\nabla_i c_{jk}, \\ \dot\nabla_i H_1^{jk} = \delta_i H_1^{jk} + \Gamma^j_{ih} H_1^{hk} + \Gamma^k_{ih} H_1^{jh} = 0, \\ \dot\nabla_i \partial_j G^1_{kl} = \delta_i \partial_j G^1_{kl} - \Gamma^h_{ij} \partial_h G^1_{kl} - \Gamma^h_{ik} \partial_j G^1_{hl} - \Gamma^h_{il} \partial_j G^1_{kh} = 0, \\ \dot\nabla_i \partial_j G^2_{kl} = \delta_i \partial_j G^2_{kl} - \Gamma^h_{ij} \partial_h G^2_{kl} - \Gamma^h_{ik} \partial_j G^2_{hl} - \Gamma^h_{il} \partial_j G^2_{kh} = a_2' g_{0j} \dot\nabla_i c_{kl}. \end{cases} \tag{2.2}$$

Proposition 3. *The Levi-Civita connection ∇ of the pseudo-Riemannian manifold (TM, G) has the following expression in the local adapted frame $(\partial_1, ..., \partial_n, \delta_1, ..., \delta_n)$*

$$\begin{aligned} \nabla_{\partial_i}\partial_j &= Q^h_{ij}\partial_h, & \nabla_{\delta_i}\partial_j &= (\Gamma^h_{ij} + \widetilde{P}^h_{ji})\partial_h + P^h_{ji}\delta_h, \\ \nabla_{\partial_i}\delta_j &= P^h_{ij}\delta_h + \widetilde{P}^h_{ij}\partial_h, & \nabla_{\delta_i}\delta_j &= (\Gamma^h_{ij} + \widetilde{S}^h_{ij})\delta_h + S^h_{ij}\partial_h, \end{aligned}$$

where the M-tensor fields Q^h_{ij}, P^h_{ij}, $\widetilde{P}^h_{ij}$, S^h_{ij}, $\widetilde{S}^h_{ij}$ are given by:

$$Q^h_{ij} = \frac{1}{2}H_1^{hk}(\partial_i G^1_{jk} + \partial_j G^1_{ik}),$$

$$P^h_{ij} = \frac{1}{2}H_1^{hk}(\partial_i G^1_{jk} - \partial_k G^1_{ij}),$$

$$\widetilde{P}^h_{ij} = \frac{1}{2}H_1^{hk}\partial_i G^2_{jk} - \frac{1}{2}H_1^{rl}(\partial_i G^1_{jl} - \partial_l G^1_{ij})G^2_{rk}H_1^{kh},$$

$$S^h_{ij} = \frac{a_2}{2}(\dot{\nabla}_i c_{jk} + \dot{\nabla}_j c_{ki} - \dot{\nabla}_k c_{ij})H_1^{kh} -$$

$$-a_1 R_{0ijk}H_1^{kh} + \frac{1}{2}H_1^{sl}(\partial_l G^2_{ij})G^2_{sk}H_1^{kh},$$

$$\widetilde{S}^h_{ij} = -\frac{1}{2}H_1^{hk}\partial_k G^2_{ij},$$

R_{lijk} denoting the local coordinate components of the Riemann-Christoffel tensor of the Levi-Civita connection $\dot{\nabla}$ on M and $R_{0ijk} = y^l R_{lijk}$

Remark. *Replacing the expressions of G^1_{ij}, G^2_{ij}, H_1^{ij}, $\partial_i G^1_{jk}$, $\partial_i G^2_{jk}$ by their local coordinate components we obtain some quite complicated expressions.*

The curvature tensor field K of the connection ∇ is defined by the well-known formula

$$K(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z, \quad X, Y, Z \in \Gamma(TM)$$

Proposition 4. *The local coordinate expression of the curvature tensor field in the adapted local frame $(\partial_1, ..., \partial_n, \delta_1, ..., \delta_n)$ is given by*

$$\begin{aligned} K(\partial_i, \partial_j)\partial_k &= YYYY^h_{kij}\partial_h, \\ K(\partial_i, \partial_j)\delta_k &= YYXY^h_{kij}\partial_h + YYXX^h_{kij}\delta_h, \\ K(\partial_i, \delta_j)\partial_k &= YXYY^h_{kij}\partial_h + YXYX^h_{kij}\delta_h, \\ K(\partial_i, \delta_j)\delta_k &= YXXY^h_{kij}\partial_h + YXXX^h_{kij}\delta_h, \\ K(\delta_i, \delta_j)\partial_k &= XXYY^h_{kij}\partial_h + XXYX^h_{kij}\delta_h, \\ K(\delta_i, \delta_j)\delta_k &= XXXY^h_{kij}\partial_h + XXXX^h_{kij}\delta_h, \end{aligned}$$

where we have denoted

$$\begin{aligned}
YYYY^h_{kij} &= \partial_i Q^h_{jk} + Q^l_{jk} Q^h_{il} - \partial_j Q^h_{ik} - Q^l_{ik} Q^h_{jl}\\
YYXY^h_{kij} &= \partial_i \widetilde{P}^h_{jk} + \widetilde{P}^h_{il} P^l_{jk} + \widetilde{P}^l_{jk} Q^h_{il} - \partial_j \widetilde{P}^h_{ik} - \widetilde{P}^h_{jl} P^l_{ik} - \widetilde{P}^l_{ik} Q^h_{jl}\\
YYXX^h_{kij} &= \partial_i P^h_{jk} + P^l_{jk} P^h_{il} - \partial_j P^h_{ik} - P^l_{ik} P^h_{jl}\\
YXYY^h_{kij} &= \partial_i \widetilde{P}^h_{kj} + \widetilde{P}^l_{kj} Q^h_{il} + \widetilde{P}^h_{il} P^l_{kj} - \widetilde{P}^h_{lj} Q^l_{ik}\\
YXYX^h_{kij} &= \partial_i P^h_{kj} + P^l_{kj} P^h_{il} - P^h_{lj} Q^l_{ik}\\
YXXY^h_{kij} &= \partial_i S^h_{jk} + S^l_{jk} Q^h_{il} + \widetilde{S}^l_{jk} \widetilde{P}^h_{il} - S^h_{jl} P^l_{ik} - \widetilde{P}^l_{ik} \widetilde{P}^h_{lj} - \dot{\nabla}_j \widetilde{P}^h_{ik}\\
YXXX^h_{kij} &= \partial_i \widetilde{S}^h_{jk} + \widetilde{S}^l_{jk} P^h_{il} - \widetilde{S}^h_{jl} P^l_{ik} - \widetilde{P}^l_{ik} P^h_{lj}\\
XXYY^h_{kij} &= \dot{\nabla}_i \widetilde{P}^h_{kj} + \widetilde{P}^l_{kj} \widetilde{P}^h_{li} + P^l_{kj} S^h_{il} - \dot{\nabla}_j \widetilde{P}^h_{ki} - \widetilde{P}^l_{ki} \widetilde{P}^h_{lj} - P^l_{ki} S^h_{jl} +\\
&\quad + R^h_{kij} + R^l_{0ij} Q^h_{lk}\\
XXYX^h_{kij} &= \widetilde{P}^l_{kj} P^h_{li} + P^l_{kj} \widetilde{S}^h_{il} - \widetilde{P}^l_{ki} P^h_{lj} - P^l_{ki} \widetilde{S}^h_{jl}\\
XXXY^h_{kij} &= \dot{\nabla}_i S^h_{jk} + S^h_{il} \widetilde{S}^l_{jk} + S^l_{jk} \widetilde{P}^h_{li} - \dot{\nabla}_j S^h_{ik} - S^h_{jl} \widetilde{S}^l_{ik} - S^l_{ik} \widetilde{P}^h_{lj} + R^l_{0ij} \widetilde{P}^h_{lk}\\
XXXX^h_{kij} &= \dot{\nabla}_i \widetilde{S}^h_{jk} + \widetilde{S}^l_{jk} \widetilde{S}^h_{il} + S^l_{jk} P^h_{li} - \dot{\nabla}_j \widetilde{S}^h_{ik} - \widetilde{S}^l_{ik} \widetilde{S}^h_{jl} - S^l_{ik} P^h_{lj} +\\
&\quad + R^h_{kij} + R^l_{0ij} P^h_{lk}
\end{aligned}$$

Remark. *Note that, as a first step, the formulae for the local expression of K also contain some other terms involving the Christoffel symbols Γ^h_{ij}. However, all of these terms are involved in the derivative $\dot{\nabla}$. For example*

$$\dot{\nabla}_i \widetilde{P}^h_{jk} = \delta_i \widetilde{P}^h_{jk} - \Gamma^l_{ij} \widetilde{P}^h_{lk} - \Gamma^l_{ik} \widetilde{P}^h_{jl} + \Gamma^h_{il} \widetilde{P}^l_{jk},$$

but using the expression of $\widetilde{P}^h_{ij}$ and taking account of relations (2.2) we obtain after a straightforward computation that

$$\dot{\nabla}_i \widetilde{P}^h_{jk} = \frac{a_2^{'}}{2} H_1^{hl} g_{0j} \dot{\nabla}_i c_{kl} - \frac{a_2}{2} H_1^{sl} (\partial_j G^1_{kl} - \partial_l G^1_{jk})(\dot{\nabla}_i c_{sr}) H_1^{rh}$$

Remark also that the terms $\dot{\nabla}_i Q^h_{jk}$ and $\dot{\nabla}_i P^h_{jk}$ do not appear because they are zero as follows from the formulae (2.2).

Now, we have to replace the expression of the M tensor fields Q^h_{ij}, P^h_{ij}, $\widetilde{P}^h_{ij}$, S^h_{ij}, $\widetilde{S}^h_{ij}$ in order to obtain the explicit expression of the components of K. However, the final expressions are quite complicated, but they may be obtained after some long and hard computation made by using the Mathematica package RICCI.

Recall that the pseudo-Riemannian manifold (TM, G) has constant sectional curvature k if its curvature tensor field K is given by

$$K(X,Y)Z = K_0(X,Y)Z = k(G(Y,Z)X - G(X,Z)Y), \forall X, Y, Z \in \Gamma(TM).$$

In order to find under which conditions (TM, G) has constant sectional curvature we shall consider the differences between the components of the tensor fields K and K_0 and we shall denote them by $Diff$. For example

$$Diff\ YYYY^h_{kij} = YYYY^h_{kij} - YYYY^h_{0kij}.$$

The explicit expression of $Diff\ YYYY^h_{kij}$ is

$$Diff\ YYYY^h_{kij} = \frac{a_1^{'} - b_1}{2(a_1 + 2tb_1)} (g_{jk} \delta^h_i - g_{ij} \delta^h_k) +$$

$$+\frac{1}{4a_1^2(a_1+2tb_1)}(3a_1a_1^{'2}-2a_1^2a_1^{''}-3a_1b_1^2+2a_1^2b_1^{'}+2a_1^{'2}b_1t-4a_1a_1^{''}b_1t-$$

$$-2b_1^3t+4a_1a_1^{'}b_1^{'}t)(\delta_k^h g_{0i}g_{0j}-\delta_i^h g_{0j}g_{0k}).$$

From Lemma 2 it follows that $Diff\ YYYY_{kij}^h=0$ if and only if $b_1=a_1^{'}$. By replacing $b_1=a_1^{'}$ in the expression of $Diff\ YYXY_{kij}^h$ we obtain

$$Diff\ YYXY_{kij}^h=-ka_1g_{jk}\delta_i^h+ka_1g_{ik}\delta_j^h+ka_1^{'}\delta_j^h g_{0i}g_{0k}-ka_1^{'}\delta_i^h g_{0j}g_{0k}.$$

Using again Lemma 2 and taking account that $a_1\neq 0$ it follows that $Diff\ YYXY_{kij}^h=0$ if and only if $k=0$. Under the conditions $b_1=a_1^{'}$ and $k=0$ we have

$$Diff\ YYXX_{kij}^h=Diff\ YXYX_{kij}^h=Diff\ XXYX_{kij}^h=0.$$

Computing $Diff\ XXXX_{kij}^h$ and taking $y=0$ it follows that $R=0$, so (M,g) is flat. Taking $y=0$ in the formulae $Diff\ YXXX_{kij}^h=0$ we obtain

$$\begin{cases} na_2^{'}(0)c_{jk}=-2b_2(0)g_{jk} \\ a_2^{'}(0)c_{jk}=-(n+1)b_2(0)g_{jk} \end{cases}$$

from which we have

$$(n^2+n-2)b_2(0)g_{jk}=0$$

Assuming that $n>1$ it follows that $b_2(0)=0$, so $a_2^{'}(0)c_{jk}=0$.

Now we may consider the following cases:

(i) $a_2^{'}=0$, $b_2=0$ so the pseudo-Riemannian metric G is given by

$$\begin{cases} G(\partial_i,\partial_j)=0, \\ G(\delta_i,\partial_j)=a_1g_{ij}+a_1^{'}g_{0i}g_{0j}, \\ G(\delta_i,\delta_j)=a_2c_{ij}, \end{cases} \tag{2.3}$$

where $a_1:[0,\infty)\to R$ is a smooth function and a_2 is a nonzero constant. Computing the remaining differences we have

$$Diff\ YXYY_{kij}^h=Diff\ YXXY_{kij}^h=Diff\ YXXX_{kij}^h=$$

$$=Diff\ XXYY_{kij}^h=Diff\ XXXX_{kij}^h=0,$$

and

$$Diff\ XXXY_{kij}^h=\frac{a_2}{2a_1}(\dot\nabla_i\dot\nabla_k c_j^h-\dot\nabla_j\dot\nabla_k c_i^h+\dot\nabla_j\dot\nabla^h c_{ik}-\dot\nabla_i\dot\nabla^h c_{jk})+$$

$$+\frac{a_1^{'}a_2}{2a_1(a_1+2ta_1^{'})}(\dot\nabla_i\dot\nabla_l c_{jk}-\dot\nabla_i\dot\nabla_k c_{lj}+\dot\nabla_j\dot\nabla_k c_{li}-\dot\nabla_j\dot\nabla_l c_{ik})y^hy^l.$$

Taking $y=0$ in $Diff\ XXXY_{kij}^h=0$ it follows that

$$\dot{\nabla}_i \dot{\nabla}_k c_j^h - \dot{\nabla}_j \dot{\nabla}_k c_i^h + \dot{\nabla}_j \dot{\nabla}^h c_{ik} - \dot{\nabla}_i \dot{\nabla}^h c_{jk} = 0.$$

Observing that the first bracket of the expression of $Diff\ XXXY_{kij}^h$ is zero if and only if the second bracket of it is zero we may state:

Theorem 5. *If the tangent bundle (TM, G) has constant sectional curvature, where G has the entries given by (2.3), then it must be flat. Moreover, (TM, G) is flat if and only if (M, g) is flat and the tensor field c satisfies the condition*

$$\dot{\nabla}_i \dot{\nabla}_l c_{jk} - \dot{\nabla}_i \dot{\nabla}_k c_{lj} + \dot{\nabla}_j \dot{\nabla}_k c_{li} - \dot{\nabla}_j \dot{\nabla}_l c_{ik} = 0. \tag{2.4}$$

A symmetric tensor field c of type $(0,2)$ on M is Codazzi tensor field if

$$(\dot{\nabla}_X c)(Y, Z) = (\dot{\nabla}_Y c)(X, Z),\ \ X, Y, Z \in \Gamma(M)$$

Note that the condition (2.4) is fulfilled if c is parallel with respect to ∇ or it is a Codazzi tensor field on M.

(ii) $a_2 = 0$, $b_2 = 0$ so the pseudo-Riemannian metric G is given by

$$\begin{cases} G(\partial_i, \partial_j) = 0, \\ G(\delta_i, \partial_j) = a_1 g_{ij} + a_1^{'} g_{0i} g_{0j}, \\ G(\delta_i, \delta_j) = 0, \end{cases} \tag{2.5}$$

where $a_1 : [0, \infty) \to R$ is a smooth function. In this case all the differences $Diff$ are zero, so we have the following

Theorem 6. *If the tangent bundle (TM, G) has constant sectional curvature, where G has the entries given by (2.5), then it must be flat. Moreover, (TM, G) is flat if and only if (M, g) is flat.*

(iii) $a_2 = 0$, so the pseudo-Riemannian metric G is given by

$$\begin{cases} G(\partial_i, \partial_j) = 0, \\ G(\delta_i, \partial_j) = a_1 g_{ij} + a_1^{'} g_{0i} g_{0j}, \\ G(\delta_i, \delta_j) = b_2 g_{0i} g_{0j}, \end{cases} \tag{2.6}$$

where $a_1, b_2 : [0, \infty) \to R$ are smooth functions, $b_2(0) = 0$. In this case we have

$$Diff\ XXYY_{kij}^h = u g_{jk} y_i y^h - u g_{ik} y_j y^h,$$

where $u = \frac{b_2(a_1 b_2 - 2a_1^{'} b_2 t + 2a_1 b_2^{'} t)}{4a_1(a_1 + 2a_1^{'} t)^2}$. From $Diff\ XXYY_{kij}^h = 0$ we have, using the remark made in the first section,

$$b_2(a_1b_2 - 2a_1^{'}b_2t + 2a_1b_2^{'}t) = 0.$$

Remark that $b_2 = 0$ is a solution of this equation. Next we shall prove that $b_2 = 0$ is the unique solution of this equation. First of all let us observe that

$$\left(\frac{tb_2^2}{a_1^2}\right)' = \frac{a_1b_2^2 + 2ta_1b_2b_2^{'} - 2a_1^{'}b_2^2t}{a_1^3} = 0, \forall t \geq 0$$

It follows that $tb_2^2a_1^{-2}$ is a constant function, but since $b_2(0) = 0$, we must have $tb_2^2(t)a_1^{-2}(t) = 0, \forall t \geq 0$, so $b_2(t) = 0$, for all $t \geq 0$. As a consequence of Theorem 6 we obtain:

Theorem 7. *If the tangent bundle (TM, G) has constant sectional curvature, where G has the entries given by (2.6), then it must be flat. Moreover, (TM, G) is flat if and only if (M, g) is flat and $b_2 = 0$.*

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