Lie algebras of a class of top spaces

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Abstract. In this paper 1-dimensional and 2-dimensional top spaces with finite numbers of identities and connected Lie group components are characterized. MF-semigroups are determined. By using of the left-invariant vector fields of top spaces and their one-parameter subgroups, a relation between the Lie algebras of a class of top spaces and the Lie algebras of a class of Lie groups is determined. As a result a solution for an open problem to a class of top spaces is presented.

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1 Introduction

Basically a top space is a smooth manifold which points can be (smoothly) multiplied together and generally its identity is a map. In this paper we are going to characterize two classes of top spaces. Then we will consider the relation between left-invariant vector fields of a top space and its one-parameter subgroups. We know that if the cardinality of the identities of a top space is finite then the set of its left-invariant vector fields under the Lie bracket is a Lie algebra. We are going to deduce a Lie group which its Lie algebra be isomorphic to the Lie algebra of a special kind of top spaces.

2 Basic notions

In this paper we assume that T is a top space [3, 5], and for all $t \in T$, the set $T_{e(t)}$ is a connected set. In [8] one can find the conditions which imply to the connectedness of $T_{e(t)}$.

Let $(\tilde{T}_{e(t)}, p_t, \tilde{e(t)})$ be a universal covering space of $(T_{e(t)}, e(t))$. Then $\tilde{T}_{e(t)}$ with the multiplication $\tilde{m}_t(\tilde{t_1}, \tilde{t_2})$ with $\tilde{t_1}, \tilde{t_2} \in \tilde{T}_{e(t)}$ such that $p_t o \tilde{m}_t(\tilde{t_1}, \tilde{t_2}) = m_t(p_t(\tilde{t_1}, \tilde{t_2})$ where m_t is the restriction of m on $T_{e(t)} \times T_{e(t)}$, is a Lie group [6].

If $\tilde{T}$ is the disjoint union of $\tilde{T}_{e(t)}$ where $t \in T$ then the product $\tilde{m}$ on $\tilde{T} \times \tilde{T}$ determines

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uniquely by the equalities $p_{st}o\tilde{m}(\tilde{s},\tilde{t}) = m(p_s(\tilde{s}), p_t(\tilde{t}))$ and $\tilde{m}(\tilde{e(s)}, \tilde{e(t)}) = \tilde{e(st)}$ [6]. Moreover $(\tilde{T}, \tilde{m})$ is a top space [6].
If $P : \tilde{T} \longrightarrow T$ is the mapping $p(\tilde{t}) = p_t(\tilde{t})$ then P is a homomorphism of top spaces, and the pair $(\tilde{T}, P)$ is called an upper top space of T. The kernel of p is called the MF-semigroup of T [6].
Theorem 2.1 [6] If $(\tilde{T}, p)$ and $(\tilde{S}, q)$ are two upper top spaces of a top space T, then $\ker p$ is isomorphic to $\ker q$.
Theorem 2.2 [6] If T is a top space and D its MF-semigroup then D is isomorphic to $\bigcup^{0}_{t\in e(T)} \pi_1(T_{e(t)}, e(t))$, where $\pi_1(T_{e(t)}, e(t))$ is the fundamental group of $T_{e(t)}$ with base point $e(t)$ and $\bigcup^{0}$ denotes the disjoint union.

As a result of Theorem 3.3, if T is a Lie group then the MF-semigroup of T is the fundamental group of T.

3 Characterization of two classes of top spaces

We begin this section with the following theorem.
Theorem 3.1 Let T be a top space and the cardinality of $e(T)$ be finite. Moreover let H be a closed submanifold generalized subgroup of T [5]. Then H is a top space.
Proof. Since the cardinality of $e(T)$ is finite then for all $t \in T$, $e^{-1}(e(t))$ is open and closed subset of T and it is a Lie group. We know that $H_{e(t)} = H \cap e^{-1}(e(t))$ is a closed subset of $e^{-1}(e(t))$. The Cartan theorem [2] implies that $H_{e(t)}$ is a Lie subgroup of $e^{-1}(e(t))$ and then $H = \bigcup_{e(t)\in T} H_{e(t)}$ is a top space. □
Corollary 3.1 Let T be a top space and the cardinality of $e(T)$ be finite. Moreover let H be a submanifold generalized subgroup of T. Then H is a top space.
Proof. Since H is a locally closed generalized subgroup of T, then H is a closed submanifold of T [7], and so H is a top space. □
Example 3.1 Let T be the top space $\mathbb{R} - \{0\}$ with the product $a.b \longmapsto a|b|$, then Corollary 3.1 implies that $H_1 = \{+1, -1\}$ and $H_2 = \{(-1)^{n+1}2^n, (-1)^n 2^n | n \in N \cup \{0\}\}$ are top spaces.

Theorem 3.2 *Suppose that T is a one-dimensional top space and the cardinality of $e(T)$ is finite, if $e^{-1}(e(t))$ is connected for all $t \in T$ then $T \cong \oplus_{card(e(T))} A_i$, where $A_i = \mathbb{R}^1$ or $A_i = S^1$.*

Proof. We know that $T = \bigcup^{0}_{t\in e(T)} e^{-1}(e(t))$ and $e^{-1}(e(t))$ is a connected Lie group. Since $e^{-1}(e(t))$ is isomorphic to $\mathbb{R}^1$ or S^1, then $T \cong \oplus_{card(e(T))} A_i$, where $A_i = \mathbb{R}^1$ or $A_i = S^1$. □

Theorem 3.3 *Let T be a top space and D be its MF-semigroup, if $|e(T)| < \infty$, and $e^{-1}(e(t))$ is a connected subset of T for all $t \in T$, then D is isomorphic to a direct sum of integer numbers.*

Proof. $D \cong \bigcup_{t\in e(T)}^{0} \pi_1(T_{e(t)}, e(t))$ where $T_{e(t)} = e^{-1}(e(t))$ and $\pi(T_{e(t)}, e(t))$ is a fundamental group of $T_{e(t)}$ with the base point $e(t)$. Since for all $a \in \mathbb{R}$ and $b \in S^1, \pi_1(S^1, b)$ and $\pi_1(\mathbb{R}, a)$ are isomorphic with $(\mathbb{Z}, +)$ and $\{e\}$ respectively, then D is isomorphic to a direct sum of integer numbers. □

Theorem 3.4 If T is a two dimensional top space and $e^{-1}(e(t))$ is a connected set, for all $t \in T$. Then $T \cong \oplus A_i$ where $A_i = \mathbb{R}^2, A_i = T^2, A_i = \mathbb{R} \times S^1$ or identity connected component T_0^t of the group of affine motions of real line on $e^{-1}(e(t))$.

Proof. Since $T = \bigcup_{t\in e(T)}^{0} e^{-1}(e(t))$ and $e^{-1}(e(t))$ is a connected Lie group, then we know that each two dimensional Lie groups is isomorphic to $\mathbb{R}^2, T^2, \mathbb{R}\times S^1$ or identity connected component T_0^t of the group of affine motions of real line on $e^{-1}(e(t))$. □

Example 3.2 If T is the top space of Example 3.1 then $e(T) = \{1, -1\}$, $e^{-1}(1) = (0, \infty)$ and $e^{-1}(-1) = (-\infty, 0)$. Thus $T \cong \mathbb{R} \oplus \mathbb{R}$ and $D \cong \{e\}$.

4 Left-invariant vector fields and one-parameter subgroups

We begin this section by the following theorem.

Theorem 4.1 [3] *Let T be a top space and let the cardinality of $e(T)$ be a natural number. Then the set of left-invariant vector fields on T [4] is a Lie algebra under the Lie bracket operation.*

Now, we consider a problem which sketched in the paper [3].
If T is a top space and $e(T)$ is a finite set, then Theorem 4.1 implies that there exists a Lie algebra corresponding to T. According to this Lie algebra there is a Lie group. Now the problem is: What is the relation between this Lie group and T?

Definition 4.1 Suppose T is a top space. A curve $\phi : \mathbb{R} \longrightarrow T$ is called one-parameter subgroup of top space T if it is satisfies the condition $\phi(t_1 + t_2) = \phi(t_1)\phi(t_2)$; for all $t_1, t_2 \in \mathbb{R}$.

Lemma 4.1 *Let $\phi : \mathbb{R} \longrightarrow T$ be a one-parameter subgroup of T, then $\phi(0) \in e(T)$. Moreover $\phi(s)\phi(-s) \in e(T)$; for all $s \in \mathbb{R}$.*

Proof. If $\phi : \mathbb{R} \longrightarrow T$ is a one-parameter subgroup of a top space T, then $\phi(0) = \phi(0 + 0) = \phi(0)\phi(0)$. If $t = \phi(0)$, then $t = tt$ and so $e(t) = t^{-1}t = t^{-1}(tt) = (t^{-1}t)t = e(t)t = t$. Thus $e(t) = t$. □

Given a one-parameter subgroup $\phi : \mathbb{R} \longrightarrow T$, then there exists a vector field X such that $\frac{d\phi^\mu(t)}{dt} = X^\mu(\phi(t))$, where X^μ denotes a component of X in a coordinate system. We show that this vector field is a left-invariant vector field. If $L_t : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $L_t(s) = t + s$; for all $s \in \mathbb{R}$, then $(L_t)_*\left(\frac{d}{dt}|_{t=0}\right) = \left(\frac{d}{dt}|_t\right)$. Next, we apply induced map $\phi_* : d_t(\mathbb{R}) \longrightarrow d_{\phi(t)}(T)$ on the vectors $\frac{d}{dt}|_{t_l}$ and $\frac{d}{dt}|_t$, (4.1) $$\phi_*\Big(\frac{d}{dt}|_{t_1}\Big) = \frac{\partial\phi^\mu(t)}{\partial t}|_{t_1}\frac{\partial}{\partial y^\mu}|_{\phi(t_1)} = X|_{\phi(t_1)}$$

(4.2) $$\phi_*\Big(\frac{d}{dt}|_{t}\Big) = \frac{\partial\phi^\mu(t)}{\partial t}|_{t}\frac{\partial}{\partial y^\mu}|_{\phi(t)} = X|_{\phi(t)}$$

(1) and (2) imply that:

(4.3) $$(\phi L_t)_*\Big(\frac{d}{dt}|_{t_1}\Big) = (\phi_*)(L_t)_*\Big(\frac{d}{dt}|_{t_1}\Big) = \phi_*\frac{d}{dt}|_{t+t_1} = X|_{\phi(t_1+t)};$$

the equality $\phi L_t = l_{\phi(t)}\phi$ implies: $(\phi L_t)_* = (l_{\phi(t)}\phi)_*$, so $\phi_*(L_t)_* = (l_{\phi(t)})_*\phi_*$ and then:

$$\phi_*(L_t)_*\Big(\frac{d}{dt}|_{t_l}\Big) = (l_{\phi(t)})_*\phi_*\Big(\frac{d}{dt}|_{t_l}\Big).$$

It follows from (3) and (1) that $X(\phi(t+t_1)) = (l_{\phi(t)})_*X|_{\phi(t_1)}$. Thus X is left-invariant vector field.

Now, let X be a left-invariant vector field on top space T, we show that there exist one-parameter subgroups on T corresponding to X. X defines a one-parameter group of transformation $\sigma(r,s); (r\in\mathbb{R}, s\in T)$ such that $\frac{d\sigma^\mu}{dt} = X^\mu$ and $\sigma(0,s)=s$, for all $s\in T$. If we define $\phi:\mathbb{R}\longrightarrow T$ by $\phi(t)=\sigma(t,\phi(0))$ and $\phi(0)\in e(T)$, then the curve ϕ becomes a one-parameter subgroup of T. To prove this, we show that $\phi(t+s)=\phi(t)\phi(s)$, for all $s,t\in\mathbb{R}$. If the parameter s is fixed and; $\overline{\sigma}:\mathbb{R}\longrightarrow T$ is the map $\overline{\sigma}(t,\phi(s))=\phi(s)\phi(t)$ then we have,

$$\begin{aligned}\overline{\sigma}(0,\phi(s)) &= \phi(s)\phi(0) = \phi(s)e(\phi(0)) = \phi(s)e(\phi(s-s))\\ &= \phi(s)e(\sigma(s-s,\phi(0))) = \phi(s)e(\phi(s)\phi(s)^{-1})\\ &= \phi(s)e(\phi(s))e(\phi(s)^{-1}) = \phi(s) = \sigma(s,\phi(0)),\end{aligned}$$

thus $\overline{\sigma}(0,\phi(s))=\phi(s)$. Also $\overline{\sigma}$ satisfies the same differential equation of σ;

$$\begin{aligned}\frac{d}{dt}\overline{\sigma}(t,\phi(s)) &= \frac{d}{dt}(\phi(s)\phi(t)) = (L_{\phi(s)})_*\Big(\frac{d}{dt}\phi(t)\Big)\\ &= (L_{\phi(s)})_*(X(\phi(t))) = X(\phi(s)\phi(t)) = X(\overline{\sigma}(t,\phi(s))).\end{aligned}$$

By the uniqueness theorem of ordinary differential equation, we conclude that:

$$\phi(t+s) = \sigma(t+s,\phi(0)) = \sigma(t,\sigma(s,\phi(0))) = \overline{\sigma}(t,\phi(s)) = \phi(s)\phi(t).$$

Note that the correspondence between one-parameter subgroups of T and left-invariant vector fields on T is not one-to-one and we can find for every left-invariant vector field $X, |e(T)|$ one-parameter subgroup of T.

Example 4.1 If $T=\mathbb{R}$, with the product $(a,b)\longmapsto a$, then we know that card $(e(T))=\infty$ and then by the previous assertion there exists infinite left-invariant vector fields on T. Note that the vector field X on T is a left-invariant vector field if and only if $X:T\longrightarrow\mathbb{R}$ is defined by $X(u)=cu$, for some constant number $c\in\mathbb{R}$. It is clearly that for every one-parameter subgroup $\phi,\phi(\mathbb{R})$ is a commutative subgroup of T. By selecting $\phi(0)\in e(T)$, we have a commutative subgroup of T. Therefore we can find a correspondence between left-invariant vector field and free commutative group $\prod^*_{\phi(0)\in e(T)} \phi(\mathbb{R})$.

Definition 4.2 Let T be a top space and let G be a topological group. Then a covering projection $P : T \longrightarrow G$ is called a top space covering projection if P satisfies the following conditions:
(i) $P(t) = e$, for all $t \in e(T)$, where e is identity element;
(ii) $P(t_1t_2) = P(t_1)P(t_2)$, for all $t_1, t_2 \in T$.

Example 4.2 Suppose that $T = \mathbb{R} - \{0\}$ with the product $(a, b) \longmapsto a|b|$, $G = \mathbb{R}^+$ with the usual product and standard topology, if $P : T \longrightarrow G$ is defined by $P(t) = |t|$, then P is a top space covering for G.

Lemma 4.2 The space $P(T)$ with the induced topology of T is a Lie group.
Proof. It is clear that $P(T)$ is a group and the following diagram is a commutative diagram:

$$\begin{array}{ccc} T \times T & \xrightarrow{\theta_1} & T \\ P \times P \downarrow & & \downarrow P \\ P(T) \times P(T) & \xrightarrow{\theta_2} & P(T) \end{array}$$

where $\theta_1(t_1, t_2) = t_1t_2^{-1}$. Since P is C^∞-map and $Po\theta_1 = \theta_2 o(P \times P)$, then $P(T)$ is a Lie group. □

Note that since P is a surjective local diffeomorphism, then $P(T) = G$.

Now, we state the main theorem of this section.

Theorem 4.2 *Let P be a top space covering projection for a top space T and a topological group G and let $|e(T)| < \infty$. Then there exists a correspondence (but not necessarily one-to-one) between one-parameter subgroups G and one parameter subgroups of T.*

Proof. It is clear that if ϕ is a one-parameter subgroup of T, then $Po\phi$ is a one-parameter subgroup of G. Now, if $\psi : \mathbb{R} \longrightarrow G$ is a one-parameter subgroup of G then $\psi(0) = e$ and there exist a connected neighborhood U of e such that it induces a diffeomorphism on each connected component of $P^{-1}(U) = \bigcup^{0}_{t\in e(T)} V_t$. We can find $\phi_t : S \longrightarrow V_t$ such that $S \subseteq \mathbb{R}$, $\phi_t(r_1 + r_2) = \phi_t(r_1)\phi_t(r_2)$, $\frac{d\phi_{t_1}}{dt} = \frac{d\phi_{t_2}}{dt}$ and $Po\psi_t = \phi$, for all $t_1, t_2, t \in e(T)$. We can extend each ϕ_t to a one-parameter subgroup $\phi_t : \mathbb{R} \longrightarrow e^{-1}(e(t))$ such that $Po\phi_t = \psi$ and $\frac{d\phi_{t_1}}{dt} = \frac{d\phi_{t_2}}{dt}$, for all $t_1, t_2 \in e(T)$. □

Corollary 4.1 *If T is a top space with $|e(T)| < \infty$, and G is a Lie group and $P : T \longrightarrow G$ a top space covering projection for G, then there exists a one-to-one correspondence between left-invariant vector field G and left invariant vector fields of T. Moreover the Lie algebra created by the left invariant vector fields of T is isomorphic to the Lie algebra of G.*

Proof. Let X be a left-invariant vector field, then there exist $|e(T)|$ one-parameter subgroups of T correspondence to X, and all of these one-parameter subgroups of T correspond to some one-parameter subgroups of G. Since G is a Lie group then there exists only one left-invariant vector field correspondence with that one-parameter subgroups. □

Note. The Lie algebra T and G are denoted by $\mathcal{T}$ and $\mathcal{G}$ respectively.

Corollary 4.2 *With the assumptions of corollary 4.1 if G is a connected set then $\tilde{\mathcal{G}}$ and $\mathcal{T}$ are isomorphic Lie algebras, where $\tilde{\mathcal{G}}$ is the Lie algebra of universal covering of G.*

Proof. Suppose that $(\tilde{G}, q, \tilde{e})$ is a universal covering of G. Since q is a homomorphism then $\tilde{\mathcal{G}} \cong \mathcal{G}$ and Corollary 4.1 implies that $\tilde{\mathcal{G}} \cong \mathcal{T}$. □

Corollary 4.3 *Let T and G be connected sets and $e(t_0) \in T$ be fixed. Moreover let $P : T \longrightarrow G$ be a top space covering projection for G. Then there exists a unique Lie group structure on T such that $e(t_0)$ is identity element and Lie algebra of T (as a Lie group) is equal to the Lie algebra of left invariant vector fields of T (as a top space).*

Proof. There exists a unique structure on T such that T is a Lie group with identity element $e(t_0)$ and P is a morphism of Lie groups. Thus Lie algebra of T (as a Lie group) is equal to Lie algebra T (as a top space) [2]. □

5 Conclusion

In this paper we solved the problem which has been sketched in [3] for a class of top spaces, but the problem is open for the other classes of top spaces. Regarding related literature, we address the reader to [1].

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