On kinematics and differential geometry of Euclidean submanifolds

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Abstract. In this study, we derive the equations of a motion model of two smooth homothetic along pole curves submanifolds M and N; the curves are trajectories of instantaneous rotation centers at the contact points of these submanifolds. We comment on the homothetic motions, which assume sliding and rolling.

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§1. Preliminaries

In ([7]), R.Müller has generalized 1-parameter motions in n-dimensional Euclidean space given the equation of the form $Y = AX + C$, and has investigated the so-called axoid surfaces. Further, K. Nomizu has defined ([8]) the 1-parameter motion model along the pole curves on the tangent plane of the sphere, by using parallel vector fields, and has obtained results in the particular cases of sliding and rolling. Then, H.H.Hacısalihoğlu has the investigated 1-parameter homothetic motion in the n-dimensional Euclidean space ([4]), and B. Karakaş has adapted K. Nomizu's kinematic model to homothetic motion, defining as well parallel vector fields along curves ([5]).

In this study, we define the kinematic model of smooth submanifolds M and N using arbitrary orthonormal frames along the pole curves and obtain the equations of this homothetic motion which assume both rolling and sliding of M upon N along these curves. Further, we obtain the equations of homothetic motion of M on N for two given arbitrary curves on M and N respectively, assuming that these curves are pole curves.

§2. Introduction

We shall use hereafter the definitions and notations from ([5]). The *homothetic motion* of smooth submanifolds M onto N in Euclidean 3-space is generated by the transformation

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(2.1)
$$\begin{array}{rcll} F:M & \to & N & \\ X(t) & \to & Y(t) & = hAX(t)+C \end{array},$$

where A is a proper orthogonal $3x3$ matrix, X and C are 3×1 vectors and $h\neq 0$ is homothetic scale. The entries of A, C and h are continuously differentiable functions of the time t, the entries of X are the coordinates of a point on M in Euclidean coordinates (x_1,x_2,x_3), and Y is a trajectory of X. We take B as hA, and by differentiating (2.1), we obtain

$$\frac{dY}{dt}=B\frac{dX}{dt}+\frac{dB}{dt}X+\frac{dC}{dt}, \tag{2.2}$$

where

$$\frac{dB}{dt}X+\frac{dC}{dt},\quad B\frac{dX}{dt},\quad \frac{dY}{dt}$$

are respectively called *sliding velocity*, *relative velocity* and *absolute velocity* of the point X. As well, we call X *center of instantaneous rotation* if its sliding velocity vanishes. If X is the center of an instantaneous rotation, then X is a pole point at the time t of the motion F given in (2.1). Since $\det(\frac{dB}{dt})\neq 0$, then every homothetic motion in E^3 is a regular motion. Consider a regular curve $X(t)$ on M, which is defined on a closed interval $I\subset\mathbb{R}$, such that all its points are pole points. In this case, we call the curves

$$X(t)=-\left(\frac{dB}{dt}\right)^{-1}\left(\frac{dC}{dt}\right)$$

and

$$Y(t)=-B\left(\frac{dB}{dt}\right)^{-1}\left(\frac{dC}{dt}\right)+C$$

moving and *fixed pole curves*, respectively, where the matrix $B\left(\frac{dB}{dt}\right)^{-1}$ is described by

$$-B\left(\frac{dB}{dt}\right)^{-1}=\left(\frac{dh}{dt}A+h\frac{dA}{dt}\right)h^{-1}A^{-1}=\underbrace{\frac{dh}{dt}\cdot h^{-1}I_3}_{\varphi}+\underbrace{\frac{dA}{dt}\cdot A^{-1}}_{S}$$

We call φ and S the *sliding part* and *the rolling part of the motion* F, respectively. Every homothetic motion in E^3 consists of both sliding and rolling. For $S\neq 0$, there is a uniquely determined vector $W(t)$ such that $S(U)$ equal to the cross product $W(t)\wedge U$ for every vector $U\in IR^3$. The vector $W(t)$ is called the angular velocity of the point $X(t)$ at the moment t ([8]).

§3. Sliding and rolling of M onto N

We consider further two smooth manifolds M and N which are tangent (inside or outside) each other and the curves $X(t)$ on M and $Y(t)$ on N as moving and fixed regular pole curves. Let Σ be the common tangent plane of M and N (tangent to $X(t)$ and to $Y(t)$) at the contact point p). We consider a Cartesian coordinate system in E^3 and let $\{e_1=(1,0,0),e_2=(0,1,0),e_3=(0,0,1)\}$ be the canonic unit basis. We denote by $\xi=\xi(t)$ and $\eta=\eta(t)$ the normal vector fields of M and N along the curves $X(t)$ and $Y(t)$, respectively. In addition, we denote by $\{T,N,B\}$ and by $\{\bar{T},\bar{N},\bar{B}\}$

the Frenet vector fields of the curves $X(t)$ and $Y(t)$, respectively, (tangent, principal normal and binormal vector fields). Since the homothetic motion $F: M \to N$ consists of rolling, then $W(t)$ lies in the tangent plane of both $X(t)$ of M and $Y(t)$ of N at each contact point ([5]), where we have $B\xi = \epsilon h\eta$. Here by ϵ we denoted the sign function such that if $\epsilon = +1$ then M moves inside of N along pole curves, and if $\epsilon = -1$ then M moves outside of N along pole curves.

Assume that $\{b_1, b_2\}$ and $\{a_1, a_2\}$ are orthonormal systems along the regular pole curves $X(t)$ and $Y(t)$, and let b_1, b_2 and a_1, a_2 transforms into each another via $b_1 = hB^{-1}a_1$ and $b_2 = hB^{-1}a_2$, respectively. Thus we call $\{b_1, b_2, \xi\}$ and $\{a_1, a_2, \eta\}$ moving and fixed orthonormal systems along the curves $(X) = X(t)$ and $(Y) = Y(t)$, respectively. Since (X) is the pole curve, we can write the equation $\frac{dY}{dt} = B\frac{dX}{dt}$ by using (2.2). If t is the arc-length parameter for the curve (X), then we can write $\frac{dY}{dt} = hAT$ and we have the following equations

$$h = \left\| \frac{dY}{dt} \right\|, \quad \bar{T} = \frac{1}{h}\frac{dY}{dt}$$

Since $\xi \in Sp\{N, B\}$, then we can write

$$\xi(t) = \cos\psi(t)N + \sin\psi(t)B \tag{3.3}$$

and we need to construct the frames $\{b_1, b_2, X\}$ and $\{a_1, a_2, e_3\}$ in order to determine the orthogonal matrix A and to set up the kinematic model. During this process, we make use of the frames $\{T, \xi\wedge T, \xi\}$ and $\{\bar{T}, \eta\wedge\bar{T}, \eta\}$, which are called *Darboux frames* along (X) and (Y) on M and N, respectively. We can find an orthogonal matrix Q by using (3.3) and

$$\begin{pmatrix} T \\ \xi\wedge T \\ \xi \end{pmatrix} = (Q)\begin{pmatrix} T \\ N \\ B \end{pmatrix}. \tag{3.4}$$

It is well known that if the system $\{T, N, B\}$ rotates relative to $\{e_1, e_2, e_3\}$, then we can write for $P \in SO(3)$ the relations

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix} = (P)\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{3.5}$$

The tangent spaces $Sp\{b_1, b_2\}$ and $Sp\{T, \xi\wedge T\}$ coincide with Σ, and hence we have

$$\begin{pmatrix} b_1 \\ b_2 \\ \xi \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{R}\begin{pmatrix} T \\ \xi\wedge T \\ \xi \end{pmatrix}, \tag{3.6}$$

where $\theta = \theta(t)$ is the angle between b_1 and T (b_2 and $\xi\wedge T$). We obtain the orthogonal matrix $A_1 = (P)^T[Q]^T[R]^T$ by using (3.4)-(3.6). The matrix A_1 transforms b_1 to e_1, b_2 to e_2 and ξ to e_3, respectively. On the other hand, we denote the skew symmetric matrix $\frac{dA_1^T}{dt}A_1$ as w_1, so that w_1 will be given by

$$w_1 = \begin{pmatrix} 0 & \theta' + k_1 \sin\psi & m_1 \\ -\theta' - k_1 \sin\psi & 0 & m_2 \\ -m_1 & -m_2 & 0 \end{pmatrix}, \tag{3.7}$$

where $k_1 = k_1(t)$ and $k_2 = k_2(t)$ are the curvature and torsion of the pole curve (X) respectively, and

$$m_1 = \epsilon k_1 \cos\theta \cos\psi + (k_2 + \psi') \sin\theta, \;\; m_2 = -\epsilon k_1 \sin\theta \cos\psi - (k_2 + \psi') \cos\theta.$$

Thus we have the following

Proposition 1. *The vector fields b_1 and b_2 are parallel according to the connection of M along the curve (X) if and only if $\theta' + k_1 \sin\psi = 0$ is satisfied.*

Proof. Let $\bar{\nabla}$ be the Levi Civita connection and S_M be the shape operator of M. We obtain b_1 by using (3.4) and (3.5),

$$b_1 = \cos\theta T + \sin\theta \sin\psi N - \sin\theta \cos\psi B$$

and from the Gauss equation, we have

$$\bar{\nabla}_T b_1 = \nabla_T b_1 + \langle S_M(T), b_1 \rangle \xi,$$

and after routine calculations, we obtain

$$\bar{\nabla}_T b_1 = -(\theta' + k_1 \sin\psi) \cdot (\sin\theta T - \sin\psi \cos\theta N + \cos\psi \cos\theta B).$$

It is easy to see that $\bar{\nabla}_T b_1 = 0$ if and only if $\theta' + k_1 \sin\psi = 0$. Hence, b_1 is a parallel vector field relative to the connection of M along the curve (X) if and only if $\theta' + k_1 \sin\psi = 0$ is satisfied. Similarly, we can easily prove that b_2 is a parallel vector field relative to the connection of M along the curve (X) if and only if $\theta' + k_1 \sin\psi = 0$ is satisfied, as well. ∎

On the other hand, since $\eta \in Sp\{\bar{N}, \bar{B}\}$ then we have

$$\eta(t) = \cos\bar{\psi}(t)\bar{N} + \sin\bar{\psi}(t)\bar{B} \tag{3.8}$$

thus we can find an orthogonal matrix $\overline{Q}$ by using (3.8) such that

$$\begin{pmatrix} \bar{T} \\ \eta \wedge \bar{T} \\ \eta \end{pmatrix} = (\bar{Q}) \begin{pmatrix} \bar{T} \\ \bar{N} \\ \bar{B} \end{pmatrix} \tag{3.9}$$

Since $\{\bar{T}, \bar{N}, \bar{B}\}$ rotates according to the orthonormal basis $\{e_1, e_2, e_3\}$, we obtain $\bar{P} \in SO(3)$ as follows

$$\begin{pmatrix} \bar{T} \\ \bar{N} \\ \bar{B} \end{pmatrix} = [\bar{P}] \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \tag{3.10}$$

and since $Sp\{a_1, a_2\}$ and $Sp\{\bar{T}, \eta \wedge \bar{T}\}$ coincide with Σ, we infer

$$\begin{pmatrix} a_1 \\ a_2 \\ \eta \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\bar{\theta} & \sin\bar{\theta} & 0 \\ -\sin\bar{\theta} & \cos\bar{\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\bar{R}} \begin{pmatrix} \bar{T} \\ \eta\wedge\bar{T} \\ \eta \end{pmatrix} \tag{3.11}$$

where $\bar{\theta} = \bar{\theta}(t)$ is the angle between a_1 and $\bar{T}$, (a_2 and $\eta\wedge T$). Thus we obtain another orthogonal matrix $A_2 = (\bar{P})^T(\bar{Q})^T(\bar{R})^T$ by using (3.9)-(3.11) such that the matrix A_2 transforms a_1, a_2, η into e_1, e_2, e_3, respectively. We denote the skew symmetric matrix $\frac{dA_2^T}{dt}A_2$ as w_2, which has the form

$$w_2 = \begin{pmatrix} 0 & \bar{\theta}' + \bar{k}_1\sin\bar{\psi} & \left\{\begin{matrix} \bar{k}_1\cos\bar{\theta}\cos\bar{\psi}+ \\ (\bar{k}_2+\bar{\psi}')\sin\bar{\theta} \end{matrix}\right\} \\ -\bar{\theta}' - \bar{k}_1\sin\bar{\psi} & 0 & -\left\{\begin{matrix} \bar{k}_1\sin\bar{\theta}\cos\bar{\psi}- \\ (\bar{k}_2+\bar{\psi}')\cos\bar{\theta} \end{matrix}\right\} \\ -\left\{\begin{matrix} \bar{k}_1\cos\bar{\theta}\cos\bar{\psi}+ \\ (\bar{k}_2+\bar{\psi}')\sin\bar{\theta} \end{matrix}\right\} & \left\{\begin{matrix} \bar{k}_1\sin\bar{\theta}\cos\bar{\psi}- \\ (\bar{k}_2+\bar{\psi}')\cos\bar{\theta} \end{matrix}\right\} & 0 \end{pmatrix} \tag{3.12}$$

where $\bar{k}_1 = \bar{k}_1(t)$ and $\bar{k}_2 = \bar{k}_2(t)$ are curvature and torsion of the pole curve (Y). Thus we have the following

Proposition 2. *The vector fields $a_1(t)$ and $a_2(t)$ are parallel according to the connection of N along curve (Y) if and only if $\overline{\theta'} + \bar{k}_1\sin\bar{\psi} = 0$ is satisfies.*

Proof. It is easy to proof similarly remark 1. ■

Therefore, we can find the main matrix A of the motion (2.1) by using A_1 and A_2 as $A = A_2A_1^T$ so that A transforms b_1 to a_1, b_2 to a_2 and ξ to $\epsilon\eta$, respectively. The skew-symmetric matrix $S = \frac{dA}{dt}A^T$ is instantaneous rotation matrix and S represents a linear isomorphism as $S : T_{Y(t)}N \longrightarrow Sp\{\eta\}$. We obtain the matrix S by using (3.7) and (3.12) as $S = A_2(-w_2 + w_1)A_2^T$. Consequently, the matrix S determines an unique vector $w_p \in Sp\{a_1, a_2, \eta\}$ as follows.

$$\begin{aligned} w_p &= \left\{\begin{matrix} -\bar{k}_1\cos\bar{\psi}\sin\bar{\theta} + (\bar{\psi}' + \bar{k}_2)\cos\bar{\theta}+ \\ \epsilon k_1\cos\psi\sin\theta - \epsilon(\psi' + k_2)\cos\theta \end{matrix}\right\} a_1|_p \\ &+ \left\{\begin{matrix} -\bar{k}_1\cos\bar{\psi}\cos\bar{\theta} - (\bar{\psi}' + \bar{k}_2)\sin\bar{\theta}+ \\ \epsilon k_1\cos\psi\cos\theta + \epsilon(\psi' + k_2)\sin\theta \end{matrix}\right\} a_2|_p \\ &- \left\{\begin{matrix} \overline{\theta'} + \bar{k}_1\sin\bar{\psi} \\ -\theta' - k_1\sin\psi \end{matrix}\right\} \eta|_p \end{aligned} \tag{3.13}$$

Thus, we obtained an important condition for the rolling part of the homothetic motion of smooth submanifolds in E^3, along regular pole curves. Hence we have proved the following

Theorem 1. *The transformation F is a rolling motion defined as $Bb_1 = ha_1$, $Bb_2 = ha_2$ and $B\xi = \epsilon h\eta$ at the centers of instantaneous rotation if and only if $\overline{\theta'} + \bar{k}_1\sin\bar{\psi} - \theta' - k_1\sin\psi = 0$ is satisfied.*

Furthermore, we can specify this result to special cases:

Corollary 1. *a) If M is a manifold in E^3 and N is a plane, then the angular velocity vector at the contact points will be*

$$w_p = \left\{ \begin{array}{c} \epsilon k_1 \cos\psi \sin\theta - \\ \epsilon(\psi' + k_2)\cos\theta \end{array} \right\} a_1|_p + \left\{ \begin{array}{c} \epsilon k_1 \cos\psi \cos\theta + \\ \epsilon(\psi' + k_2)\sin\theta \end{array} \right\} a_2|_p - \left\{ \begin{array}{c} \overline{\theta'} + \bar{k}_1 - \\ \theta' - k_1 \sin\psi \end{array} \right\} \eta|_p$$

In this case, F is a rolling motion if and only if $\overline{\theta'} + \bar{k}_1 - \theta' - k_1 \sin\psi = 0$ is satisfied.

b) If M is unit sphere and N is a plane then angular velocity vector at the contact points will be as follows,

$$w_p = \epsilon k_1 \cos\psi \sin\theta a_{1p} + \epsilon k_1 \cos\psi \cos\theta a_{2p} - \left\{ \bar{\theta}' + \bar{k}_1 - \theta' - k_1 \sin\psi \right\} \eta_p$$

In this case, F is a rolling motion if and only if $\overline{\theta'} + \bar{k}_1 - \theta' - k_1 \sin\psi = 0$ is satisfied.

Corollary 2. *a) In case that $b_1(t)$, $b_2(t)$ and $a_1(t)$, $a_2(t)$ are parallel vector fields along the pole curves, then we obtain all of the results in [5].*

b) In case that $b_1(t)$, $b_2(t)$ and $a_1(t)$, $a_2(t)$ are parallel vector fields along the pole curves and $h = 1$ then we obtain all of the results in [8].

c) If (X) and (Y) are geodesics of M and N, respectively, then M rolling on (or in) N along the pole curves.

As well, we can state the following

Proposition 3. *Let (X) and (Y) be geodesics on M and N with the same curvatures and same torsions. Then M both slides and rolls on N or M slides without rolls inside of N along the pole curves.*

Proof. Since (X) and (Y) are geodesics on M and N with same curvatures and torsions, then we take $\theta = \bar{\theta} = \psi = \bar{\psi} = 0$, $\bar{k}_1 = k_1 = \lambda$ and $\bar{k}_2 = k_2 = \mu$. Substituting in (3.13), we obtain

$$w_p = (1 - \epsilon) \left\{ \mu \bar{T}_p - \lambda(\eta \wedge \bar{T})_p \right\}$$

In the case $\epsilon = -1$, the vectors ξ and η are opposite at the contact point p, and thus $w_p \neq 0$ and w_p is tangent to both (X) and (Y). This means that M slides and rolls outside of N along the pole-geodesic curves. In the case $\epsilon = 1$, the vectors ξ and η have the same orientation at the contact points p, we infer $w_p = 0$, which means that M slides without rolling inside N along the pole-geodesic curves. ∎

Theorem 2. *Let M and N be two submanifolds and $X(t)$ and $Y(t)$ be smooth pole curves on M and N respectively, which satisfy the conditions of Theorem 1, and which are tangent to each other at the contact points. Then we can find a unique homothetic motion F of M upon N along pole curves.*

Theorem 3. *Let S_M and S_N be the shape operators of M and N along the curves (X) and (Y), respectively. If*

$$h^{-1} S_M(\frac{dX}{dt}) = S_N(\frac{dY}{dt}),$$

then F is a sliding motion without rolling.

Proof. We can write the following equations along the curves (X) and (Y), respectively.

$$S_M(\frac{dX}{dt}) = \frac{d\xi}{dt} \quad \text{and} \quad S_N(\frac{dY}{dt}) = \frac{d\eta}{dt}.$$

Differentiating (3.3) and using (3.6) we yield

$$\frac{d\epsilon\xi}{dt} = -\epsilon \left\{ \begin{array}{c} k_1 \cos\theta \cos\psi \\ +(k_2 + \psi') \sin\theta \end{array} \right\} b_1 + \epsilon \left\{ \begin{array}{c} k_1 \sin\theta \cos\psi \\ -(k_2 + \psi') \cos\theta \end{array} \right\} b_2$$

and since $b_1 = hB^{-1}a_1$, $b_2 = hB^{-1}a_2$ and $\xi = \epsilon hB^{-1}\eta$,

$$h^{-1}B(\frac{d\epsilon\xi}{dt}) = -\epsilon \left\{ \begin{array}{c} k_1 \cos\theta \cos\psi + \\ (k_2 + \psi') \sin\theta \end{array} \right\} a_1 + \epsilon \left\{ \begin{array}{c} k_1 \sin\theta \cos\psi - \\ (k_2 + \psi') \cos\theta \end{array} \right\} a_2,$$

we obtain by differentiating (3.8) and using (3.11),

$$\frac{d\eta}{dt} = - \left\{ \begin{array}{c} \bar{k}_1 \cos\bar{\theta} \cos\bar{\psi} + \\ (\bar{k}_2 + \bar{\psi}') \sin\bar{\theta} \end{array} \right\} a_1 + \left\{ \begin{array}{c} -\bar{k}_1 \sin\bar{\theta} \cos\bar{\psi} + \\ (\bar{k}_2 + \bar{\psi}') \cos\bar{\theta} \end{array} \right\} a_2$$

Since $h^{-1}B(\frac{d\xi}{dt}) = \frac{d\eta}{dt}$, we obtain

$$\epsilon \left\{ k_1 \sin\theta \cos\psi - (k_2 + \psi') \cos\theta \right\} = \left\{ -\bar{k}_1 \sin\bar{\theta} \cos\bar{\psi} + (\bar{k}_2 + \bar{\psi}') \cos\bar{\theta} \right\}$$

$$\epsilon \left\{ k_1 \cos\theta \cos\psi + (k_2 + \psi') \sin\theta \right\} = \left\{ \bar{k}_1 \cos\bar{\theta} \cos\bar{\psi} + (\bar{k}_2 + \bar{\psi}') \sin\bar{\theta} \right\}$$

Substituting these equations into (3.13), we obtain $w_p = 0$. Hence F is a sliding motion without rolling. ■

Proposition 4. *If F is a sliding and rolling motion then the shape operators of M and N satisfy following inequality*

$$h^{-1}S_M(\frac{dX}{dt}) \neq S_N(\frac{dY}{dt}).$$

Example 1. Assume $\epsilon = -1$. Let $X(t) = (\sin t, 0, \cos t)$, $t \in [0, \pi]$ be a unit speed curve on the unit sphere $\phi(u, v) = (\sin v \sin u, \sin v \cos u, \cos v)$ and $Y(t) = (\sin t, -t, \cos t - 2)$ be a helix on the cylinder $x^2 + (z + 2)^2 = 1$. We obtain the unit normal vector fields of M and N, ψ, $\bar{\psi}$, θ, $\bar{\theta}$ and the Frenet vector fields and curvatures of the curves $X(t)$ and $Y(t)$ as follows

$$T = (\sin t, 0, -\cos t), \quad N = (-\sin t, 0, -\cos t), \quad B = (0, 1, 0)$$

$$k_1 = 1, \quad k_2 = 0, \quad \psi = \pi, \quad \xi(t) = (\sin t, 0, \cos t)$$

and

$$\bar{T} = \left(\frac{\sqrt{2}}{2} \cos t, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \sin t \right), \quad \bar{N} = (-\sin t, 0, -\cos t)$$

$$\bar{B} = \left(\frac{\sqrt{2}}{2} \cos t, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \sin t \right), \quad \eta(t) = -(\sin t, 0, \cos t)$$

$$\bar{k}_1 = \frac{\sqrt{2}}{2},\ \ \bar{k}_2 = -\frac{\sqrt{2}}{2},\ \ \bar{\psi} = \pi$$

Since $\|dY/dt\| = h$, we find $h = \sqrt{2}$ and we obtain $\bar{\theta}(t) = \theta(t) = 0$ by using $\frac{dY}{dt} = B\frac{dX}{dt}$. Hence the motion will be described by

$$(3.14)\quad \mathrm{Y(t)} = \begin{pmatrix} \cos^2 t - \sqrt{2}\sin^2 t & \cos t & (1+\sqrt{2})\cos t\sin t \\ -\cos t & 1 & \sin t \\ -(1+\sqrt{2})\cos t\sin t & -\sin t & \sin^2 t - \sqrt{2}\cos^2 t \end{pmatrix} \mathrm{X(t)} + \begin{pmatrix} (1+\sqrt{2})\sin t \\ -t \\ (1+\sqrt{2})\cos t - 2 \end{pmatrix}$$

The matrix S and the vector w_p are
(3.15)

$$S = \begin{pmatrix} 0 & \frac{\sqrt{2}}{2}\sin t & 1+\frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2}\sin t & 0 & \frac{-\sqrt{2}}{2}\cos t \\ -1-\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\cos t & 0 \end{pmatrix},\ w_p = \left(\frac{\sqrt{2}}{2}\cos t, 1+\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2}\sin t\right)$$

Both (X) and (Y) satisfy (2.2), and these curves are the moving and the fixed pole curves of the motion (3.14), and the vector fields given in (3.7), (3.12) and (3.15) lie in the common tangent spaces $T_{X(t)}(M)$ and $T_{Y(t)}(N)$ at the contact points. In addition, since the vector w_p satisfies the condition given in Theorem 1, the motion (3.14) is a rolling motion. Since $\epsilon = -1$, then the unit sphere is rolling on the cylinder.

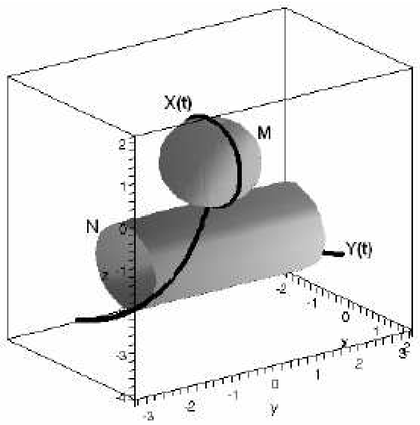

Figure 1: Unit sphere rolling on the cylinder along the curves (X) and (Y)

Example 2. Assume $\epsilon = 1$. Let $X(t) = (\sin t, 0, \cos t - 1)$, $t \in [0, \pi]$ be a unit speed curve on the unit sphere $\phi(u,v) = (\sin v\sin u, \sin v\cos u, \cos v - 1)$ and let $Y(t) = (2\sin t, -t, 2\cos t - 2)$ be a helix on the cylinder $x^2 + (z+2)^2 = 4$. We obtain the unit normal vector fields of M and N, ψ, $\bar{\psi}$, θ, $\bar{\theta}$ and the Frenet vector fields and curvatures of the curves $X(t)$ and $Y(t)$ as follows

$$T = (\sin t, 0, -\cos t),\ \ N = (-\sin t, 0, -\cos t),\ \ B = (0,1,0)$$

$$k_1 = 1, \quad k_2 = 0, \quad \psi = \pi, \quad \xi(t) = (\sin t, 0, \cos t)$$

and

$$\bar{T} = \left(\frac{2\sqrt{5}}{5}\cos t, \frac{-\sqrt{5}}{5}, \frac{-2\sqrt{5}}{5}\sin t \right), \quad \bar{N} = (-\sin t, 0, -\cos t)$$

$$\bar{B} = \left(\frac{\sqrt{5}}{5}\cos t, \frac{2\sqrt{5}}{5}, \frac{-\sqrt{5}}{5}\sin t \right), \quad \eta(t) = (\sin t, 0, \cos t)$$

$$\bar{k}_1 = \frac{2\sqrt{5}}{5}, \quad \bar{k}_2 = -\frac{\sqrt{5}}{5}, \quad \bar{\psi} = \pi$$

Since $\|dY/dt\| = h$, we find $h = \sqrt{2}$ and we obtain $\bar{\theta}(t) = \theta(t) = 0$ by using $\frac{dY}{dt} = B\frac{dX}{dt}$. Hence the motion will be described by

$$\text{Y(t)=} \begin{pmatrix} 2\cos^2 t + \sqrt{5}\sin^2 t & \cos t & (\sqrt{5}-2)\cos t \sin t \\ -\cos t & 2 & \sin t \\ (\sqrt{5}-2)\cos t \sin t & -\sin t & 2\sin^2 t + \sqrt{5}\cos^2 t \end{pmatrix} \text{X(t)}$$

(3.16)

$$+ \begin{pmatrix} \left\{ \begin{array}{c} (\frac{\sqrt{5}-2}{2})\sin 2t + \\ (2-\sqrt{5})\sin t \end{array} \right\} \\ (\sin t - t) \\ \left\{ \begin{array}{c} (\frac{\sqrt{5}-2}{2})(\cos 2t + 1) \\ +(2-\sqrt{5})\cos t \end{array} \right\} \end{pmatrix}$$

The matrix S and the vector w_p be as follows.

(3.17)

$$S = \begin{pmatrix} 0 & -\frac{\sqrt{5}}{5}\sin t & 1 - \frac{2\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5}\sin t & 0 & \frac{\sqrt{5}}{5}\cos t \\ -1 + \frac{2\sqrt{5}}{5} & \frac{-\sqrt{5}}{5}\cos t & 0 \end{pmatrix}, \quad w_p = \left(\frac{-\sqrt{5}}{5}\cos t, 1 - \frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\sin t \right)$$

Both (X) and (Y) satisfy (2.2). Then these curves are the moving and the fixed pole curves of the motion (3.16) respectively, and the vector fields given in (3.7), (3.12) and (3.17) lie in the common tangent spaces $T_{X(t)}(M)$ and $T_{Y(t)}(N)$ at the contact points. Moreover, the vector w_p satisfies the condition given in Theorem 1, and thus the motion (3.16) is a rolling motion. Since $\epsilon = 1$, then the unit sphere is rolling in the cylinder.

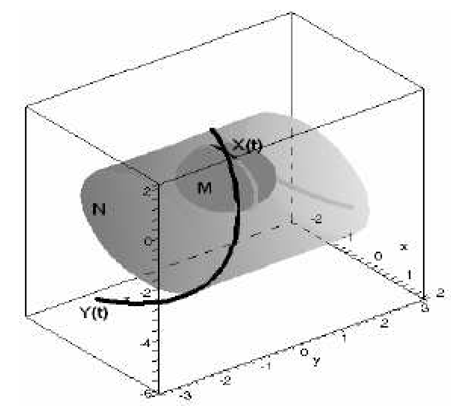

Figure 2: Unit sphere rolling in the cylinder along the curves (X) and (Y)

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