# Geometry of parallelizable manifolds in the context of generalized Lagrange spaces

M.I. Wanas, Nabil L. Youssef and A.M. Sid-Ahmed

**Abstract.** In this paper, we deal with a generalization of the geometry of parallelizable manifolds, or the absolute parallelism (AP-) geometry, in the context of generalized Lagrange spaces. All geometric objects defined in this geometry are not only functions of the positional argument $x$, but also depend on the directional argument $y$. In other words, instead of dealing with geometric objects defined on the manifold $M$, as in the case of classical AP-geometry, we are dealing with geometric objects in the pullback bundle $\pi^{-1}(TM)$ (the pullback of the tangent bundle $TM$ by $\pi : TM \longrightarrow M$). Many new geometric objects, which have no counterpart in the classical AP-geometry, emerge in this more general context. We refer to such a geometry as generalized AP-geometry (GAP-geometry). In analogy to AP-geometry, we define a $d$-connection in $\pi^{-1}(TM)$ having remarkable properties, which we call the canonical $d$-connection, in terms of the unique torsion-free Riemannian $d$-connection. In addition to these two $d$-connections, two more $d$-connections are defined, the dual and the symmetric $d$-connections. Our space, therefore, admits twelve curvature tensors (corresponding to the four defined $d$-connections), three of which vanish identically. Simple formulae for the nine non-vanishing curvatures tensors are obtained, in terms of the torsion tensors of the canonical $d$-connection. The different $W$-tensors admitted by the space are also calculated. All contractions of the $h$- and $v$-curvature tensors and the $W$-tensors are derived. Second rank symmetric and skew-symmetric tensors, which prove useful in physical applications, are singled out.

**M.S.C. 2000**: 53B40, 53A40, 53B50
**Key words**: Parallelizable manifold, Generalized Lagrange space, AP-geometry, GAP-geometry, Canonical $d$-connection, $W$-tensor.

Balkan Journal of Geometry and Its Applications, Vol.13, No.2, 2008, pp. 120-139.

# 1 Introduction

The geometry of parallelizable manifolds or the absolute parallelism geometry (AP-geometry) ([5], [10], [14], [15]) has many advantages in comparison to Riemannian geometry. Unlike Riemannian geometry, which has ten degrees of freedom (corresponding to the metric components for $n = 4$), AP-geometry has sixteen degrees of freedom (corresponding to the number of components of the four vector fields defining the parallelization). This makes AP-geometry a potential candidate for describing physical phenomena other than gravity. Moreover, as opposed to Riemannian geometry, which admits only one symmetric linear connection, AP-geometry admits at least four natural (built-in) linear connections, two of which are non-symmetric and three of which have non-vanishing curvature tensors. Last, but not least, associated with an AP-space, there is a Riemannian structure defined in a natural way. Thus, AP-geometry contains within its geometrical structure all the mathematical machinery of Riemannian geometry. Accordingly, a comparison between the results obtained in the context of AP-geometry and general relativity, which is based on Riemannian geometry, can be carried out.

In this paper, we study AP-geometry in the wider context of a generalized Lagrange space ([7], [9], [11], [12]). All geometric objects defined in this space are not only functions of the positional argument $x$, but also depend on the directional argument $y$. In other words, instead of dealing with geometric objects defined on the manifold $M$, as in the case of classical AP-space, we are dealing with geometric objects in the pullback bundle $\pi^{-1}(TM)$ (the pullback of the tangent bundle $TM$ by the projection $\pi : TM \longrightarrow M$) [1]. Many new geometric objects, which have no counterpart in the classical AP-space, emerge in this more general context. We refer to such a space as a $d$-parallelizable manifold or a generalized absolute parallelism space (GAP-space).

The paper is organized in the following manner. In section 2, following the introduction, we give a brief account of the basic concepts and definitions that will be needed in the sequel, introducing the notion of a non-linear connection $N^{\alpha}_{\mu}$. In section 3, we consider an $n$-dimensional $d$-parallelizable manifold $M$ ([2], [11]) on which we define a metric in terms of the $n$ independent $\pi$-vector fields $\underset{i}{\lambda}$ defining the parallelization on $\pi^{-1}(TM)$. Thus, our parallelizable manifold becomes a generalized Lagrange space, which is a generalization of the classical AP-space. We then define the canonical $d$-connection $D$, relative to which the $h$- and $v$-covariant derivatives of the vector fields $\underset{i}{\lambda}$ vanish. We end this section with a comparison between the classical AP-space and the GAP-space. In section 4, commutation formulae are recalled and some identities obtained. We then introduce, in analogy to the AP-space, two other $d$-connections: the dual $d$-connection and the symmetric $d$-connection. The nine non-vanishing curvature tensors, corresponding to the dual, symmetric and Riemannian $d$-connections are then calculated, expressed in terms of the torsion tensors of the canonical $d$-connection. In section 5, a summary of the fundamental symmetric and skew symmetric second rank tensors is given, together with the symmetric second rank tensors of zero trace. In section 6, all possible contractions of the $h$- and $v$-curvature tensors are obtained and the contracted curvature tensors are expressed in terms of the fundamental tensors given in section 5. In section 7, we study the different $W$-tensors corresponding to the different $d$-connections defined in the space, again expressed in terms of the torsion tensors of the canonical $d$-connection. Contractions

of the different $W$-tensors and the relations between them are then derived. Finally, we end this paper by some concluding remarks.

## 2 Fundamental preliminaries

Let $M$ be a differential manifold of dimension $n$ of class $C^\infty$. Let $\pi : TM \to M$ be its tangent bundle. If $(U,\ x^\mu)$ is a local chart on $M$, then $(\pi^{-1}(U),\ (x^\mu, y^\mu))$ is the corresponding local chart on $TM$. The coordinate transformation law on $TM$ is given by:

$$x^{\mu'} = x^{\mu'}(x^\nu), \quad y^{\mu'} = p^{\mu'}_\nu y^\nu,$$

where $p^{\mu'}_\nu = \frac{\partial x^{\mu'}}{\partial x^\nu}$ and $\det(p^{\mu'}_\nu) \neq 0$.

**Definition 2.1.** A non-linear connection $N$ on $TM$ is a system of $n^2$ functions $N^\alpha_\beta(x,\ y)$ defined on every local chart $\pi^{-1}(U)$ of TM which have the transformation law

$$N^{\alpha'}_{\beta'} = p^{\alpha'}_\alpha p^\beta_{\beta'} N^\alpha_\beta + p^{\alpha'}_\epsilon p^\epsilon_{\beta'\sigma'} y^{\sigma'}, \tag{2.1}$$

where $p^\epsilon_{\beta'\sigma'} = \frac{\partial p^\epsilon_{\beta'}}{\partial x^{\sigma'}} = \frac{\partial^2 x^\epsilon}{\partial x^{\beta'} \partial x^{\sigma'}}$.

The non-linear connection $N$ leads to the direct sum decomposition

$$T_u(TM) = H_u(TM) \oplus V_u(TM), \quad \forall\ u \in \mathcal{T}M = TM \setminus \{0\},$$

where $H_u(TM)$ is the *horizontal* space at $u$ associated with $N$ supplementary to the *vertical* space $V_u(TM)$. If $\delta_\mu := \partial_\mu - N^\alpha_\mu \dot{\partial}_\alpha$, where $\partial_\mu := \frac{\partial}{\partial x^\mu}$, $\dot{\partial}_\mu := \frac{\partial}{\partial y^\mu}$, then $(\dot{\partial}_\mu)$ is the natural basis of $V_u(TM)$ and $(\delta_\mu)$ is the natural basis of $H_u(TM)$ adapted to $N$.

**Definition 2.2.** A distinguished connection (d-connection) on $M$ is a triplet $D = (N^\alpha_\mu, \Gamma^\alpha_{\mu\nu}, C^\alpha_{\mu\nu})$, where $N^\alpha_\mu(x,y)$ is a non-linear connection on $TM$ and $\Gamma^\alpha_{\mu\nu}(x,y)$ and $C^\alpha_{\mu\nu}(x,y)$ transform according to the following laws:

$$\Gamma^{\alpha'}_{\mu'\nu'} = p^{\alpha'}_\alpha p^\mu_{\mu'} p^\nu_{\nu'} \Gamma^\alpha_{\mu\nu} + p^{\alpha'}_\epsilon p^\epsilon_{\mu'\nu'}, \tag{2.2}$$

$$C^{\alpha'}_{\mu'\nu'} = p^{\alpha'}_\alpha p^\mu_{\mu'} p^\nu_{\nu'} C^\alpha_{\mu\nu}. \tag{2.3}$$

In other words, $\Gamma^\alpha_{\mu\nu}$ transform as the coefficients of a linear connection, whereas $C^\alpha_{\mu\nu}$ transform as the components of a tensor.

**Definition 2.3.** The horizontal (h-) and vertical (v-) covariant derivatives with respect to the d-connection $D$ (of a tensor field $A^\alpha_\mu$) are defined respectively by:

$$A^\alpha_{\mu|\nu} := \delta_\nu A^\alpha_\mu + A^\epsilon_\mu \Gamma^\alpha_{\epsilon\nu} - A^\alpha_\epsilon \Gamma^\epsilon_{\mu\nu}; \tag{2.4}$$

$$A^\alpha_{\mu||\nu} := \dot{\partial}_\nu A^\alpha_\mu + A^\epsilon_\mu C^\alpha_{\epsilon\nu} - A^\alpha_\epsilon C^\epsilon_{\mu\nu}. \tag{2.5}$$

**Definition 2.4.** A symmetric and non-degenerate tensor field $g_{\mu\nu}(x, y)$ of type (0, 2) is called a generalized Lagrange metric on the manifold $M$. The pair $(M,\ g)$ is called a generalized Lagrange space.

**Definition 2.5.** Let $(M, g)$ be a generalized Lagrange space equipped with a non-linear connection $N^\alpha_\mu$. Then a d -connection $D = (N^\alpha_\mu, \Gamma^\alpha_{\mu,\nu}, C^\alpha_{\mu\nu})$ is said to be metrical with respect to $g$ if

$$g_{\mu\nu|\alpha} = 0, \quad g_{\mu\nu||\alpha} = 0. \tag{2.6}$$

The following remarkable result was proved by R. Miron [8]. It guarantees the existence of a unique *torsion-free* metrical $d$-connection on any generalized Lagrange space equipped with a non-linear connection. More precisely:

**Theorem 2.6.** *Let $(M, g)$ be a generalized Lagrange space. Let $N^\alpha_\mu$ be a given non-linear connection on $TM$. Then there exists a unique metrical d-connection $\overset{\circ}{D} = (N^\alpha_\mu, \overset{\circ}{\Gamma}{}^\alpha_{\mu\nu}, \overset{\circ}{C}{}^\alpha_{\mu\nu})$ such that $\overset{\circ}{\Lambda}{}^\alpha_{\mu\nu} := \overset{\circ}{\Gamma}{}^\alpha_{\mu\nu} - \overset{\circ}{\Gamma}{}^\alpha_{\nu\mu} = 0$ and $\overset{\circ}{T}{}^\alpha_{\mu\nu} := \overset{\circ}{C}{}^\alpha_{\mu\nu} - \overset{\circ}{C}{}^\alpha_{\nu\mu} = 0$. This d-connection is given by $N^\alpha_\mu$ and the generalized Christoffel symbols:*

$$\overset{\circ}{\Gamma}{}^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\epsilon}(\delta_\mu g_{\nu\epsilon} + \delta_\nu g_{\mu\epsilon} - \delta_\epsilon g_{\mu\nu}), \tag{2.7}$$

$$\overset{\circ}{C}{}^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\epsilon}(\dot\partial_\mu g_{\nu\epsilon} + \dot\partial_\nu g_{\mu\epsilon} - \dot\partial_\epsilon g_{\mu\nu}). \tag{2.8}$$

This connection will be referred to as the Riemannian $d$-connection.

# 3 $d$-Parallelizable manifolds (GAP-spaces)

The Riemannian $d$-connection mentioned in Theorem 2.6 plays the key role in our generalization of the AP-space, which, as will be revealed, appears natural. However, it is to be noted that the close resemblance of the two spaces is deceptive; as they are similar in form. However, the extra degrees of freedom in the generalized AP-space makes it richer in content and different in its geometric structure (see Remark 3.6).

We start with the concept of $d$-parallelizable manifolds.

**Definition 3.1.** An $n$-dimensional manifold $M$ is called d-parallelizable, or generalized absolute parallelism space (GAP-space), if the pull-back bundle $\pi^{-1}(TM)$ admits $n$ global linearly independent sections ($\pi$-vector fields) $\underset{i}{\lambda}(x,\ y)$, $i = 1, ..., n$.

If $\underset{i}{\lambda} = (\underset{i}{\lambda}{}^\alpha)$, $\alpha = 1, ..., n$, then

$$\underset{i}{\lambda}{}^\alpha \underset{i}{\lambda}_\beta = \delta^\alpha_\beta, \quad \underset{i}{\lambda}{}^\alpha \underset{j}{\lambda}_\alpha = \delta_{ij}, \tag{3.1}$$

where $(\underset{i}{\lambda}_\alpha)$ denotes the inverse of the matrix $(\underset{i}{\lambda}{}^\alpha)$.

Einstein summation convention is applied on both Latin (mesh) indices and Greek (world) indices, where all Latin indices are written in a lower position.

In the sequel, we will simply use the symbol $\lambda$ (without a mesh index) to denote any one of the vector fields $\underset{i}{\lambda}$ $(i = 1, ..., n)$ and in most cases, when mesh indices appear they will be in pairs, meaning summation.

We shall often use the expression GAP-space (resp. GAP-geometry) instead of $d$-parallelizable manifold (resp. geometry of $d$-parallelizable manifolds) for its typographical simplicity.

**Theorem 3.2.** *A GAP-space is a generalized Lagrange space.*

In fact, the covariant tensor field $g_{\mu\nu}(x,y)$ of order 2 given by

$$g_{\mu\nu}(x,y) := \underset{i}{\lambda}_{\mu}\,\underset{i}{\lambda}_{\nu}, \tag{3.2}$$

defines a metric in the pull-back bundle $\pi^{-1}(TM)$ with inverse given by

$$g^{\mu\nu}(x,y) = \underset{i}{\lambda}^{\mu}\,\underset{i}{\lambda}^{\nu} \tag{3.3}$$

Assume that $M$ is a GAP-space equipped with a non-linear connection $N^{\alpha}_{\mu}$. By Theorem 2.6, there exists on $(M,g)$ a unique torsion-free metrical $d$-connection $\overset{\circ}{D} = (N^{\alpha}_{\mu}, \overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu}, \overset{\circ}{C}{}^{\alpha}_{\mu\nu})$ (the Riemannian $d$-connection). We define another $d$-connection $D = (N^{\alpha}_{\mu}, \Gamma^{\alpha}_{\mu\nu}, C^{\alpha}_{\mu\nu})$ in terms of $\overset{\circ}{D}$ by:

$$\Gamma^{\alpha}_{\mu\nu} := \overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu} + \underset{i}{\lambda}^{\alpha}\,\underset{i}{\lambda}_{\mu\,\overset{\circ}{|}\,\nu}, \tag{3.4}$$

$$C^{\alpha}_{\mu\nu} := \overset{\circ}{C}{}^{\alpha}_{\mu\nu} + \underset{i}{\lambda}^{\alpha}\,\underset{i}{\lambda}_{\mu\,\overset{\circ}{||}\,\nu}. \tag{3.5}$$

Here, "$\overset{\circ}{|}$" and "$\overset{\circ}{||}$" denote the $h$- and $v$-covariant derivatives with respect to the Riemannian $d$-connection $\overset{\circ}{D}$. If "$|$" and "$||$" denote the $h$- and $v$-covariant derivatives with respect to the $d$-connection $D$, then

$$\lambda^{\alpha}{}_{|\mu} = 0, \quad \lambda^{\alpha}{}_{||\mu} = 0. \tag{3.6}$$

This can be shown as follows: $\lambda^{\alpha}{}_{|\mu} = \delta_{\mu}\lambda^{\alpha} + \lambda^{\epsilon}\Gamma^{\alpha}_{\epsilon\mu} = \delta_{\mu}\lambda^{\alpha} + \lambda^{\epsilon}(\overset{\circ}{\Gamma}{}^{\alpha}_{\epsilon\mu} + \underset{j}{\lambda}^{\alpha}\,\underset{j}{\lambda}_{\epsilon\,\overset{\circ}{|}\,\mu}) = (\delta_{\mu}\lambda^{\alpha} + \lambda^{\epsilon}\overset{\circ}{\Gamma}{}^{\alpha}_{\epsilon\mu}) - \underset{j}{\lambda}^{\alpha}{}_{\overset{\circ}{|}\,\mu}(\underset{i}{\lambda}^{\epsilon}\,\underset{j}{\lambda}_{\epsilon}) = 0$. In exactly the same way, it can be shown that $\lambda^{\alpha}{}_{||\mu} = 0$. Hence, we obtain the following

**Theorem 3.3.** *Let $(M, \underset{i}{\lambda}(x,y))$ be a GAP-space equipped with a non-linear connection $N^{\alpha}_{\mu}$. There exists a unique $d$-connection $D = (N^{\alpha}_{\mu}, \Gamma^{\alpha}_{\mu\nu}, C^{\alpha}_{\mu\nu})$, such that $\lambda^{\alpha}{}_{|\mu} = \lambda^{\alpha}{}_{||\mu} = 0$. This connection is given by $N^{\alpha}_{\beta}$, (3.4) and (3.5). Consequently, $D$ is metrical: $g_{\mu\nu|\sigma}\, g_{\mu\nu||\sigma} = 0$.*

This connection will be referred to as the canonical $d$-connection.

It is to be noted that relations (3.6) are in accordance with the classical AP-geometry in which the covariant derivative of the vector fields $\lambda$ with respect to the canonical connection $\Gamma^{\alpha}_{\mu\nu} = \underset{i}{\lambda}^{\alpha}(\partial_{\nu}\,\underset{i}{\lambda}_{\mu})$ vanishes [15].

**Theorem 3.4.** *Let $(M, \underset{i}{\lambda}(x,y))$ be a $d$-parallelizable manifold equipped with a non-linear connection $N^{\alpha}_{\mu}$. The canonical $d$-connection $D = (N^{\alpha}_{\mu}, \Gamma^{\alpha}_{\mu\nu}, C^{\alpha}_{\mu\nu})$ is explicitly expressed in terms of $\lambda$ in the form*

$$\Gamma^{\alpha}_{\mu\nu} = \underset{i}{\lambda}^{\alpha}(\delta_{\nu}\,\underset{i}{\lambda}_{\mu}), \quad C^{\alpha}_{\mu\nu} = \underset{i}{\lambda}^{\alpha}(\dot{\partial}_{\nu}\,\underset{i}{\lambda}_{\mu}). \tag{3.7}$$

*Proof.* Since $\lambda^\alpha{}_{|\nu} = 0$, we have $\delta_\nu \lambda^\alpha = -\lambda^\epsilon \Gamma^\alpha_{\epsilon\nu}$. Multiplying both sides by $\lambda_\mu$, taking into account the fact that $\overset{\lambda}{i}{}^\alpha \overset{\lambda}{i}{}_\mu = \delta^\alpha_\mu$, we get $\Gamma^\alpha_{\mu\nu} = -\overset{\lambda}{i}{}_\mu(\delta_\nu \overset{\lambda}{i}{}^\alpha) = \overset{\lambda}{i}{}^\alpha(\delta_\nu \overset{\lambda}{i}{}_\mu)$. The proof of the second relation is exactly similar and we omit it. □

It is to be noted that the components of the canonical $d$-connection are similar in form to the components of the canonical connection in the classical AP-context [15], noting that $\partial_\nu$ is replaced by $\delta_\nu$ (for the $h$-counterpart) and by $\dot\partial_\nu$ (for the $v$-counterpart) respectively (See Table 1). The above expressions for the canonical connection seem therefore like a natural generalization of the classical AP case.

By (3.4) and (3.5), in view of the above theorem, we have the following

**Corollary 3.5.** *The Reimannian d-connection* $\overset{\circ}{D} = (N^\alpha_\mu, \overset{\circ}{\Gamma}{}^\alpha_{\mu\nu}, \overset{\circ}{C}{}^\alpha_{\mu\nu})$ *is explicitly expressed in terms of* $\overset{\lambda}{i}$ *in the form*

$$\overset{\circ}{\Gamma}{}^\alpha_{\mu\nu} = \overset{\lambda}{i}{}^\alpha(\delta_\nu \overset{\lambda}{i}{}_\mu - \overset{\lambda}{i}{}_{\mu\,|}{}^{\circ}{}_\nu), \quad \overset{\circ}{C}{}^\alpha_{\mu\nu} = \overset{\lambda}{i}{}^\alpha(\dot\partial_\nu \overset{\lambda}{i}{}_\mu - \overset{\lambda}{i}{}_{\mu\,||}{}^{\circ}{}_\nu). \tag{3.8}$$

$A^\circ$. s a result of the dependence of $\lambda$ on the velocity vector $y$, the $n^3$ functions $\overset{\lambda}{i}{}^\alpha(\partial_\nu \overset{\lambda}{i}{}_\mu)$, as opposed to the classical AP-space, do not transform as the coefficients of a linear connection, but transform according to the rule

$$\overset{\lambda}{i}{}^{\alpha'}(\partial_{\nu'} \overset{\lambda}{i}{}_{\mu'}) = p^{\alpha'}_\alpha p^\mu_{\mu'} p^\nu_{\nu'} \overset{\lambda}{i}{}^\alpha(\partial_\nu \overset{\lambda}{i}{}_\mu) + p^{\alpha'}_\epsilon p^\epsilon_{\mu'\nu'} + p^{\alpha'}_\alpha p^\mu_{\mu'} p^\nu_{\nu'\epsilon'} y^{\epsilon'} C^\alpha_{\mu\nu}. \tag{3.9}$$

Similarly, it can be shown that, in general, tensors in the context of the classical AP-space do not transform like tensors in the wider context of the GAP-space; their dependence on the velocity vector $y$ spoils their tensor character. In other words, tensors in the classical AP-context do not necessarily behave like tensors *when they are regarded as functions of position $x$ and velocity vector $y$*. This means that though the classical AP-space and the GAP-space appear similar in form, they differ radically in their geometric structures. We now introduce some tensors that will prove useful later on. Let

$$\gamma^\alpha_{\mu\nu} := \overset{\lambda}{i}{}^\alpha \overset{\lambda}{i}{}_{\mu\,|}{}^{\circ}{}_\nu = \Gamma^\alpha_{\mu\nu} - \overset{\circ}{\Gamma}{}^\alpha_{\mu\nu}, \qquad G^\alpha_{\mu\nu} := \overset{\lambda}{i}{}^\alpha \overset{\lambda}{i}{}_{\mu\,||}{}^{\circ}{}_\nu = C^\alpha_{\mu\nu} - \overset{\circ}{C}{}^\alpha_{\mu\nu}. \tag{3.10}$$

In analogy to the AP-space, we refer to $\gamma^\alpha_{\mu\nu}$ and $G^\alpha_{\mu\nu}$ as the $h$- and $v$-contortion tensors respectively.

Let

$$\Lambda^\alpha_{\mu\nu} := \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu} = \gamma^\alpha_{\mu\nu} - \gamma^\alpha_{\nu\mu}. \tag{3.11}$$

be the torsion tensor of the canonical connection $\Gamma^\alpha_{\mu\nu}$ and

$$\Omega^\alpha_{\mu\nu} := \gamma^\alpha_{\mu\nu} + \gamma^\alpha_{\nu\mu}. \tag{3.12}$$

Similarly, let

$$T^\alpha_{\mu\nu} := C^\alpha_{\mu\nu} - C^\alpha_{\nu\mu} = G^\alpha_{\mu\nu} - G^\alpha_{\nu\mu} \tag{3.13}$$

be what we may call the torsion tensor of $C^\alpha_{\mu\nu}$ and

$$D^\alpha_{\mu\nu} := G^\alpha_{\mu\nu} + G^\alpha_{\nu\mu}. \tag{3.14}$$

Now, if $\gamma_{\sigma\mu\nu} := g_{\epsilon\sigma}\gamma^\epsilon_{\mu\nu}$ and $G_{\sigma\mu\nu} := g_{\epsilon\sigma} G^\epsilon_{\mu\nu}$, then $\gamma_{\sigma\mu\nu}$ and $G_{\sigma\mu\nu}$ are skew symmetric in the first pair of indices. This, in turn, implies that

$$\gamma^\epsilon_{\epsilon\nu} = G^\epsilon_{\epsilon\nu} = 0. \tag{3.15}$$

Hence, if

$$\beta_\mu := \gamma^\epsilon_{\mu\epsilon}, \quad B_\mu := G^\epsilon_{\mu\epsilon},$$

then

$$\Lambda^\epsilon_{\mu\epsilon} = \gamma^\epsilon_{\mu\epsilon} = \beta_\mu, \quad T^\epsilon_{\mu\epsilon} = G^\epsilon_{\mu\epsilon} = B_\mu. \tag{3.16}$$

**Table 1. Comparison between the classical AP-geometry and the GAP-geometry**

| | Classical AP-geometry | GAP-geometry |
|---|---|---|
| Building blocks | $\lambda^{\alpha}(x)$ | $\lambda^{\alpha}(x,y)$ |
| Metric | $g_{\mu\nu}(x) = \underset{i}{\lambda_{\mu}}(x)\underset{i}{\lambda_{\nu}}(x)$ | $g_{\mu\nu}(x,y) = \underset{i}{\lambda_{\mu}}(x,y)\underset{i}{\lambda_{\nu}}(x,y)$ |
| Riemannian connection | $\overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\epsilon}\{\partial_{\mu}g_{\nu\epsilon} + \partial_{\nu}g_{\mu\epsilon} + \partial_{\epsilon}g_{\mu\nu}\}$ | $\overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\epsilon}\{\delta_{\mu}g_{\nu\epsilon} + \delta_{\nu}g_{\mu\epsilon} + \delta_{\epsilon}g_{\mu\nu}\}$<br>$\overset{\circ}{C}{}^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\epsilon}\{\dot{\partial}_{\mu}g_{\nu\epsilon} + \dot{\partial}_{\nu}g_{\mu\epsilon} + \dot{\partial}_{\epsilon}g_{\mu\nu}\}$ |
| Canonical connection | $\Gamma^{\alpha}_{\mu\nu} = \underset{i}{\lambda^{\alpha}}(\partial_{\nu}\underset{i}{\lambda_{\mu}})$ | $\Gamma^{\alpha}_{\mu\nu} = \underset{i}{\lambda^{\alpha}}(\delta_{\nu}\underset{i}{\lambda_{\mu}})$ ($h$-counterpart)<br>$C^{\alpha}_{\mu\nu} = \underset{i}{\lambda^{\alpha}}(\dot{\partial_{\nu}}\underset{i}{\lambda_{\mu}})$ ($v$-counterpart) |
| AP-condition | $\lambda^{\alpha}{}_{\mid\mu} = 0$ | $\lambda^{\alpha}{}_{\mid\mu} = 0$ ($h$-covariant derivative)<br>$\lambda^{\alpha}{}_{\mid\mid\mu} = 0$ ($v$-covariant derivative) |
| Torsion | $\Lambda^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu}$ | $\Lambda^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu}$ ($h$-counterpart)<br>$T^{\alpha}_{\mu\nu} = C^{\alpha}_{\mu\nu} - C^{\alpha}_{\nu\mu}$ ($v$-counterpart) |
| Contorsion | $\gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu}$ | $\gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu}$ ($h$-counterpart)<br>$G^{\alpha}_{\mu\nu} = C^{\alpha}_{\mu\nu} - \overset{\circ}{C}{}^{\alpha}_{\mu\nu}$ ($v$-counterpart) |
| Basic vector | $\beta_{\mu} = \Lambda^{\alpha}_{\mu\alpha} = \gamma^{\alpha}_{\mu\alpha}$ | $\beta_{\mu} = \Lambda^{\alpha}_{\mu\alpha} = \gamma^{\alpha}_{\mu\alpha}$ ($h$-counterpart)<br>$B_{\mu} = T^{\alpha}_{\mu\alpha} = G^{\alpha}_{\mu\alpha}$ ($v$-counterpart) |

Finally, it can be shown, in analogy to the classical AP-space [3], that the contortion tensors $\gamma_{\mu\nu\sigma}$ and $G_{\mu\nu\sigma}$ can be expressed in terms of the torsion tensors in the form

$$\gamma_{\mu\nu\sigma} = \frac{1}{2}(\Lambda_{\mu\nu\sigma} + \Lambda_{\sigma\nu\mu} + \Lambda_{\nu\sigma\mu}) \tag{3.17}$$

$$G_{\mu\nu\sigma} = \frac{1}{2}(T_{\mu\nu\sigma} + T_{\sigma\nu\mu} + T_{\nu\sigma\mu}), \tag{3.18}$$

where $\Lambda_{\mu\nu\sigma} := g_{\epsilon\mu}\Lambda^{\epsilon}_{\nu\sigma}$ and $T_{\mu\nu\sigma} := g_{\epsilon\mu}T^{\epsilon}_{\nu\sigma}$. It is clear by (3.11), (3.13), (3.17) and (3.18) that the torsion tensors vanish if and only if the contortion tensors vanish.

Table 1 gives a comparison between the fundamental geometric objects in the classical AP-geometry and the GAP-geometry. Similar objects of the two spaces will be denoted by the same symbol. As previously mentioned, "$h$" stands for "horizontal" whereas "$v$" stands for "vertical".

# 4 Curvature tensors in Generalized AP-space

Owing to the existence of two types of covariant derivatives with respect to the canonical connection $D$, we have essentially three commutation formulae and consequently three curvature tensors.

**Lemma 4.1.** *Let* $[\delta_\sigma, \delta_\mu] := \delta_\sigma\delta_\mu - \delta_\mu\delta_\sigma$ *and let* $[\delta_\sigma, \dot\partial_\mu]$ *be similarly defined. Then*

$$[\delta_\sigma, \delta_\mu] = R^\epsilon_{\sigma\mu}\ \dot\partial_\epsilon, \quad [\delta_\sigma, \dot\partial_\mu] = (\dot\partial_\mu N^\epsilon_\sigma)\ \dot\partial_\epsilon, \tag{4.1}$$

*where* $R^\alpha_{\sigma\mu} := \delta_\mu N^\alpha_\sigma - \delta_\sigma N^\alpha_\mu$ *is the curvature tensor of the non-linear connection* $N^\alpha_\mu$.

**Theorem 4.2.** *The three commutation formulae of* $\lambda^\alpha_{\ i}$ *corresponding to the canonical connection* $D = (N^\alpha_\mu, \Gamma^\alpha_{\mu\nu}, C^\alpha_{\mu\nu})$ *are given by*

**(a)** $\lambda^\alpha{}_{|\mu\sigma} - \lambda^\alpha{}_{|\sigma\mu} = \lambda^\epsilon\, R^\alpha_{\epsilon\mu\sigma} + \lambda^\alpha{}_{|\epsilon}\, \Lambda^\epsilon_{\sigma\mu} + \lambda^\alpha{}_{||\epsilon}\, R^\epsilon_{\sigma\mu}$

**(b)** $\lambda^\alpha{}_{||\mu\sigma} - \lambda^\alpha{}_{||\sigma\mu} = \lambda^\epsilon S^\alpha_{\epsilon\mu\sigma} + \lambda^\alpha{}_{||\epsilon}\, T^\epsilon_{\sigma\mu}$

**(c)** $\lambda^\alpha{}_{||\mu|\sigma} - \lambda^\alpha{}_{|\sigma||\mu} = \lambda^\epsilon P^\alpha_{\epsilon\mu\sigma} + \lambda^\alpha{}_{|\epsilon}\, C^\epsilon_{\sigma\mu} + \ \lambda^\alpha{}_{||\epsilon}\, P^\epsilon_{\sigma\mu}$,

*where*

$$\begin{aligned}
R^\alpha_{\nu\mu\sigma} &:= (\delta_\sigma\Gamma^\alpha_{\nu\mu} - \delta_\mu\Gamma^\alpha_{\nu\sigma}) + (\Gamma^\epsilon_{\nu\mu}\Gamma^\alpha_{\epsilon\sigma} - \Gamma^\epsilon_{\nu\sigma}\Gamma^\alpha_{\epsilon\mu}) + L^\alpha_{\nu\mu\sigma}, && \textit{(h-curvature)}\\
S^\alpha_{\nu\mu\sigma} &:= \dot\partial_\sigma C^\alpha_{\nu\mu} - \dot\partial_\mu C^\alpha_{\nu\sigma} + C^\epsilon_{\nu\mu}C^\alpha_{\epsilon\sigma} - C^\epsilon_{\nu\sigma}C^\alpha_{\epsilon\mu}, && \textit{(v-curvature)}\\
P^\alpha_{\nu\mu\sigma} &:= C^\alpha_{\nu\mu|\sigma} - \dot\partial_\mu\Gamma^\alpha_{\nu\sigma} - P^\epsilon_{\sigma\mu}C^\alpha_{\nu\epsilon}, && \textit{(hv-curvature)}
\end{aligned}$$

*given that* $L^\alpha_{\nu\mu\sigma} := C^\alpha_{\nu\epsilon}\, R^\epsilon_{\mu\sigma}$ *and* $P^\nu_{\sigma\mu} := \dot\partial_\mu N^\nu_\sigma - \Gamma^\nu_{\mu\sigma}$.

A direct consequence of the above commutation formulae, together with the fact that $\lambda^\alpha{}_{|\mu} = \lambda^\alpha{}_{||\mu} = 0$, is the following

**Corollary 4.3.** *The three curvature tensors* $R^\alpha_{\nu\mu\sigma}$, $S^\alpha_{\nu\mu\sigma}$ *and* $P^\alpha_{\nu\mu\sigma}$ *of the canonical connection* $D = (N^\alpha_\mu, \Gamma^\alpha_{\mu\nu}, C^\alpha_{\mu\nu})$ *vanish identically.*

It is to be noted that the above result is a natural generalization of the corresponding result of the classical AP-geometry [15].

The Bianchi identities [4] for the canonical $d$-connection $(N^\alpha_\mu, \Gamma^\alpha_{\mu\nu}, C^\alpha_{\mu\nu})$ gives

**Proposition 4.4.** *The following identities hold*

**(a)** $\mathfrak{S}_{\nu,\mu,\sigma}\Lambda^\alpha_{\nu\mu|\sigma} = \mathfrak{S}_{\nu,\mu,\sigma}(\Lambda^\alpha_{\mu\epsilon}\Lambda^\epsilon_{\nu\sigma} + L^\alpha_{\mu\nu\sigma})$

**(b)** $\mathfrak{S}_{\nu,\mu,\sigma}T^\alpha_{\nu\mu||\sigma} = \mathfrak{S}_{\nu,\mu,\sigma}(T^\alpha_{\mu\epsilon}T^\epsilon_{\nu\sigma})$,
*where* $\mathfrak{S}_{\nu,\mu,\sigma}$ *denotes a cyclic permutation on* $\nu, \mu, \sigma$.

**Corollary 4.5.** *The following identities hold:*

**(a)** $\Lambda^\epsilon_{\mu\nu|\epsilon} = \beta_{\mu|\nu} - \beta_{\nu|\mu} + \beta_\epsilon\Lambda^\epsilon_{\mu\nu} + \mathfrak{S}_{\epsilon,\nu,\mu}L^\epsilon_{\epsilon\nu\mu}$.

**(b)** $T^\epsilon_{\mu\nu||\epsilon} = B_{\mu||\nu} - B_{\nu||\mu} + B_\epsilon T^\epsilon_{\mu\nu}$,

*Proof.* Both identities follow by contracting the indices $\alpha$ and $\sigma$ in the identities (a) and (b) of Proposition 4.4, taking into account that $\beta_\mu = \Lambda^\epsilon_{\mu\epsilon}$, $B_\mu = T^\epsilon_{\mu\epsilon}$ and $L^\alpha_{\mu\nu\sigma} = -L^\alpha_{\mu\sigma\nu}$. □

In addition to the Riemannian and the canonical $d$-connections, our space admits at least two other natural $d$-connections. In analogy to the classical AP-space, we define the dual $d$-connection $\widetilde{D} = (N^\alpha_\mu, \widetilde{\Gamma}^\alpha_{\mu\nu}, \widetilde{C}^\alpha_{\mu\nu})$ by

$$\widetilde{\Gamma}^\alpha_{\mu\nu} := \Gamma^\alpha_{\nu\mu}, \quad \widetilde{C}^\alpha_{\mu\nu} := C^\alpha_{\nu\mu} \tag{4.2}$$

and the symmetric $d-$connection $\widehat{D} = (N^\alpha_\mu, \widehat{\Gamma}^\alpha_{\mu\nu}, \widehat{C}^\alpha_{\mu\nu})$ by

$$\widehat{\Gamma}^\alpha_{\mu\nu} := \frac{1}{2}(\Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\nu\mu}), \quad \widehat{C}^\alpha_{\mu\nu} := \frac{1}{2}(C^\alpha_{\mu\nu} + C^\alpha_{\nu\mu}). \tag{4.3}$$

Covariant differentiation with respect to $\widetilde{\Gamma}^\alpha_{\mu\nu}$ and $\widehat{\Gamma}^\alpha_{\mu\nu}$ will be denoted by "$\widetilde{|}$" and "$\widehat{|}$" respectively.

Now, corresonding to each of the four $d$-connections there are three curvature tensors. Therefore, we have a total of twelve curvature tensors three of which, as already mentioned, vanish identically. The vanishing of the curvature tensors of the canonical $d$-connection allows us to express, in a relatively compact form, six of the other curvature tensors (the $h$- and $v$-curvature tensors) corresponding to the Riemannian, symmetric and the dual $d$-connections. These curvature tensors are expressed in terms of the torsion tensors $\Lambda^\alpha_{\mu\nu}$, $T^\alpha_{\mu\nu}$ and their covariant derivatives with respect to the canonical $d$-connection, together with the curvature $R^\alpha_{\mu\nu}$ of the non-linear connection $N^\alpha_\mu$. The other three $hv$-curvature tensors are calculated, though their expressions are more complicated. This is to be expected since the expression obtained for the $hv$-curvature tensor of the canonical $d$-connection lacks the symmetry properties enjoyed by the $h$- and $v$-curvature tensors.

**Theorem 4.6.** *The h-, v- and hv-curvature tensors of the dual d-connection* $\widetilde{D} = (N^\alpha_\mu, \widetilde{\Gamma}^\alpha_{\mu\nu}, \widetilde{C}^\alpha_{\mu\nu})$ *can be expressed in the form:*

**(a)** $\widetilde{R}^\alpha_{\mu\sigma\nu} = \Lambda^\alpha_{\sigma\nu|\mu} + C^\alpha_{\epsilon\mu} R^\epsilon_{\sigma\nu} + L^\alpha_{\sigma\nu\mu} + L^\alpha_{\nu\mu\sigma}.$

**(b)** $\widetilde{S}^\alpha_{\mu\sigma\nu} = T^\alpha_{\sigma\nu||\mu}.$

**(c)** $\widetilde{P}^\alpha_{\nu\mu\sigma} = T^\alpha_{\mu\nu|\sigma} - \Lambda^\alpha_{\sigma\nu||\mu} + T^\epsilon_{\mu\nu}\Lambda^\alpha_{\sigma\epsilon} - T^\alpha_{\mu\epsilon}\Lambda^\epsilon_{\sigma\nu} - \Lambda^\alpha_{\epsilon\nu}C^\epsilon_{\sigma\mu} - P^\epsilon_{\sigma\mu}T^\alpha_{\epsilon\nu}.$

*The corresponding curvature tensors of the symmetric d-connection* $\widehat{D} = (N^\alpha_\mu, \widehat{\Gamma}^\alpha_{\mu\nu}, \widehat{C}^\alpha_{\mu\nu})$ *can be expressed in the form:*

**(d)** $\widehat{R}^\alpha_{\mu\sigma\nu} = \frac{1}{2}(\Lambda^\alpha_{\mu\nu|\sigma} - \Lambda^\alpha_{\mu\sigma|\nu}) + \frac{1}{4}(\Lambda^\epsilon_{\mu\nu}\Lambda^\alpha_{\sigma\epsilon} - \Lambda^\epsilon_{\mu\sigma}\Lambda^\alpha_{\nu\epsilon}) + \frac{1}{2}(\Lambda^\epsilon_{\sigma\nu}\Lambda^\alpha_{\epsilon\mu}) + \frac{1}{2}(T^\alpha_{\epsilon\mu}R^\epsilon_{\sigma\nu}).$

**(e)** $\widehat{S}^\alpha_{\mu\sigma\nu} = \frac{1}{2}(T^\alpha_{\mu\nu||\sigma} - T^\alpha_{\mu\sigma||\nu}) + \frac{1}{4}(T^\epsilon_{\mu\nu}T^\alpha_{\sigma\epsilon} - T^\epsilon_{\mu\sigma}T^\alpha_{\nu\epsilon}) + \frac{1}{2}(T^\epsilon_{\sigma\nu}T^\alpha_{\epsilon\mu}).$

**(f)** $\widehat{P}^\alpha_{\nu\mu\sigma} = \frac{1}{2}\ (\Lambda^\alpha_{\mu\nu|\sigma} - \Lambda^\alpha_{\sigma\nu||\mu}) + \frac{1}{4}\ \Lambda^\epsilon_{\sigma\mu}T^\alpha_{\epsilon\nu} - \frac{1}{2}\ \Lambda^\alpha_{\epsilon\nu}C^\epsilon_{\sigma\mu} + \frac{1}{4}\ \mathfrak{S}_{\mu,\nu,\sigma}\Lambda^\epsilon_{\mu\nu}\Lambda^\alpha_{\sigma\epsilon} - \frac{1}{2}\ P^\epsilon_{\sigma\mu}T^\alpha_{\epsilon\nu}.$

*The corresponding curvature tensors of the Riemannian d-connection* $\overset{\circ}{D} = (N^\alpha_\mu, \overset{\circ}{\Gamma}{}^\alpha_{\mu\nu}, \overset{\circ}{C}{}^\alpha_{\mu\nu})$ *can be expressed in the form*

**(g)** $\overset{\circ}{R}{}^\alpha_{\mu\sigma\nu} = \gamma^\alpha_{\mu\nu|\sigma} - \gamma^\alpha_{\mu\sigma|\nu} + \gamma^\epsilon_{\mu\sigma}\gamma^\alpha_{\epsilon\nu} - \gamma^\epsilon_{\mu\nu}\gamma^\alpha_{\epsilon\sigma} + \gamma^\alpha_{\mu\epsilon}\Lambda^\epsilon_{\nu\sigma} + G^\alpha_{\mu\epsilon}R^\epsilon_{\nu\sigma}.$

**(h)** $\overset{\circ}{S}{}^{\alpha}_{\mu\sigma\nu} = G^{\alpha}_{\mu\nu||\sigma} - G^{\alpha}_{\mu\sigma||\nu} + G^{\epsilon}_{\mu\sigma}G^{\alpha}_{\epsilon\nu} - G^{\epsilon}_{\mu\nu}G^{\alpha}_{\epsilon\sigma} + G^{\alpha}_{\mu\epsilon}T^{\epsilon}_{\nu\sigma}$.

**(i)** $\overset{\circ}{P}{}^{\alpha}_{\nu\mu\sigma} = \dot{\partial}_u\gamma^{\alpha}_{\nu\sigma} - G^{\alpha}_{\nu\mu|\sigma} + (G^{\epsilon}_{\nu\mu} - C^{\epsilon}_{\nu\mu})\gamma^{\alpha}_{\epsilon\sigma} - (G^{\alpha}_{\epsilon\mu} - C^{\alpha}_{\epsilon\mu})\gamma^{\epsilon}_{\nu\sigma} + P^{\epsilon}_{\sigma\mu}G^{\alpha}_{\nu\epsilon}$.

*Proof.* We prove (a) and (c) only. The proof of the other parts is similar.

**(a)** We have

$$\begin{aligned}
\widetilde{R}^{\alpha}_{\mu\sigma\nu} &= \delta_\nu\widetilde{\Gamma}^{\alpha}_{\mu\sigma} - \delta_\sigma\widetilde{\Gamma}^{\alpha}_{\mu\nu} + \widetilde{\Gamma}^{\epsilon}_{\mu\sigma}\widetilde{\Gamma}^{\alpha}_{\epsilon\nu} - \widetilde{\Gamma}^{\epsilon}_{\mu\nu}\widetilde{\Gamma}^{\alpha}_{\epsilon\sigma} + \widetilde{C}^{\alpha}_{\mu\epsilon}R^{\epsilon}_{\sigma\nu} \\
&= \delta_\nu\Gamma^{\alpha}_{\sigma\mu} - \delta_\sigma\Gamma^{\alpha}_{\nu\mu} + \Gamma^{\epsilon}_{\sigma\mu}\Gamma^{\alpha}_{\nu\epsilon} - \Gamma^{\epsilon}_{\nu\mu}\Gamma^{\alpha}_{\sigma\epsilon} + C^{\alpha}_{\epsilon\mu}R^{\epsilon}_{\sigma\nu} \\
&= \{\delta_\nu\Gamma^{\alpha}_{\sigma\mu} + \Gamma^{\epsilon}_{\sigma\mu}(\Lambda^{\alpha}_{\nu\epsilon} + \Gamma^{\alpha}_{\epsilon\nu})\} - \{\delta_\sigma\Gamma^{\alpha}_{\nu\mu} + \Gamma^{\epsilon}_{\nu\mu}(\Lambda^{\alpha}_{\sigma\epsilon} + \Gamma^{\alpha}_{\epsilon\sigma})\} \\
&\quad + C^{\alpha}_{\epsilon\mu}R^{\epsilon}_{\sigma\nu} \\
&= (\delta_\nu\Gamma^{\alpha}_{\sigma\mu} + \Gamma^{\epsilon}_{\sigma\mu}\Gamma^{\alpha}_{\epsilon\nu}) - (\delta_\sigma\Gamma^{\alpha}_{\nu\mu} + \Gamma^{\epsilon}_{\nu\mu}\Gamma^{\alpha}_{\epsilon\sigma}) - (\Gamma^{\epsilon}_{\sigma\mu}\Lambda^{\alpha}_{\epsilon\nu} + \Gamma^{\epsilon}_{\nu\mu}\Lambda^{\alpha}_{\sigma\epsilon}) \\
&\quad + C^{\alpha}_{\epsilon\mu}R^{\epsilon}_{\sigma\nu} \\
&= (R^{\alpha}_{\sigma\mu\nu} - C^{\alpha}_{\sigma\epsilon}R^{\epsilon}_{\mu\nu} + \delta_\mu\Gamma^{\alpha}_{\sigma\nu} + \Gamma^{\epsilon}_{\sigma\nu}\Gamma^{\alpha}_{\epsilon\mu}) - (R^{\alpha}_{\nu\mu\sigma} - C^{\alpha}_{\nu\epsilon}R^{\epsilon}_{\mu\sigma} \\
&\quad + \delta_\mu\Gamma^{\alpha}_{\nu\sigma} + \Gamma^{\epsilon}_{\nu\sigma}\Gamma^{\alpha}_{\epsilon\mu}) - (\Gamma^{\epsilon}_{\sigma\mu}\Lambda^{\alpha}_{\epsilon\nu} + \Gamma^{\epsilon}_{\nu\mu}\Lambda^{\alpha}_{\sigma\epsilon}) + C^{\alpha}_{\epsilon\mu}R^{\epsilon}_{\sigma\nu}. \\
&= \delta_\mu\Lambda^{\alpha}_{\sigma\nu} + \Gamma^{\alpha}_{\epsilon\mu}\Lambda^{\epsilon}_{\sigma\nu} - \Gamma^{\epsilon}_{\sigma\mu}\Lambda^{\alpha}_{\epsilon\nu} - \Gamma^{\epsilon}_{\nu\mu}\Lambda^{\alpha}_{\sigma\epsilon} + C^{\alpha}_{\epsilon\mu}R^{\epsilon}_{\sigma\nu} + C^{\alpha}_{\sigma\epsilon}R^{\epsilon}_{\nu\mu} + C^{\alpha}_{\nu\epsilon}R^{\epsilon}_{\mu\sigma} \\
&= \Lambda^{\alpha}_{\sigma\nu|\mu} + C^{\alpha}_{\epsilon\mu}R^{\epsilon}_{\sigma\nu} + L^{\alpha}_{\sigma\nu\mu} + L^{\alpha}_{\nu\mu\sigma}.
\end{aligned}$$

**(c)** We have

$$\begin{aligned}
\widetilde{P}^{\alpha}_{\nu\mu\sigma} &= C^{\alpha}_{\mu\widetilde{\nu|\sigma}} - \dot{\partial}_\mu\Gamma^{\alpha}_{\sigma\nu} - (\dot{\partial}_\mu N^{\epsilon}_{\sigma} - \Gamma^{\epsilon}_{\sigma\mu})C^{\alpha}_{\epsilon\nu} \\
&= C^{\alpha}_{\nu\mu|\sigma} + (C^{\alpha}_{\mu\widetilde{\nu|\sigma}} - C^{\alpha}_{\nu\mu|\sigma}) - \dot{\partial}_\mu\Lambda^{\alpha}_{\sigma\nu} - \dot{\partial}_\mu\Gamma^{\alpha}_{\nu\sigma} - \dot{\partial}_\mu N^{\epsilon}_{\sigma}(T^{\alpha}_{\epsilon\nu} + C^{\alpha}_{\nu\epsilon}) \\
&\quad + (\Lambda^{\epsilon}_{\sigma\mu} + \Gamma^{\epsilon}_{\mu\sigma})(T^{\alpha}_{\epsilon\nu} + C^{\alpha}_{\nu\epsilon}) \\
&= P^{\alpha}_{\nu\mu\sigma} - (\dot{\partial}_\mu N^{\epsilon}_{\sigma} - \Gamma^{\epsilon}_{\mu\sigma})T^{\alpha}_{\epsilon\nu} - \dot{\partial}_\mu\Lambda^{\alpha}_{\sigma\nu} + \Lambda^{\epsilon}_{\sigma\mu}C^{\alpha}_{\epsilon\nu} + (C^{\alpha}_{\mu\widetilde{\nu|\sigma}} - C^{\alpha}_{\nu\mu|\sigma}) \\
&= (C^{\alpha}_{\mu\widetilde{\nu|\sigma}} - C^{\alpha}_{\nu\mu|\sigma}) + \Lambda^{\epsilon}_{\sigma\mu}C^{\alpha}_{\epsilon\nu} - \dot{\partial}_\mu\Lambda^{\alpha}_{\sigma\nu} - P^{\epsilon}_{\sigma\mu}T^{\alpha}_{\epsilon\nu} \\
&= T^{\alpha}_{\mu\nu|\sigma} + C^{\epsilon}_{\mu\nu}\Lambda^{\alpha}_{\sigma\epsilon} - C^{\alpha}_{\mu\epsilon}\Lambda^{\epsilon}_{\sigma\nu} - \dot{\partial}_\mu\Lambda^{\alpha}_{\sigma\nu} - P^{\epsilon}_{\sigma\mu}T^{\alpha}_{\epsilon\nu} \\
&= T^{\alpha}_{\mu\nu|\sigma} - \dot{\partial}_\mu\Lambda^{\alpha}_{\sigma\nu} + (T^{\epsilon}_{\mu\nu} + C^{\epsilon}_{\nu\mu})\Lambda^{\alpha}_{\sigma\epsilon} - (T^{\alpha}_{\mu\epsilon} + C^{\alpha}_{\epsilon\mu})\Lambda^{\epsilon}_{\sigma\nu} - P^{\epsilon}_{\sigma\mu}T^{\alpha}_{\epsilon\nu} \\
&= T^{\alpha}_{\mu\nu|\sigma} - \Lambda^{\alpha}_{\sigma\nu||\mu} + T^{\epsilon}_{\mu\nu}\Lambda^{\alpha}_{\sigma\epsilon} - T^{\alpha}_{\mu\epsilon}\Lambda^{\epsilon}_{\sigma\nu} - \Lambda^{\alpha}_{\epsilon\nu}C^{\epsilon}_{\sigma\mu} - P^{\epsilon}_{\sigma\mu}T^{\alpha}_{\epsilon\nu}.
\end{aligned}$$

which are the required formulae. □

## 5 Fundamental second rank tensors

Due to the importance of second order symmetric and skew-symmetric tensors in physical applications, we here list such tensors in Table 2 below. We regard these tensors as *fundamental* since their counterparts in the classical AP-context *play a key role in physical applications.* Moreover, in the AP-geometry, most second rank tensors which have physical significance can be expressed as a linear combination of these fundamental tensors. The Table is constructed as similar as possible to that given by Mikhail (cf. [5], Table 2), to facilitate comparison with the case of the classical AP-geometry which has many physical applications [14]. Corresponding "horizontal"

and "vertical" tensors are denoted by the same symbol with the "vertical" tensors barred. It is to be noted that all "vertical " tensors have no counterpart in the classical AP-context.

**Table 2. Summary of the fundamental symmetric and skew-symmetric second rank tensors**

| Horizontal | | Vertical | |
|---|---|---|---|
| Skew-Symmetric | Symmetric | Skew-Symmetric | Symmetric |
| $\xi_{\mu\nu} := \gamma_{\mu\nu}{}^{\alpha}{}_{\vert\alpha}$ | | $\overline{\xi}_{\mu\nu} := G_{\mu\nu}{}^{\alpha}{}_{\vert\alpha}$ | |
| $\gamma_{\mu\nu} := \beta_{\alpha}\gamma_{\mu\nu}{}^{\alpha}$ | | $\overline{\gamma}_{\mu\nu} := B_{\alpha}G_{\mu\nu}{}^{\alpha}$ | |
| $\eta_{\mu\nu} := \beta_{\epsilon}\,\Lambda^{\epsilon}_{\mu\nu}$ | $\phi_{\mu\nu} := \beta_{\epsilon}\,\Omega^{\epsilon}_{\mu\nu}$ | $\overline{\eta}_{\mu\nu} := B_{\epsilon}\,T^{\epsilon}_{\mu\nu}$ | $\overline{\phi}_{\mu\nu} := B_{\epsilon}\,D^{\epsilon}_{\mu\nu}$ |
| $\chi_{\mu\nu} := \Lambda^{\alpha}_{\mu\nu\vert\alpha}$ | $\psi_{\mu\nu} := \Omega^{\epsilon}_{\mu\nu\vert\epsilon}$ | $\overline{\chi}_{\mu\nu} := T^{\alpha}_{\mu\nu\vert\vert\alpha}$ | $\overline{\psi}_{\mu\nu} := D^{\alpha}_{\mu\nu\vert\vert\alpha}$ |
| $\epsilon_{\mu\nu} := \frac{1}{2}(\beta_{\mu\vert\nu} - \beta_{\nu\vert\mu})$ | $\theta_{\mu\nu} := \frac{1}{2}(\beta_{\mu\vert\nu} + \beta_{\nu\vert\mu})$ | $\overline{\epsilon}_{\mu\nu} := \frac{1}{2}(B_{\mu\vert\vert\nu} - B_{\nu\vert\vert\mu})$ | $\overline{\theta}_{\mu\nu} := \frac{1}{2}(B_{\mu\vert\vert\nu} + B_{\nu\vert\vert\mu})$ |
| $k_{\mu\nu} :=$ $\gamma^{\epsilon}_{\alpha\mu}\gamma^{\alpha}_{\nu\epsilon} - \gamma^{\epsilon}_{\mu\alpha}\gamma^{\alpha}_{\epsilon\nu}$ | $h_{\mu\nu} :=$ $\gamma^{\epsilon}_{\alpha\mu}\gamma^{\alpha}_{\nu\epsilon} + \gamma^{\epsilon}_{\mu\alpha}\gamma^{\alpha}_{\epsilon\nu}$ | $\overline{k}_{\mu\nu} :=$ $G^{\epsilon}_{\alpha\mu}G^{\alpha}_{\nu\epsilon} - G^{\epsilon}_{\mu\alpha}G^{\alpha}_{\epsilon\nu}$ | $\overline{h}_{\mu\nu} :=$ $G^{\epsilon}_{\alpha\mu}G^{\alpha}_{\nu\epsilon} + G^{\epsilon}_{\mu\alpha}G^{\alpha}_{\epsilon\nu}$ |
| | $\sigma_{\mu\nu} := \gamma^{\epsilon}_{\alpha\mu}\gamma^{\alpha}_{\epsilon\nu}$ | | $\overline{\sigma}_{\mu\nu} := G^{\epsilon}_{\alpha\mu}G^{\alpha}_{\epsilon\nu}$ |
| | $\omega_{\mu\nu} := \gamma^{\epsilon}_{\mu\alpha}\gamma^{\alpha}_{\nu\epsilon}$ | | $\overline{\omega}_{\mu\nu} := G^{\epsilon}_{\mu\alpha}G^{\alpha}_{\nu\epsilon}$ |
| | $\alpha_{\mu\nu} := \beta_{\mu}\beta_{\nu}$ | | $\overline{\alpha}_{\mu\nu} := B_{\mu}B_{\nu}$ |

Due to the metricity condition in Theorem 3.3, one can use the metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ to perform the operations of lowering and raising tensor indices under the $h$- and $v$- covariant derivatives relative to the canonical $d$-connection.

Thus, contraction with the metric tensor of the above fundamental tensors gives the following table of scalars:

**Table 3. Summary of the fundamental scalars**

| | | | | |
|---|---|---|---|---|
| **Horizontal** | $\alpha := \beta_{\mu}\beta^{\mu}$ | $\theta := \beta^{\mu}{}_{\vert\mu}$ | $\phi := \beta_{\epsilon}\,\Omega^{\epsilon\mu}{}_{\mu}$ | $\psi := \Omega^{\alpha\mu}{}_{\mu\vert\alpha}$ |
| | $\omega := \gamma^{\epsilon\mu}{}_{\alpha}\,\gamma^{\alpha}_{\mu\epsilon}$ | $\sigma := \gamma^{\epsilon}{}_{\alpha}{}^{\mu}\,\gamma^{\alpha}_{\epsilon\mu}$ | $h := 2\gamma^{\alpha\mu}{}_{\epsilon}\,\gamma^{\epsilon}_{\alpha\mu}$ | |
| **Vertical** | $\overline{\alpha} := B_{\mu}B^{\mu}$ | $\overline{\theta} := B^{\mu}{}_{\vert\mu}$ | $\overline{\phi} := B_{\epsilon}\,D^{\epsilon\mu}{}_{\mu}$ | $\overline{\psi} := D^{\alpha\mu}{}_{\mu\vert\alpha}$ |
| | $\overline{\omega} := G^{\epsilon\mu}{}_{\alpha}\,G^{\alpha}_{\mu\epsilon}$ | $\overline{\sigma} := G^{\epsilon}{}_{\alpha}{}^{\mu}\,G^{\alpha}_{\epsilon\mu}$ | $\overline{h} := 2G^{\alpha\mu}{}_{\epsilon}\,G^{\epsilon}_{\alpha\mu}$ | |

In physical applications, second order symmetric tensors of zero trace have special importance. For example, in the case of electromagnetism, the tensor characterizing the electro-magnetic energy is a second order symmetric tensor having zero trace. So it is of interest to search for such tensors. The Table below gives some of the second rank tensors of zero trace.

**Table 4. Summary of the fundamental tensors of zero trace**

| **Horizontal** | **Vertical** |
|---|---|
| $\phi_{\mu\nu} + 2\alpha_{\mu\nu}$ | $\overline{\phi}_{\mu\nu} + 2\overline{\alpha}_{\mu\nu}$ |
| $\psi_{\mu\nu} + 2\theta_{\mu\nu}$ | $\overline{\psi}_{\mu\nu} + 2\overline{\theta}_{\mu\nu}$ |
| $h_{\mu\nu} + 2\omega_{\mu\nu}$ | $\overline{h}_{\mu\nu} + 2\overline{\omega}_{\mu\nu}$ |
| $\frac{1}{2}(\phi_{\mu\nu} - \psi_{\mu\nu}) + \theta_{\mu\nu} - \alpha_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\beta^{\alpha}{}_{\vert\alpha}$ | $\frac{1}{2}(\overline{\phi}_{\mu\nu} - \overline{\psi}_{\mu\nu}) + \overline{\theta}_{\mu\nu} - \overline{\alpha}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}B^{\alpha}{}_{\vert\vert\alpha}$ |

We now consider some useful second rank tensors which are not expressible in terms of the fundamental tensors appearing in Table 2. Unlike the tensors of Table 2, some of the tensors to be defined below have no horizontal and vertical counterparts. To this end, let

$$L_{\mu\nu} := L^{\alpha}_{\alpha\mu\nu} = C^{\alpha}_{\alpha\epsilon}R^{\epsilon}_{\mu\nu},\ \ M_{\mu\nu} := L^{\alpha}_{\mu\alpha\nu} = C^{\alpha}_{\mu\epsilon}\,R^{\epsilon}_{\alpha\nu},\ \ N_{\mu\nu} := C^{\alpha}_{\epsilon\mu}\,R^{\epsilon}_{\alpha\nu},\ \ F_{\mu\nu} := \overset{\circ}{C}{}^{\alpha}_{\epsilon\mu}\,R^{\epsilon}_{\alpha\nu}.$$

Then, clearly

$$T_{\mu\nu} := M_{\mu\nu} - N_{\mu\nu} = T^{\alpha}_{\mu\epsilon}\,R^{\epsilon}_{\alpha\nu},\ \ G_{\mu\nu} := M_{\mu\nu} - F_{\mu\nu} = G^{\alpha}_{\mu\epsilon}\,R^{\epsilon}_{\alpha\nu},\ \ G_{\mu\nu} - T_{\mu\nu} = G^{\alpha}_{\epsilon\mu}\,R^{\epsilon}_{\alpha\nu}.$$

Finally, let $T := g^{\mu\nu}T_{\mu\nu}$ and $G := g^{\mu\nu}G_{\mu\nu}$. By the above, we have the following:
Symmetric second rank tensors: $M_{(\mu\nu)},\ \ N_{(\mu\nu)},\ \ F_{(\mu\nu)}$.
Skew-symmetric second rank tensors: $M_{[\mu\nu]},\ \ N_{[\mu\nu]},\ \ F_{[\mu\nu]},\ \ L_{\mu\nu}$.

# 6 Contracted curvatures and curvature scalars

It may be convenient, for physical reasons, to consider second rank tensors derived from the curvature tensors by contractions. It is also of interest to reduce the number of these tensors to a minimum which is fundamental (cf. Propositions 6.1 and 6.2).

Contracting the indices $\alpha$ and $\mu$ in the expressions obtained for the $h$- and $v$-curvature tensors in Theorem 4.6, taking into account Corollary 4.5, we obtain

**Proposition 6.1.** *Let* $\widetilde{\mathcal{R}}_{\sigma\nu} := \widetilde{R}^{\alpha}_{\alpha\sigma\nu}$, $\widehat{\mathcal{R}}_{\sigma\nu} := \widehat{R}^{\alpha}_{\alpha\sigma\nu}$ *and* $\overset{\circ}{\mathcal{R}}_{\sigma\nu} := \overset{\circ}{R}{}^{\alpha}_{\alpha\sigma\nu}$ *with similar expressions for* $\widetilde{\mathcal{S}}_{\sigma\nu}$, $\widehat{\mathcal{S}}_{\sigma\nu}$ *and* $\overset{\circ}{\mathcal{S}}_{\sigma\nu}$. *Then, we have*

**(a)** $\widetilde{\mathcal{R}}_{\sigma\nu} = \beta_{\sigma|\nu} - \beta_{\nu|\sigma} + \beta_{\epsilon}\Lambda^{\epsilon}_{\sigma\nu} + B_{\epsilon}R^{\epsilon}_{\sigma\nu}$,

**(b)** $\widetilde{\mathcal{S}}_{\sigma\nu} = B_{\sigma||\nu} - B_{\nu||\sigma} + B_{\epsilon}T^{\epsilon}_{\sigma\nu}$,

**(c)** $\widehat{\mathcal{R}}_{\sigma\nu} = \frac{1}{2}\widetilde{\mathcal{R}}_{\sigma\nu}$,

**(d)** $\widehat{\mathcal{S}}_{\sigma\nu} = \frac{1}{2}\widetilde{\mathcal{S}}_{\sigma\nu}$,

**(e)** $\overset{\circ}{\mathcal{R}}_{\sigma\nu} = \overset{\circ}{\mathcal{S}}_{\sigma\nu} = 0.$

**Proposition 6.2.** *Let* $\widetilde{R}_{\mu\sigma} := \widetilde{R}^{\alpha}_{\mu\sigma\alpha}$, $\widehat{R}_{\mu\sigma} := \widehat{R}^{\alpha}_{\mu\sigma\alpha}$ *and* $\overset{\circ}{R}_{\mu\sigma} := \overset{\circ}{R}{}^{\alpha}_{\mu\sigma\alpha}$ *with similar expressions for* $\widetilde{S}_{\mu\sigma}$, $\widehat{S}_{\mu\sigma}$ *and* $\overset{\circ}{S}_{\mu\sigma}$. *Then, we have*

**(a)** $\widetilde{R}_{\mu\sigma} = \beta_{\sigma|\mu} + C^{\alpha}_{\epsilon\mu} R^{\epsilon}_{\sigma\alpha} + L^{\alpha}_{\sigma\alpha\mu} + L^{\alpha}_{\alpha\mu\sigma},$

**(b)** $\widetilde{S}_{\mu\sigma} = B_{\sigma||\mu},$

**(c)** $\widehat{R}_{\mu\sigma} = \frac{1}{2}\widetilde{R}_{\mu\sigma} + \frac{1}{4}\{\beta_{\epsilon}\Lambda^{\epsilon}_{\sigma\mu} + \Lambda^{\epsilon}_{\alpha\sigma}\Lambda^{\alpha}_{\mu\epsilon}\},$

**(d)** $\widehat{S}_{\mu\sigma} = \frac{1}{2}\widetilde{S}_{\mu\sigma} + \frac{1}{4}\{B_{\epsilon}T^{\epsilon}_{\sigma\mu} + T^{\epsilon}_{\alpha\sigma}T^{\alpha}_{\mu\epsilon}\},$

**(e)** $\overset{\circ}{R}_{\mu\sigma} = \beta_{\mu|\sigma} - \gamma^{\alpha}_{\mu\sigma|\alpha} + \beta_{\epsilon}\gamma^{\epsilon}_{\mu\sigma} - \gamma^{\alpha}_{\mu\epsilon}\gamma^{\epsilon}_{\sigma\alpha} + G^{\alpha}_{\mu\epsilon}R^{\epsilon}_{\alpha\sigma},$

**(f)** $\overset{\circ}{S}_{\mu\sigma} := \overset{\circ}{S}{}^{\alpha}_{\mu\sigma\alpha} = B_{\mu||\sigma} - G^{\alpha}_{\mu\sigma||\alpha} + B_{\epsilon}G^{\epsilon}_{\mu\sigma} - G^{\alpha}_{\mu\epsilon}G^{\epsilon}_{\sigma\alpha}.$

**Proposition 6.3.** *The following holds.*

**(a)** $\widetilde{R}_{[\mu\sigma]} = \frac{1}{2}\{\beta_{\sigma|\mu} - \beta_{\mu|\sigma}\} + C^{\epsilon}_{\epsilon\alpha}R^{\alpha}_{\mu\sigma} + C^{\epsilon}_{(\alpha\sigma)}R^{\alpha}_{\epsilon\mu} - C^{\epsilon}_{(\alpha\mu)}R^{\alpha}_{\epsilon\sigma},$

**(b)** $\widetilde{R}_{(\mu\sigma)} = \frac{1}{2}\{\beta_{\sigma|\mu} + \beta_{\mu|\sigma} + T^{\epsilon}_{\alpha\mu}R^{\alpha}_{\sigma\epsilon} + T^{\epsilon}_{\alpha\sigma}R^{\alpha}_{\mu\epsilon}\},$

**(c)** $\widetilde{S}_{[\mu\sigma]} = \frac{1}{2}\{B_{\sigma||\mu} - B_{\mu||\sigma}\},$

**(d)** $\widetilde{S}_{(\mu\sigma)} = \frac{1}{2}\{B_{\sigma||\mu} + B_{\mu||\sigma}\},$

**(e)** $\widehat{R}_{[\mu\sigma]} = \frac{1}{2}\widetilde{R}_{[\mu\sigma]} + \frac{1}{4}\beta_{\epsilon}\,\Lambda^{\epsilon}_{\sigma\mu},$

**(f)** $\widehat{R}_{(\mu\sigma)} = \frac{1}{2}\widetilde{R}_{(\mu\sigma)} + \frac{1}{4}\Lambda^{\epsilon}_{\alpha\sigma}\,\Lambda^{\alpha}_{\mu\epsilon},$

**(g)** $\widehat{S}_{[\mu\sigma]} = \frac{1}{2}\widetilde{S}_{[\mu\sigma]} + \frac{1}{4}B_{\epsilon}\,T^{\epsilon}_{\sigma\mu},$

**(h)** $\widehat{S}_{(\mu\sigma)} = \frac{1}{2}\widetilde{S}_{(\mu\sigma)} + \frac{1}{4}T^{\epsilon}_{\alpha\sigma}\,T^{\alpha}_{\mu\epsilon},$

**(i)** $\overset{\circ}{R}_{[\mu\sigma]} = \frac{1}{2}\{L^{\alpha}_{\alpha\mu\sigma} + \overset{\circ}{C}{}^{\alpha}_{\sigma\epsilon}\,R^{\epsilon}_{\alpha\mu} - \overset{\circ}{C}{}^{\alpha}_{\mu\epsilon}\,R^{\epsilon}_{\alpha\sigma}\},$

**(j)** $\overset{\circ}{R}_{(\mu\sigma)} = \frac{1}{2}\{(\beta_{\mu|\sigma} + \beta_{\sigma|\mu}) - \Omega^{\alpha}_{\mu\sigma|\alpha} + \beta_{\epsilon}\,\Omega^{\epsilon}_{\mu\sigma}\} - \gamma^{\alpha}_{\mu\epsilon}\,\gamma^{\epsilon}_{\sigma\alpha} + \frac{1}{2}\{G^{\alpha}_{\mu\epsilon}\,R^{\epsilon}_{\alpha\sigma} + G^{\alpha}_{\sigma\epsilon}\,R^{\epsilon}_{\alpha\mu}\},$

**(k)** $\overset{\circ}{S}_{[\mu\sigma]} = 0,$

**(l)** $\overset{\circ}{S}_{(\mu\sigma)} = \frac{1}{2}\{(B_{\mu||\sigma} + B_{\sigma||\mu}) - D^{\alpha}_{\mu\sigma||\alpha} + B_{\epsilon}\,D^{\epsilon}_{\mu\sigma}\} - G^{\alpha}_{\mu\epsilon}\,G^{\epsilon}_{\sigma\alpha}.$

**Corollary 6.4.** *The following holds:*

**(a)** $\widetilde{R}^{\sigma}_{\sigma} := g^{\mu\sigma}\widetilde{R}_{\mu\sigma} = \beta^{\sigma}{}_{|\sigma} + T^{\epsilon\sigma}{}_{\alpha}\,R^{\alpha}_{\epsilon\sigma},$

**(b)** $\widetilde{S}^{\sigma}_{\sigma} := g^{\mu\sigma}\widetilde{S}_{\mu\sigma} = B^{\sigma}{}_{||\sigma},$

**(c)** $\widehat{R}^{\sigma}_{\sigma} := g^{\mu\sigma}\widehat{R}_{\mu\sigma} = \frac{1}{2}\{\beta^{\sigma}{}_{|\sigma} + T^{\epsilon\sigma}{}_{\alpha}\, R^{\alpha}_{\epsilon\sigma}\} + \frac{1}{4}\Lambda^{\epsilon\sigma}{}_{\alpha}\,\Lambda^{\alpha}_{\epsilon\sigma}$,

**(d)** $\widehat{S}^{\sigma}_{\sigma} := g^{\mu\sigma}\widehat{S}_{\mu\sigma} = \frac{1}{2}B^{\sigma}{}_{||\sigma} + \frac{1}{4}T^{\epsilon\sigma}{}_{\alpha}\, T^{\alpha}_{\epsilon\sigma}$,

**(e)** $\overset{\circ}{R}{}^{\sigma}{}_{\sigma} := g^{\mu\sigma}\overset{\circ}{R}_{\mu\sigma} = \beta^{\sigma}{}_{|\sigma} - \frac{1}{2}\Omega^{\alpha\sigma}{}_{\sigma|\alpha} + \frac{1}{2}\beta_{\alpha}\,\Omega^{\alpha\sigma}{}_{\sigma} - \gamma^{\alpha\sigma}{}_{\epsilon}\,\gamma^{\epsilon}_{\sigma\alpha} + G^{\alpha\sigma}{}_{\epsilon}\,R^{\epsilon}_{\alpha\sigma}$,

**(f)** $\overset{\circ}{S}{}^{\sigma}{}_{\sigma} := g^{\mu\sigma}\overset{\circ}{S}_{\mu\sigma} = B^{\sigma}{}_{||\sigma} - \frac{1}{2}D^{\alpha\sigma}{}_{\sigma||\alpha} + \frac{1}{2}B_{\alpha}\,D^{\alpha\sigma}{}_{\sigma} - G^{\alpha\sigma}{}_{\epsilon}\,G^{\epsilon}_{\sigma\alpha}$.

We now apply a different method for calculating both $\overset{\circ}{R}_{\mu\sigma}$ and $\overset{\circ}{S}_{\mu\sigma}$, now expressed in terms of the covariant derivative of the contorsion tensors with respect to the *Riemannian d*-connection. Then we obtain

**Proposition 6.5.** *The "Ricci" tensors* $\overset{\circ}{R}_{\mu\sigma}$ *and* $\overset{\circ}{S}_{\mu\sigma}$ *can be expressed in the form*

**(a)** $\overset{\circ}{R}_{\mu\sigma} = \beta_{\mu\,\overset{\circ}{|}\,\sigma} - \gamma^{\alpha}_{\mu\sigma\,\overset{\circ}{|}\,\alpha} - \beta_{\epsilon}\gamma^{\epsilon}_{\mu\sigma} + \gamma^{\epsilon}_{\mu\alpha}\gamma^{\alpha}_{\epsilon\sigma} + G^{\alpha}_{\mu\epsilon}R^{\epsilon}_{\alpha\sigma}$.

**(b)** $\overset{\circ}{S}_{\mu\sigma} = B_{\mu\,\overset{\circ}{||}\,\sigma} - G^{\alpha}_{\mu\sigma\,\overset{\circ}{||}\,\alpha} - B_{\epsilon}G^{\epsilon}_{\mu\sigma} + G^{\epsilon}_{\mu\alpha}G^{\alpha}_{\epsilon\sigma}$.

*Proof.* We prove **(a)** only; the proof of **(b)** is similar.
We have

$$\begin{aligned}
0 = R^{\alpha}_{\mu\sigma\alpha} &= (\delta_{\alpha}\Gamma^{\alpha}_{\mu\sigma} - \delta_{\sigma}\Gamma^{\alpha}_{\mu\alpha}) + (\Gamma^{\epsilon}_{\mu\sigma}\Gamma^{\alpha}_{\epsilon\alpha} - \Gamma^{\epsilon}_{\mu\alpha}\Gamma^{\alpha}_{\epsilon\sigma}) + R^{\epsilon}_{\sigma\alpha}C^{\alpha}_{\mu\epsilon} \\
&= \delta_{\alpha}(\overset{\circ}{\Gamma}{}^{\alpha}_{\mu\sigma} + \gamma^{\alpha}_{\mu\sigma}) - \delta_{\sigma}(\overset{\circ}{\Gamma}{}^{\alpha}_{\mu\alpha} + \gamma^{\alpha}_{\mu\alpha}) + (\overset{\circ}{\Gamma}{}^{\epsilon}_{\mu\sigma} + \gamma^{\epsilon}_{\mu\sigma})(\overset{\circ}{\Gamma}{}^{\alpha}_{\epsilon\alpha} + \gamma^{\alpha}_{\epsilon\alpha}) \\
&\quad - (\overset{\circ}{\Gamma}{}^{\epsilon}_{\mu\alpha} + \gamma^{\epsilon}_{\mu\alpha})(\overset{\circ}{\Gamma}{}^{\alpha}_{\epsilon\sigma} + \gamma^{\alpha}_{\epsilon\sigma}) + R^{\epsilon}_{\sigma\alpha}C^{\alpha}_{\mu\epsilon} \\
&= \overset{\circ}{R}_{\mu\sigma} - (\delta_{\sigma}\gamma^{\alpha}_{\mu\alpha} - \gamma^{\alpha}_{\epsilon\alpha}\overset{\circ}{\Gamma}{}^{\epsilon}_{\mu\sigma}) + (\delta_{\alpha}\gamma^{\alpha}_{\mu\sigma} + \gamma^{\epsilon}_{\mu\sigma}\overset{\circ}{\Gamma}{}^{\alpha}_{\epsilon\alpha} - \gamma^{\alpha}_{\epsilon\sigma}\overset{\circ}{\Gamma}{}^{\epsilon}_{\mu\alpha} \\
&\quad - \gamma^{\alpha}_{\mu\epsilon}\overset{\circ}{\Gamma}{}^{\epsilon}_{\sigma\alpha}) + \ R^{\epsilon}_{\sigma\alpha}(C^{\alpha}_{\mu\epsilon} - \overset{\circ}{C}{}^{\alpha}_{\mu\epsilon}) + \ \gamma^{\epsilon}_{\mu\sigma}\gamma^{\alpha}_{\epsilon\alpha} - \gamma^{\epsilon}_{\mu\alpha}\gamma^{\alpha}_{\epsilon\sigma}.
\end{aligned}$$

Consequently,

$$\overset{\circ}{R}_{\mu\sigma} = \beta_{\mu\,\overset{\circ}{|}\,\sigma} - \gamma^{\alpha}_{\mu\sigma\,\overset{\circ}{|}\,\alpha} - \beta_{\epsilon}\gamma^{\epsilon}_{\mu\sigma} + \gamma^{\epsilon}_{\mu\alpha}\gamma^{\alpha}_{\epsilon\sigma} + G^{\alpha}_{\mu\epsilon}R^{\epsilon}_{\alpha\sigma},$$

which is the required formula. □

In view of Proposition 6.2 (e) and (f) and Proposition 6.5, we obtain

**Corollary 6.6.** *The following identities holds:*

**(a)** $(\beta_{\mu|\sigma} - \beta_{\mu\,\overset{\circ}{|}\,\sigma}) - (\gamma^{\alpha}_{\mu\sigma|\alpha} - \gamma^{\alpha}_{\mu\sigma\,\overset{\circ}{|}\,\alpha}) = (\gamma^{\epsilon}_{\mu\alpha}\Omega^{\alpha}_{\sigma\epsilon} - 2\beta_{\epsilon}\gamma^{\epsilon}_{\mu\sigma})$

**(b)** $(B_{\mu||\sigma} - B_{\mu\,\overset{\circ}{||}\,\sigma}) - (G^{\alpha}_{\mu\sigma||\alpha} - G^{\alpha}_{\mu\sigma\,\overset{\circ}{||}\,\alpha}) = (G^{\epsilon}_{\mu\alpha}D^{\alpha}_{\sigma\epsilon} - 2B_{\epsilon}G^{\epsilon}_{\mu\sigma})$.

The next two tables summarize the results obtained in this section, where the contracted curvatures are expressed in terms of the fundamental tensors.

**Table 5 (a). Second rank curvature tensors**

| | Skew-symmetric | Symmetric |
|---|---|---|
| Dual | $\widetilde{R}_{[\mu\sigma]} = \epsilon_{\sigma\mu} - L_{\sigma\mu} + M_{[\sigma\mu]} + N_{[\sigma\mu]}$ | $\widetilde{R}_{(\mu\sigma)} = \theta_{\mu\sigma} + M_{(\mu\sigma)} - N_{(\mu\sigma)}$ |
| | $\widetilde{S}_{[\mu\sigma]} = \overline{\epsilon}_{\sigma\mu}$ | $\widetilde{S}_{(\mu\sigma)} = \overline{\theta}_{\mu\sigma}$ |
| Symmetric | $\widehat{R}_{[\mu\sigma]} = \frac{1}{2}\widetilde{R}_{[\mu\sigma]} + \frac{1}{4}\eta_{\sigma\mu}$ | $\widehat{R}_{(\mu\sigma)} = \frac{1}{2}\widetilde{R}_{(\mu\sigma)} + \frac{1}{4}\{h_{\mu\sigma} - \omega_{\mu\sigma} - \sigma_{\mu\sigma}\}$ |
| | $\widehat{S}_{[\mu\sigma]} = \frac{1}{2}\widetilde{S}_{[\mu\sigma]} + \frac{1}{4}\overline{\eta}_{\sigma\mu}$ | $\widehat{S}_{(\mu\sigma)} = \frac{1}{2}\widetilde{S}_{(\mu\sigma)} + \frac{1}{4}\{\overline{h}_{\mu\sigma} - \overline{\omega}_{\mu\sigma} - \overline{\sigma}_{\mu\sigma}\}$ |
| Riemannian | $\overset{\circ}{R}_{[\mu\sigma]} = \frac{1}{2}L_{\mu\sigma} - F_{[\mu\sigma]}$ | $\overset{\circ}{R}_{(\mu\sigma)} = \theta_{\mu\sigma} - \frac{1}{2}(\psi_{\mu\sigma} - \phi_{\mu\sigma}) - \omega_{\mu\sigma} + M_{(\mu\sigma)} - F_{(\mu\sigma)}$ |
| | $\overset{\circ}{S}_{[\mu\sigma]} = 0$ | $\overset{\circ}{S}_{(\mu\sigma)} = \overline{\theta}_{\mu\sigma} - \frac{1}{2}(\overline{\psi}_{\mu\sigma} - \overline{\phi}_{\mu\sigma}) - \overline{\omega}_{\mu\sigma}$ |

**Table 5 (b). h- and v-scalar curvature tensors**

| | **h-scalar curvature** | **v-scalar curvature** |
|---|---|---|
| **Dual** | $\widetilde{R}^{\sigma}_{\sigma} = \theta + T$ | $\widetilde{S}^{\sigma}_{\sigma} = \overline{\theta}$ |
| **Symmetric** | $\widehat{R}^{\sigma}_{\sigma} = \frac{1}{2}(\theta + T) - \frac{1}{4}(3\omega + \sigma)$ | $\widehat{S}^{\sigma}_{\sigma} = \frac{1}{2}\overline{\theta} - \frac{1}{4}(3\overline{\omega} + \overline{\sigma})$ |
| **Riemannian** | $\overset{\circ}{R}{}^{\sigma}_{\sigma} = \theta - \frac{1}{2}(\psi - \phi) - \omega + G$ | $\overset{\circ}{S}{}^{\sigma}_{\sigma} = \overline{\theta} - \frac{1}{2}(\overline{\psi} - \overline{\phi}) - \overline{\omega}$ |

# 7 The $W$-tensors

The $W$-tensor was first defined by M. Wanas in 1975 [13] and has been used by F. Mikhail and M. Wanas [6] to construct a geometric theory unifying gravity and electromagnetism. Recently, two of the authors of this paper studied some of the properties of this tensor in the context of the classical AP-space [15].

**Definition 7.1.** Let $(M, \lambda)$ be a generalized AP-space. For a given $d$-connection $D = (N^{\alpha}_{\beta}, \Gamma^{\alpha}_{\mu\nu}, C^{\alpha}_{\mu\nu})$, the horizontal $W$-tensor ($hW$-tensor) $H^{\alpha}_{\mu\nu\sigma}$ is defined by the formula

$$\lambda_{\mu|\nu\sigma} - \lambda_{\mu|\sigma\nu} = \lambda_{\epsilon} H^{\epsilon}_{\mu\nu\sigma},$$

whereas the vertical $W$-tensor ($vW$-tensor) $V^{\alpha}_{\mu\nu\sigma}$ is defined by the formula

$$\lambda_{\mu||\nu\sigma} - \lambda_{\mu||\sigma\nu} = \lambda_{\epsilon} V^{\epsilon}_{\mu\nu\sigma},$$

where " $|$" and " $||$" are the horizontal and the vertical covariant derivatives with respect to the connection $D$.

We now carry out the task of calculating the different $W$-tensors. As opposed to the classical AP-space, which admits essentially one $W$-tensor corresponding to the dual connection, we here have 4 distinct $W$-tensors: the horizontal and vertical $W$-tensors corresponding to the dual $d$-connection, the horizontal $W$-tensor corresponding to the symmetric $d$-connection and, finally, the horizontal $W$-tensor corresponding to the Riemannian $d$-connection. The remaining $W$-tensors coincide with the corresponding curvature tensors.

It is to be noted that some of the expressions obtained for the $W$-tensors are relatively more compact than those obtained for the corresponding curvature tensors.

**Theorem 7.2.** *The hW-tensor $\widetilde{H}^{\alpha}_{\mu\nu\sigma}$, the vW-tensor $\widetilde{V}^{\alpha}_{\mu\nu\sigma}$, the hW-tensor $\widehat{H}^{\alpha}_{\mu\nu\sigma}$ and the hW-tensor $\overset{\circ}{H}{}^{\alpha}_{\mu\nu\sigma}$ corresponding to the dual, symmetric and the Riemannian d-connections respectively can be expressed in the form:*

**(a)** $\widetilde{H}^{\alpha}_{\mu\nu\sigma} = \Lambda^{\alpha}_{\sigma\nu|\mu} + \Lambda^{\epsilon}_{\nu\sigma}\Lambda^{\alpha}_{\mu\epsilon} + \mathfrak{S}_{\mu,\nu,\sigma} L^{\alpha}_{\mu\sigma\nu}.$

**(b)** $\widetilde{V}^{\alpha}_{\mu\nu\sigma} = T^{\alpha}_{\sigma\nu||\mu} + T^{\epsilon}_{\nu\sigma}T^{\alpha}_{\mu\epsilon}.$

**(c)** $\widehat{H}^{\alpha}_{\mu\nu\sigma} = \frac{1}{2}(\Lambda^{\alpha}_{\mu\nu|\sigma} - \Lambda^{\alpha}_{\mu\sigma|\nu}) + \frac{1}{4}(\Lambda^{\epsilon}_{\mu\nu}\Lambda^{\alpha}_{\sigma\epsilon} - \Lambda^{\epsilon}_{\mu\sigma}\Lambda^{\alpha}_{\nu\epsilon}) + \frac{1}{2}(\Lambda^{\epsilon}_{\sigma\nu}\Lambda^{\alpha}_{\epsilon\mu}).$

**(d)** $\overset{\circ}{H}{}^{\alpha}_{\mu\nu\sigma} = \gamma^{\alpha}_{\mu\nu|\sigma} - \gamma^{\alpha}_{\mu\sigma|\nu} + \gamma^{\epsilon}_{\mu\sigma}\gamma^{\alpha}_{\epsilon\nu} - \gamma^{\epsilon}_{\mu\nu}\gamma^{\alpha}_{\epsilon\sigma} + \Lambda^{\epsilon}_{\nu\sigma}\gamma^{\alpha}_{\mu\epsilon}.$

*Proof.* We prove **(a)** only. The proof of the other parts is similar. We have

$$\lambda_{\epsilon}\widetilde{H}^{\epsilon}_{\mu\nu\sigma} = \lambda_{\epsilon}\widetilde{R}^{\epsilon}_{\mu\sigma\nu} + \lambda_{\underset{|\epsilon}{\mu}}\widetilde{\Lambda}^{\epsilon}_{\sigma\nu} + \lambda_{\underset{||\epsilon}{\mu}}R^{\epsilon}_{\sigma\nu}.$$

Hence, taking into account Theorem 4.6 (a), we obtain

$$\begin{aligned}
\widetilde{H}^{\alpha}_{\mu\nu\sigma} &= \widetilde{R}^{\alpha}_{\mu\sigma\nu} + \underset{i}{\lambda}{}^{\alpha}(\delta_{\epsilon}\underset{i}{\lambda}_{\mu} - \underset{i}{\lambda}_{\beta}\Gamma^{\beta}_{\epsilon\mu})\widetilde{\Lambda}^{\epsilon}_{\sigma\nu} + \underset{i}{\lambda}{}^{\alpha}(\dot{\partial}_{\epsilon}\underset{i}{\lambda}_{\mu} - \underset{i}{\lambda}_{\beta}C^{\beta}_{\epsilon\mu})R^{\epsilon}_{\sigma\nu} \\
&= \widetilde{R}^{\alpha}_{\mu\sigma\nu} + \Lambda^{\epsilon}_{\nu\sigma}(\Gamma^{\alpha}_{\mu\epsilon} - \Gamma^{\alpha}_{\epsilon\mu}) + R^{\epsilon}_{\sigma\nu}(C^{\alpha}_{\mu\epsilon} - C^{\alpha}_{\epsilon\mu}) \\
&= \Lambda^{\alpha}_{\sigma\nu|\mu} + C^{\alpha}_{\epsilon\mu}R^{\epsilon}_{\sigma\nu} + L^{\alpha}_{\sigma\nu\mu} + L^{\alpha}_{\nu\mu\sigma} + \Lambda^{\epsilon}_{\nu\sigma}\Lambda^{\alpha}_{\mu\epsilon} + T^{\alpha}_{\mu\epsilon}R^{\epsilon}_{\sigma\nu} \\
&= \Lambda^{\alpha}_{\sigma\nu|\mu} + T^{\alpha}_{\epsilon\mu}R^{\epsilon}_{\sigma\nu} + C^{\alpha}_{\mu\epsilon}R^{\epsilon}_{\sigma\nu} + L^{\alpha}_{\sigma\nu\mu} + L^{\alpha}_{\nu\mu\sigma} + \Lambda^{\epsilon}_{\nu\sigma}\Lambda^{\alpha}_{\mu\epsilon} + \ T^{\alpha}_{\mu\epsilon}R^{\epsilon}_{\sigma\nu} \\
&= \Lambda^{\alpha}_{\sigma\nu|\mu} + \Lambda^{\epsilon}_{\nu\sigma}\Lambda^{\alpha}_{\mu\epsilon} + \mathfrak{S}_{\mu,\nu,\sigma} L^{\alpha}_{\mu\sigma\nu}.
\end{aligned}$$

□

**Proposition 7.3.** *Let $\widetilde{\mathcal{H}}_{\nu\sigma} := \widetilde{H}^{\alpha}_{\alpha\nu\sigma}$, $\widehat{\mathcal{H}}_{\nu\sigma} := \widehat{H}^{\alpha}_{\alpha\nu\sigma}$ and $\overset{\circ}{\mathcal{H}}_{\nu\sigma} := \overset{\circ}{H}{}^{\alpha}_{\alpha\nu\sigma}$ with similar expression for $\widetilde{\mathcal{V}}_{\nu\sigma}$. Then, we have*

**(a)** $\widetilde{\mathcal{H}}_{\nu\sigma} = \beta_{\sigma|\nu} - \beta_{\nu|\sigma} + 2\beta_{\epsilon}\Lambda^{\epsilon}_{\sigma\nu},$

**(b)** $\widetilde{\mathcal{V}}_{\nu\sigma} = B_{\sigma||\nu} - B_{\nu||\sigma} + 2B_{\epsilon}T^{\epsilon}_{\sigma\nu},$

**(c)** $\widehat{\mathcal{H}}_{\nu\sigma} = \frac{1}{2}\{\widetilde{\mathcal{H}}_{\nu\sigma} + \beta_{\epsilon}\Lambda^{\epsilon}_{\nu\sigma}\},$

**(d)** $\overset{\circ}{\mathcal{H}}_{\nu\sigma} = 0.$

**Proposition 7.4.** *Let* $\widetilde{H}_{\mu\sigma} := \widetilde{H}^{\alpha}_{\mu\alpha\sigma}$, $\widehat{H}_{\mu\sigma} := \widehat{H}^{\alpha}_{\mu\alpha\sigma}$ *and* $\overset{\circ}{H}_{\mu\sigma} := \overset{\circ}{H}{}^{\alpha}_{\mu\alpha\sigma}$ *with similar expressions for* $\widetilde{V}_{\mu\sigma}$. *Then, we have*

**(a)** $\widetilde{H}_{\mu\sigma} = \beta_{\sigma|\mu} + \Lambda^{\epsilon}_{\alpha\sigma}\Lambda^{\alpha}_{\mu\epsilon} + \mathfrak{S}_{\alpha,\mu,\sigma} L^{\alpha}_{\alpha\mu\sigma}$,

**(b)** $\widetilde{V}_{\mu\sigma} = B_{\sigma||\mu} + T^{\epsilon}_{\alpha\sigma}T^{\alpha}_{\mu\epsilon}$,

**(c)** $\widehat{H}_{\mu\sigma} = \frac{1}{2}\widetilde{H}_{\mu\sigma} + \frac{1}{4}(\beta_{\epsilon}\Lambda^{\epsilon}_{\sigma\mu} + \Lambda^{\epsilon}_{\sigma\alpha}\Lambda^{\alpha}_{\mu\epsilon})$,

**(d)** $\overset{\circ}{H}_{\mu\sigma} = \beta_{\mu|\sigma} - \gamma^{\alpha}_{\mu\sigma|\alpha} + \beta_{\epsilon}\gamma^{\epsilon}_{\mu\sigma} - \gamma^{\epsilon}_{\sigma\alpha}\gamma^{\alpha}_{\mu\epsilon}$.

**Proposition 7.5.** *The following holds:*

**(a)** $\widetilde{H}_{[\mu\sigma]} = \frac{1}{2}\{\beta_{\sigma|\mu} - \beta_{\mu|\sigma}\} + \mathfrak{S}_{\alpha,\mu,\sigma} L^{\alpha}_{\alpha\mu\sigma}$,

**(b)** $\widetilde{H}_{(\mu\sigma)} = \frac{1}{2}\{\beta_{\sigma|\mu} + \beta_{\mu|\sigma}\} + \Lambda^{\epsilon}_{\alpha\sigma}\Lambda^{\alpha}_{\mu\epsilon}$,

**(c)** $\widetilde{V}_{[\mu\sigma]} = \frac{1}{2}\{B_{\sigma||\mu} - B_{\mu||\sigma}\}$,

**(d)** $\widetilde{V}_{(\mu\sigma)} = \frac{1}{2}\{B_{\sigma||\mu} + B_{\mu||\sigma}\} + T^{\epsilon}_{\alpha\sigma}T^{\alpha}_{\mu\epsilon}$,

**(e)** $\widehat{H}_{[\mu\sigma]} = \frac{1}{2}\widetilde{H}_{[\mu\sigma]} + \frac{1}{4}\beta_{\epsilon}\Lambda^{\epsilon}_{\sigma\mu}$,

**(f)** $\widehat{H}_{(\mu\sigma)} = \frac{1}{2}\widetilde{H}_{(\mu\sigma)} + \frac{1}{4}\Lambda^{\epsilon}_{\sigma\alpha}\Lambda^{\alpha}_{\mu\epsilon}$,

**(g)** $\overset{\circ}{H}_{[\mu\sigma]} = \frac{1}{2}\mathfrak{S}_{\alpha\mu\sigma} L^{\alpha}_{\alpha\mu\sigma}$,

**(h)** $\overset{\circ}{H}_{(\mu\sigma)} = \frac{1}{2}\{(\beta_{\mu|\sigma} + \beta_{\sigma|\mu}) - \Omega^{\alpha}_{\mu\sigma|\alpha} + \beta_{\epsilon}\Omega^{\epsilon}_{\mu\sigma}\} - \gamma^{\alpha}_{\mu\epsilon}\gamma^{\epsilon}_{\sigma\alpha}$.

**Corollary 7.6.** *the following holds:*

**(a)** $\widetilde{H}^{\alpha}_{\alpha} = \beta^{\alpha}{}_{|\alpha} + \Lambda^{\epsilon\mu}{}_{\alpha}\Lambda^{\alpha}_{\epsilon\mu}$,

**(b)** $\widetilde{V}^{\alpha}_{\alpha} = B^{\alpha}{}_{||\alpha} + T^{\epsilon\mu}{}_{\alpha}T^{\alpha}_{\epsilon\mu}$,

**(c)** $\widehat{H}^{\alpha}_{\alpha} = \frac{1}{2}\beta^{\alpha}{}_{|\alpha} + \frac{1}{4}\Lambda^{\epsilon\mu}{}_{\alpha}\Lambda^{\alpha}_{\epsilon\mu}$,

**(d)** $\overset{\circ}{H}{}^{\sigma}{}_{\sigma} = \beta^{\sigma}{}_{|\sigma} - \frac{1}{2}\Omega^{\alpha\sigma}{}_{\sigma|\alpha} + \frac{1}{2}\beta_{\alpha}\Omega^{\alpha\sigma}{}_{\sigma} - \gamma^{\alpha\sigma}{}_{\epsilon}\gamma^{\epsilon}_{\sigma\alpha}$.

Taking into account Proposition 4.4, Theorem 7.2 and the Bianchi identity [4] for the Riemannian d-connection, we get the following

**Proposition 7.7.** *The hW-tensors* $\widetilde{H}^{\alpha}_{\mu\nu\sigma}$, $\widehat{H}^{\alpha}_{\mu\nu\sigma}$, $\overset{\circ}{H}{}^{\alpha}_{\mu\nu\sigma}$ *and the vW-tensors* $\widetilde{V}^{\alpha}_{\mu\nu\sigma}$ *satisfy the following identities:*

**(a)** $\mathfrak{S}_{\mu,\nu,\sigma}\, \widetilde{H}^{\alpha}_{\mu\nu\sigma} = 2\mathfrak{S}_{\mu,\nu,\sigma}(\Lambda^{\alpha}_{\mu\epsilon}\, \Lambda^{\epsilon}_{\nu\sigma} + L^{\alpha}_{\mu\sigma\nu})$.

**(b)** $\mathfrak{S}_{\mu,\nu,\sigma}\, \widetilde{V}^{\alpha}_{\mu\nu\sigma} = 2\mathfrak{S}_{\mu,\nu,\sigma}(T^{\alpha}_{\mu\epsilon}\, T^{\epsilon}_{\nu\sigma})$.

**(c)** $\mathfrak{S}_{\mu,\nu,\sigma}\, \widehat{H}^{\alpha}_{\mu\nu\sigma} = \mathfrak{S}_{\mu,\nu,\sigma}\, L^{\alpha}_{\mu\sigma\nu}$.

**(d)** $\mathfrak{S}_{\mu,\nu,\sigma} \overset{\circ}{H}{}^{\alpha}_{\mu\nu\sigma} = \mathfrak{S}_{\mu,\nu,\sigma} L^{\alpha}_{\mu\sigma\nu}$.

We collect the results obtained in this section in the following tables, where the contracted $W$-tensors are expressed in terms of the fundamental tensors.

**Table 6 (a). Second rank W-tensors**

| | **Skew-symmetric** | **Symmetric** |
|---|---|---|
| **Dual** | $\widetilde{H}_{[\mu\sigma]} = \epsilon_{\sigma\mu} - L_{\sigma\mu} + 2M_{[\sigma\mu]}$ | $\widetilde{H}_{(\mu\sigma)} = \theta_{\mu\sigma} - (\omega_{\mu\sigma} + \sigma_{\mu\sigma} - h_{\mu\sigma})$ |
| | $\widetilde{V}_{[\mu\sigma]} = \bar{\epsilon}_{\sigma\mu}$ | $\widetilde{V}_{(\mu\sigma)} = \bar{\theta}_{\mu\sigma} - (\bar{\omega}_{\mu\sigma} + \bar{\sigma}_{\mu\sigma} - \bar{h}_{\mu\sigma})$ |
| **Symmetric** | $\widehat{H}_{[\mu\sigma]} = \frac{1}{2}\widetilde{H}_{[\mu\sigma]} + \frac{1}{4}\eta_{\sigma\mu}$ | $\widehat{H}_{(\mu\sigma)} = \frac{1}{2}\widetilde{H}_{(\mu\sigma)} + \frac{1}{4}\{\omega_{\mu\sigma} + \sigma_{\mu\sigma} - h_{\mu\sigma}\}$ |
| **Riemannian** | $\overset{\circ}{H}{}_{[\mu\sigma]} = \frac{1}{2}L_{\mu\sigma} - M_{[\mu\sigma]}$ | $\overset{\circ}{H}{}_{(\mu\sigma)} = \theta_{\mu\sigma} - \frac{1}{2}(\psi_{\mu\sigma} - \phi_{\mu\sigma}) - \omega_{\mu\sigma}$ |

**Table 6 (b). Scalar W-tensors**

| | **h-scalar W-tensors** | **v-scalar W-tensors** |
|---|---|---|
| **Dual** | $\widetilde{H}^{\sigma}_{\sigma} = \theta - (3\omega + \sigma)$ | $\widetilde{V}^{\sigma}_{\sigma} = \bar{\theta} - (3\bar{\omega} + \bar{\sigma})$ |
| **Symmetric** | $\widehat{H}^{\sigma}_{\sigma} = \frac{1}{2}\theta - \frac{1}{4}(3\omega + \sigma)$ | |
| **Riemannian** | $\overset{\circ}{H}{}^{\sigma}_{\sigma} = \theta - \frac{1}{2}(\psi - \phi) - \omega$ | |

# Concluding remarks

In the present article, we have developed a parallelizable structure in the context of a generalized Lagrange space. Four distinguished connections, depending on one non-linear connection, are used to explore the properties of this space. Different curvature tensors characterizing this structure are calculated. The contracted curvature tensors necessary for physical applications are given and compared (Tables 5(a)). The traces of these tensors are derived and compared (Table 5(b)). Finally, the different $W$-tensors with their contractions and traces are also derived (Tables 6(a) and 6(b)).

On the present work, we have the following comments and remarks:

1. Existing theories of gravity suffer from some problems connected to recent observed astrophysical phenomena, especially those admitting **anisotropic** behavior of the system concerned (e.g. the flatness of the rotation curves of spiral galaxies). So, theories in which the gravitational potential depends *on both position and direction* are needed. Such theories are to be constructed in spaces admitting this dependence. This is one of the aims motivating the present work.

2. Among the advantages of the AP-geometry are the ease in calculations and the diverse and its thorough applications. In this work, we have kept as close as possible to the classical AP-case. However, the extra degrees of freedom in our GAP-geometry have created an abundance of geometric objects which have no counterpart in the classical AP-geometry. Since the physical meaning of most of the geometric objects of the classical AP-structure is clear, we hope to attribute physical meaning to the new geometric objects appearing in the present work, especially the vertical quantities.

3. Due to the wealth of the GAP-geometry, one is faced with the problem of choosing geometric objects that represent true physical quantities. As a first step to solve this problem, we have written all second order tensors in terms of the fundamental tensors defined in section 5. This is done to facilitate comparison between these tensors and to be able to choose the most appropriate for physical application. The same procedure has been used for scalars.

4. The paper is not intended to be an end in itself. In it, we try to construct a geometric framework capable of dealing with and describing physical phenomena. The success of the classical AP-geometry in physical applications made us choose this geometry as a guide line.

   The physical interpretation of the geometric objects existing in the GAP-geometry and not in the AP-geometry will be further investigated in a forthcoming paper.

This paper is not an end in itself, but rather the beginning of a research direction. The physical interpretation of the geometric objects in the GAP-space that have no counterpart in the classical AP-space will be further investigated in forthcoming papers.

**Acknowledgement.** This paper was presented in " The International Conference on Finsler Extensions of Relativity Theory" held at Cairo, Egypt, November 4-10, 2006. ArXiv Number: 0704.2001.

# References

[1] D. Bao, S. Chern and Z. Shen, *An introduction to Riemann-Finsler Geometry*, Graduate Texts in Mathematics, Springer, 2000.

[2] F. Brickell and R. S Clark, *Differentiable manifolds*, Van Nostrand Reinhold Co., 1970.

[3] K. Hayashi and T. Shirafuji, *New general relativity*, Phys. Rev. D 19 (1979), 3524-3554.

[4] M. Matsumoto, *Foundations of Finsler Geometry and special Finsler spaces*, Kaiseisha Press, Otsu, Japan, 1986.

[5] F. I. Mikhail, *Tedrad vector fields and generalizing the theory of relativity*, Ain Shams Sc. Bull., No. 6 (1962), 87-111.

[6] F. I. Mikhail and M.I Wanas, *A generalized field theory. I. Field equations*, Proc. R. Soc. London, A. 356 (1977), 471-481.

[7] R. Miron, *A Lagrangian theory of relativity*, Sem de Geom. si Top., 84, Timisoara, Romania, 1985.

[8] R. Miron, *Metrical Finsler structures and special Finsler spaces*, J. Math. Kyoto Univ., 23 (1983), 219-224.

[9] R. Miron, *Compendium on the Geometry of Lagrange Spaces*, Handbook of Differential Geometry, Vol. II (2006), 437-512.

[10] H. P. Robertson, *Groups of motion in spaces admitting absolute parallelism*, Ann. Math, Princeton (2), 33 (1932), 496 - 520.

[11] T. Sakaguchi, *Parallelizable generalized Lagrange spaces*, Anal. Sti. Univ. AI. I. Cuza, Iasi, Mat., 33, 2 (1987), 127-136.

[12] T. Sakaguchi, *Invariant theory of parallelizable Lagrange spaces*, Tensor, N. S., 46 (1987), 137-144.

[13] M. I. Wanas, *A generalized field theory and its applications in cosmology*, Ph. D. Thesis, Cairo University, 1975.

[14] M. I. Wanas, *Absolute parallelism geometry: Developments, applications and problems*, Stud. Cercet, Stiin. Ser. Mat. Univ. Bacau, No. 10 (2001), 297-309.

[15] N. L. Youssef and A. M. Sid-Ahmed, *Linear connections and curvature tensors in the geometry of parallelizable manifolds*, Submitted.

*Authors' addresses:*

M. I. Wanas,
Cairo University, Faculty of Science,
Department of Astronomy, Cairo, Egypt.
E-mail: wanas@frcu.eun.eg

Nabil L. Youssef,
Cairo University, Faculty of Science,
Department of Mathematics, Cairo, Egypt.
E-mail: nyoussef@frcu.eun.eg, nlyoussef2003@yahoo.fr

A. M. Sid-Ahmed,
Cairo University, Faculty of Science,
Department of Mathematics, Cairo, Egypt.
E-mail address: amrs@mailer.eun.eg