Free fall motion in an invariant field of forces: the 2D-case

Cristina-Liliana Pripoae, Gabriel-Teodor Pripoae

Abstract. Following [12] and [14], the Lie groups may be classified into four families, with respect to the property of the invariant vector fields to be geodesic. A detailed study in dimensions 2 and 3 ([13]) led to many examples of *particular* invariant vector fields ξ geometrized, in this manner, by means of *some properly chosen* invariant Riemannian metrics g. In this paper we extend the previous investigation, and analyse the behavior of *all* the invariant vector fields ξ and of *all* their associated invariant Riemannian metrics g. The 2-dimensional case is completely solved: given a left invariant field of forces ξ on a 2-dimensional Lie group G, we determine *all* the metrics $g \in \mathrm{Riem}(G, \xi)$ such that ξ becomes geodesic in (G, g) and *all* the possible trajectories of free fall particles moving in the "Universe" (G, g).

M.S.C. 2000: 53C22, 22E15, 37C10, 15A03.
Key words: geodesic vector fields, trajectories, moduli spaces, geodesics.

1 Introduction

The starting point of this study was the need of geometrizing the trajectories of a left invariant vector field ξ on a Lie group G, by interpreting the trajectories of ξ as geodesics of a suitable chosen left invariant Riemannian metric g. (In this case, we call ξ a *geodesic* vector field). Such problems appeared first in Theoretical Physics, especially in Fluid Mechanics ([1], [3]), but they are quite general, whenever we have to consider symmetry groups associated to some PDE's. (For example, applications in Robotics may be found in [15]; for other different but related viewpoints, see [6], [19], [20]).

The existence problem was solved in ([12]), by the following two theorems:

Theorem 1. *Let ξ be a non-vanishing left invariant vector field on a Lie group G. Then there exists a left invariant Riemannian metric g on G, such that the trajectories of ξ be geodesics of g, if and only if*

Balkan Journal of Geometry and Its Applications, Vol.14, No.1, 2009, pp. 72-83.

$$\xi \notin L_\xi(L(G)) \tag{1.1}$$

Theorem 2. *Let ξ be a non-vanishing left invariant vector field on a Lie group G.*
(i) *If*

$$\xi \in L_\xi(L(G)) \tag{1.2}$$

then there exists a left invariant indefinite metric g on G, such that ξ be light-like and the trajectories of ξ be geodesics of g.
(ii) *Suppose* $\dim L_\xi(L(G)) = \dim G - 1$. *If there exists a left invariant Lorentzian metric g on G, such that the trajectories of ξ be geodesics of g and ξ be light-like, then the relation* (1.2) *holds.*

The possible different behaviors of the left invariant vector fields (with respect to the properties (1.1) and (1.2)) lead to an elementary "classification" of the Lie groups in 4 types ([12], [14]):

I. *the ones in which* every *left invariant vector field has the property* (1.1).

II. *the ones in which* there exist *left invariant vector fields ξ having the property* (1.1), *η, $\theta \in L(G)$ with the property* (1.2) *and* $\dim L_\eta(L(G)) = n-1$, $\dim L_\theta(L(G)) \neq n-1$;

III. *the ones in which* there exist *left invariant vector fields ξ having the property* (1.1), *$\eta \in L(G)$ with the property* (1.2) *and* $\dim L_\eta(L(G)) = n-1$, *and* there does not exist *$\theta \in L(G)$ with the property* (1.2) *and* $\dim L_\theta(L(G)) \neq n-1$;

IV. *the ones in which* there exist *left invariant vector fields ξ having the property* (1.1), *$\eta \in L(G)$ with the property* (1.2) *and* $\dim L_\eta(L(G)) \neq n-1$, *and* there does not exist *$\theta \in L(G)$ with the property* (1.2) *and* $\dim L_\theta(L(G)) = n-1$.

For a given non-null left invariant vector field ξ on an n-dimensional Lie group G, we denoted by $\mathrm{Riem}(G,\xi)$ the set of all the left invariant Riemannian metrics g on G, such that the trajectories of ξ be geodesics of g ([14]). We called $\mathrm{Riem}(G,\xi)$ the moduli space of (associated metrics of) ξ in G and proved that it may be put into a one-to-one correspondence with a convex cone in a linear space, of dimension at most $n(n+1)/2$ and at least $n(n-1)/2+1$ (which will be called the "dimension" of $\mathrm{Riem}(G,\xi)$).This number may vary from one vector field ξ to another.

The maximal number of linearly independent vector fields satisfying (1.1) equals the dimension of G. The "dimensions" of all the moduli spaces $\mathrm{Riem}(G,\xi)$ gives an algebraic "signature" of the Lie group G. Our hope is to extract a complete set of (algebraic) invariants for a classification of Lie algebras, up to an isomorphism.

Any new classification must first take into account the examples in low dimensions. In [13] we investigated the algebraic classifications of the 2-dimensional and 3-dimensional Lie algebras, and established the type of each one, as follows:

- the abelian 2-dimensional Lie algebra has type I;

- the non-abelian 2-dimensional Lie algebra has type III.

- the abelian 3-dimensional Lie algebra, $so(3)$, the Heisenberg algebra, the $e(2)$ algebra of Euclidean rigid motions in $\mathbb{R}^2$ have type I.

- the $e(1,1)$ algebra of rigid motions in the Minkowsi space $\mathbb{R}^2$ has type IV.

- the $sl(2)$ algebra has type III.

- the "generic" non-unimodular 3-dimensional Lie algebras have type II.

- the exceptional non-unimodular 3-dimensional Lie algebra (of class σ) has type IV.

In each case, for some *particular* left invariant vector fields, we constructed effectively the metrics which made them geodesic. In some very special cases, whenever the calculations were not tedious, we did the same thing for *all* the left invariant vector fields.

The previous arguments aimed to convince the reader that characterizations and properties of $\mathrm{Riem}(G,\xi)$ might be important.Unfortunately,the degree of complexity for computations in L(G) increases dramatically with the dimension of G. Until now, we succeeded to do it only for the 2-dimensional Lie groups and the results we obtained form the matter of this paper. We analyze the behavior of *all* the left invariant vector fields ξ and of *all* their associated left invariant Riemannian metrics $g \in \mathrm{Riem}(G,\xi)$. We determine (§3) *all* the geodesics of a generic element in $\mathrm{Riem}(G,\xi)$: we find exponential parametric geodesics, lines, but also a specific family of geodesics, showing a quite strange behavior. This explains/proves why (and how) a "symmetric" ("regular") field of forces on a "symmetric" space may produce "free falling" trajectories with (broken) symmetry. Consequently, the section §4 is devoted to some more speculative comments, based on interesting examples from experimental sciences.

2 The spaces $\mathrm{Riem}(G,\xi)$ in dimension 2

Let G be a 2-dimensional Lie group. There are only two non-isomorphic classes of Lie algebras $L(G)$, with the following representatives:

(i) the abelian Lie algebra $\mathbb{R}^2$; in this case, G belongs to the type I. Any left invariant vector field $\xi \in L(G)$ is a geodesic one, with respect to any left invariant semi-Riemannian metric on G. It follows that the moduli space $\mathrm{Riem}(G,\xi)$ has maximal "dimension" 3.

(ii) a non-abelian (solvable) Lie algebra with a basis $\{E_1, E_2\}$, with $[E_1, E_2] = E_2$. Expressed in canonical coordinates (x^1, x^2), we may write $E_1 = \frac{\partial}{\partial x^1}$ and $E_2 = e^{x^1}\frac{\partial}{\partial x^2}$.
In this case, E_1 satisfies the relation (1) and E_2 satisfies the relation (1.2); moreover, $\dim L_{E_2}(L(G)) = \dim G - 1$.

Consider $\xi = aE_1 + bE_2$ be an arbitrary non-null left invariant vector field, with $a, b \in \mathbb{R}$. Then ([13]): ξ satisfies (1.1) if and only if $a \neq 0$; any $\xi \in L(G)$ satisfying (1.2) must be collinear with E_2. It follows that G belongs to the type III.

The following three propositions depict some geometric properties for elements in $\mathrm{Riem}(G,\xi)$.

Proposition 1. *Let $\xi = aE_1 + bE_2$ be a left invariant vector field, with $a \neq 0$. Then, in the canonical basis, a metric g in* $\mathrm{Riem}(G,\xi)$ *has the components:*

$$g_{11} = \alpha \quad , \quad g_{12} = -ba^{-1}\beta \quad , \quad g_{22} = \beta \tag{2.1}$$

where α and β are positive constants, satisfying the relation $a^2\alpha > b^2\beta$.

Proof. The condition that ξ satisfies (1.1) expresses as

$$g(\xi, [\xi, E_1]) = g(\xi, [\xi, E_2]) = 0$$

It follows that, in the given basis, the components of g write as in (2.1). In local coordinates, we have $\xi = a\frac{\partial}{\partial x^1} + be^{x^1}\frac{\partial}{\partial x^2}$, and the components of g are: $g_{11} = \alpha$, $g_{12} = -ba^{-1}e^{-x^1}\beta$, $g_{22} = e^{-2x^1}\beta$. $\square$

Remarks. (i) From Proposition 1, it follows that, for an arbitrary given ξ, the moduli space $\mathrm{Riem}(G, \xi)$ depends generically on two parameters, so its "dimension" is 2.

(ii) Consider two collinear vector fields in $L(G)$, $\xi = aE_1 + bE_2$ and $\tilde{\xi} = \tilde{a}E_1 + \tilde{b}E_2$. Then $ba^{-1} = \tilde{b}\tilde{a}^{-1}$; relation (2.1) implies $\mathrm{Riem}(G, \xi) = \mathrm{Riem}(G, \tilde{\xi})$. So, collinear left invariant vector fields have the same moduli space. As the converse is also true, we deduce that the left invariant directions are in a 1:1 correspondence with the moduli spaces.

The Lie algebras are purely algebraic objects, and their classifications may arise only from algebraic constructions and invariants (as proved by the practices of the 20-th century). The moduli spaces we defined are also (in essence) algebraic objects, but their elements are geometric ones. We believe that geometric invariants for metrics in the moduli spaces $\mathrm{Riem}(G, \xi)$ (as curvature or geodesics) may provide insights for new classifications of Lie algebras.

Straightforward calculations prove

Proposition 2. *Let $\xi = aE_1 + bE_2$ be a left invariant vector field, with $a \neq 0$ and $g \in \mathrm{Riem}(G, \xi)$, with the components given by (2.1). Then the Gaussian curvature of g is*

$$k = -a^2(a^2\alpha - b^2\beta)$$

Proposition 3. *Let $\xi = aE_1 + bE_2$ be a left invariant vector field, with $a \neq 0$ and $g \in \mathrm{Riem}(G, \xi)$, with the components given by (2.1). Then:*

(i) the potentials of ξ with respect to g are of the form

$$f(x^1, x^2) = \Delta a^{-1}x^1 + \gamma$$

where $\Delta = a^2\alpha - b^2\beta$ and γ is an arbitrary real parameter.

(ii) for the functions in (i), the Hessian with respect to g has the components

$$Hess_{11} = -\beta b^2 a^{-1} \quad , \quad Hess_{12} = \beta b e^{-x^1} \quad , \quad Hess_{22} = -\beta a e^{-2x^1}$$

Moreover, this Hessian is semi-negative defined .

3 Geodesics of the metrics in $\mathrm{Riem}(G,\xi)$

For the 2-dimensional non-abelian Lie group G, consider a basis $\{E_1,E_2\}$ in $L(G)$, with $[E_1,E_2]=E_2$, as in §2. Let $\xi=aE_1+bE_2$ be an arbitrary non-null left invariant vector field, with $a,b\in\mathbb{R}$ and $a\neq 0$. Consider an arbitrary metric $g\in\mathrm{Riem}(G,\xi)$, of the form (2.1). The trajectories of ξ are geodesics of g, but there exist many other geodesics of g as well. So, it is of interest to determine *all* the geodesics of g, thus describing all the possible trajectories of free falling particles moving under the "field of forces" ξ in the "Universe" G.

Proposition 4. *Let $\xi=aE_1+bE_2$ be a left invariant vector field, with $a\neq 0$ and $g\in\mathrm{Riem}(G,\xi)$, with the components given by* (2.1)*. Then the geodesics of g are given by the equations:*

$$\Delta(x^1)''+b^2\beta((x^1)')^2-2ab\beta e^{-x^1}(x^1)'(x^2)'+a^2\beta e^{-2x^1}((x^2)')^2=0 \tag{3.1}$$

$$\Delta(x^2)''+ab\alpha\, e^{x^1}((x^1)')^2+ab\beta\, e^{-x^1}((x^2)')^2-2a^2\alpha(x^1)'(x^2)'=0$$

where $\Delta=a^2\alpha-b^2\beta$.

Proof. From the proof of Proposition 1, we know the coefficients of g, written in local coordinates. We calculate the Christoffel's coefficients:

$$\Gamma^1_{11}=b^2\beta\Delta^{-1}\quad,\quad\Gamma^1_{22}=a^2\beta\Delta^{-1}e^{-2x^1}\quad,\quad\Gamma^1_{12}=\Gamma^1_{21}=-ab\beta\Delta^{-1}e^{-x^1}$$

$$\Gamma^2_{11}=ab\alpha\Delta^{-1}e^{x^1}\quad,\quad\Gamma^2_{22}=ab\beta\Delta^{-1}e^{-x^1}\quad,\quad\Gamma^2_{12}=\Gamma^2_{21}=-a^2\alpha\Delta^{-1}$$

Then, the equations of the geodesics: $\ddot{x}^i+\Gamma^i_{jk}(x^j)'(x^k)'=0$, for $i\in\{1,2\}$, provide the required system (3.1).□

Now, we shall integrate the geodesics system of equations (3.1). Denote $x:=x^1$ and $y:=x^2$. Without restraining the generality, it is sufficient to consider two cases:

Case I: ($a=1$, $b=0$; $\beta=1$ and arbitrary positive α). It follows that $\Delta=\alpha$.

The system (3.1) becomes

$$\alpha x''+e^{-2x}(y')^2=0\quad,\quad\alpha y''-2\alpha x'y'=0 \tag{3.2}$$

Suppose that (generically) $y'=0$. Then $x''=0$ and we obtain the obvious general solution of the form

$$t\to(x(t),y(t))=(k_1t+k_2,k_3) \tag{3.3}$$

where k_1, k_2, k_3 are arbitrary constants.

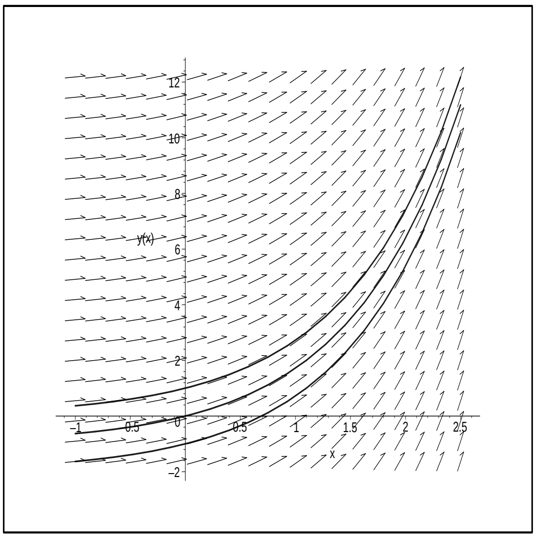

Figure 1:

We have $\xi = E_1$;the trajectories of ξ are exactly the (particular) geodesics given by (3.3) and depicted in Figure 1.

Suppose that (generically) $y' \neq 0$. From the second equation in (3.2) we obtain $y' = \pm e^{2x+k_1}$, where k_1 is an arbitrary real constant. In what follows, we detail the case $y' = +e^{2x+k_1}$. We introduce y' in the first equation (3.2) and deduce $x'' = \gamma e^{2x}$, where we denoted $\gamma := -\alpha^{-1}e^{2k_1}$. This last equation has the solution $x = x(t)$ satisfying the relation

$$\int (k_2 + \gamma e^{2x})^{-1} dx = k_3 \pm t$$

We integrate and obtain two families of curves:

$$x(t) = \frac{1}{2}\ln\{k_2\gamma^{-1}[k_2(\frac{e^{2k_2(k_3\pm t)}+1}{1-e^{2k_2(k_3\pm t)}})^2 - 1]\} \tag{3.4}$$

$$y(t) = \alpha k_2 e^{-k_1}(1-k_2)t \mp 2\alpha k_2^2 e^{-k_1}\frac{1}{1-e^{2k_2(k_3\pm t)}} + k_4$$

and

$$x(t) = \frac{1}{2}\ln\{k_2\gamma^{-1}[k_2(\frac{1-e^{2k_2(k_3\pm t)}}{1+e^{2k_2(k_3\pm t)}})^2 - 1]\} \tag{3.5}$$

$$y(t) = \alpha k_2 e^{-k_1}(1-k_2)t \mp 2\alpha k_2^2 e^{-k_1}\frac{1}{1+e^{2k_2(k_3\pm t)}} + k_4$$

In a similar manner, we treat the case $y' = -e^{2x+k_1}$, and derive the geodesics

$$x(t) = \frac{1}{2}\ln\{k_2\gamma^{-1}[k_2(\frac{e^{2k_2(k_3\pm t)}+1}{1-e^{2k_2(k_3\pm t)}})^2 - 1]\} \tag{3.6}$$

$$y(t) = -\alpha k_2 e^{-k_1}(1-k_2)t \pm 2\alpha k_2^2 e^{-k_1}\frac{1}{1-e^{2k_2(k_3\pm t)}} + k_4$$

and

$$x(t) = \frac{1}{2}\ln\{k_2\gamma^{-1}[k_2(\frac{1-e^{2k_2(k_3\pm t)}}{1+e^{2k_2(k_3\pm t)}})^2 - 1]\} \tag{3.7}$$

$$y(t) = -\alpha k_2 e^{-k_1}(1-k_2)t \pm 2\alpha k_2^2 e^{-k_1}\frac{1}{1+e^{2k_2(k_3\pm t)}} + k_4$$

respectively, where k_1, k_2, k_3, k_4 are arbitrary real constants.

The formulas (3.3)-(3.7) provide all the parametrized geodesics of (G, g), in the case I.

Case II: ($a = 1$, $b = 1$; $\beta = 1$ and arbitrary $\alpha > 1$). It follows that $\Delta = \alpha - 1$. The system (3.1) becomes

$$(\alpha-1)x'' + (x')^2 - 2e^{-x}x'y' + e^{-2x}(y')^2 = 0 \tag{3.8}$$

$$(\alpha-1)y'' + \alpha e^{x}(x')^2 - 2\alpha x'y' + e^{-x}(y')^2 = 0 \tag{3.9}$$

We multiply equation (3.4) with e^{-x}, substract from the first equation and get

$$y'' - 2x'y' = e^{x}(x'' - (x')^2)$$

We multiply equation (3.8) with α, we multiply equation (3.9) with e^{-x}, substract and simplify by $(\alpha - 1)$; we get

$$\alpha x'' - e^{-x}y'' + e^{-2x}(y')^2 = 0$$

We make the change of variables: $u := e^{-x}y'$ and $v := x'$. The last two equations give the system

$$\alpha v' = u' + uv - u^2 \quad , \quad v' = v^2 + u' - uv \tag{3.10}$$

We distinguish now (generically) two subcases:

Subcase II-1: $u = v$. It follows from (3.10) that $u = v = 1$. Hence $u' = u^2 - u$ and $v' = 0$. We deduce that the geodesics are of the form

$$t \to (x(t), y(t) = (k_1 t + k_2, e^{k_1 t + k_2} + k_3)$$

where k_1, k_2, k_3 are arbitrary real constants. The respective curves are parametrized lines and exponentials, or degenerate geodesics (”points”).

Subcase II-2: $u \neq v$ (the ”generic” case). It follows that

$$u' = \frac{1}{1-\alpha}(v-u)(\alpha v - u) \quad , \quad v' = \frac{1}{1-\alpha}(u-v)^2$$

We divide the two equations member by member and derive

$$\frac{du}{dv} = \frac{\alpha v - u}{v - u}$$

(remember that $\alpha > 1$). We integrate

$$\frac{du}{dv} = (\alpha - 1)\frac{v}{v - u} + 1$$

and obtain

$$u^2 + \alpha v^2 - 2uv - k_1 = 0 \tag{3.11}$$

where k_1 is an arbitrary real non-negative constant.

The determinant of the attached bilinear form is $\alpha - 1 > 0$, so (3.11) is the implicit equation of a family of ellipses, parametrized by two parameters (α and k_1). Elementary calculations reduce the ellipses equation to

$$u(t) = k_1 \ \cos t + \frac{1}{\sqrt{\alpha - 1}} \, k_1 \ \sin t$$

$$v(t) = \frac{1}{\sqrt{\alpha - 1}} \, k_1 \ \sin t$$

We return to the initial coordinates (x, y) and deduce the general parametric form of the geodesics, in the case II-2 :

$$x(t) = \frac{1}{\sqrt{\alpha - 1}} \, k_1 \ \cos t + k_2 \tag{3.12}$$

$$y(t) = k_1 \int \cos t \, e^{\frac{1}{\sqrt{\alpha-1}} \ k_1 \ \cos t + k_2} dt - e^{\frac{1}{\sqrt{\alpha-1}} \ k_1 \ \cos t + k_2} + k_3$$

where $k_1 \geq 0$, k_2 and k_3 are arbitrary constants. (The last integral may be integrated by Taylor (expansion and) approximation).

Unlike the previous cases, the "generic" family of geodesics $t \to (x(t), y(t))$ given by (3.12) contains not only lines and exponential curves, but also some quite strange ones. The following example provide geodesics of (G, g), with an unexpected behavior.

Consider in the preceding formulae: $\alpha := 5$, $k_1 = 1$, $k_2 = k_3 := 0$. Using the Taylor decomposition for y', we derive the (approximative) parametric equations of the geodesic

$$x(t) = 0.5 \ \cos t$$

$$y(t) = -e^{0.5 \ \cos t} + 0.0721315556 t^5 - 0.4121803177 t^3 + 1.6487212707 t.$$

For different intervals of definition (different resolution/scale), the (Maple) graphics of this geodesic are given in the figures 2,3 and 4.

Remark. The preceding example has the following interpretation: consider the 2-dimensional non-commutative Lie group G as "Universe" (configurations space), subject to a unique and global field of *forces* $\xi := E_1 + E_2$, invariant under the left translations. Consider a left invariant Riemannian metric $g \in \mathrm{Riem}(G, \xi)$, as a "measure instrument" for the "*matter*" of G. A (virtual) free falling particle in (G, g) may have a regular trajectory, for example when this one is an integral curve of ξ (Figure 1). Surprisingly, in this highly invariant (regular) dynamical framework, *chaotic-like free falling particles may also appear*, as Figures 2,3,4 suggest.

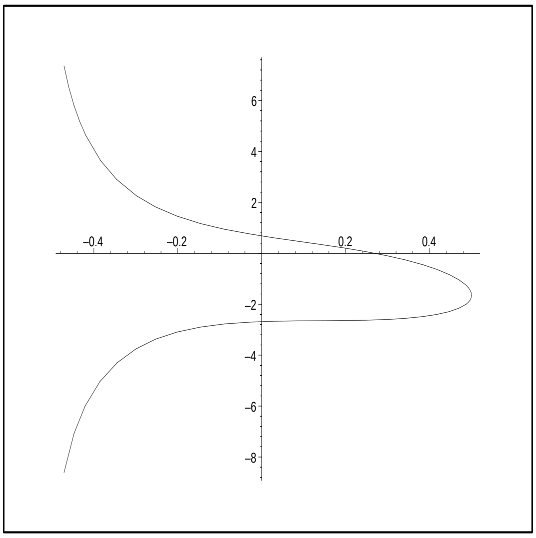

Figure 2: Definition interval [-2.8;2.8]

4 Various possible applications

From Proposition 2, we see that the (negative !) Gaussian curvature k of (G, g) is proportional with the energy of the vector field ξ,with respect to g, calculated as $\Delta := g(\xi, \xi) = a^2\alpha - b^2\beta$ (i.e. $k = -a^2\Delta$). Moreover, the energy of ξ varies following variations of its E_1-projection, but is insensitive to variations of its E_2-projection.

This mathematical results express a hidden relationship between curvature and energy, highly plausible to be useful in modeling real life facts as suggested by many empirical observations and experimental facts. The strong connections between energy and (various geometrical notions related to) curvature, encountered in Theoretical Physics, especially in Relativity, are well-known (see [7]). In what follows, we give several examples from various other domains:

(i) In [4], the authors study the geometric properties of the energy landscape of coarse-grained, off-lattice models of polymers; they use a suitable metric on the the configuration space, depending on the potential energy function, and suppose the dynamical trajectories are the geodesics of the respective metric. By numerical simulations, they show correlations between fluctuations of the curvature and the folding transition, which allows distinguishing different families of polymers.

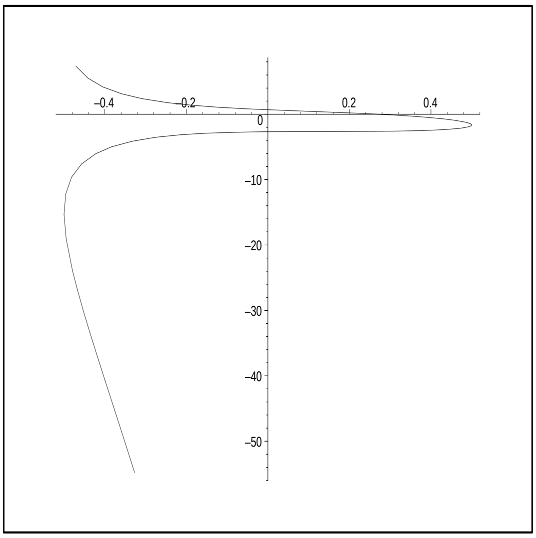

Figure 3: Definition interval [-4; 3]

(ii) In [17], some Nature's geometries are studied, which include wavy (or fractal) edges, where a pattern repeats on different scales. One family of such patterns includes the complex wavy structures that are found along the edges of thin living tissues (flowers,leaves,etc).This complexity usually is considered to have genetic roots. However, by applying simple growth laws and principles from physics and geometry and testing their ideas with flexible synthetic membranes (and computer simulations), the authors have found how variations of some surface metrics (of negative (!) Gaussian curvature) lead to wavy shape variations (related to membrane energy).

(iii) In [2], one studies the subcellular protein localization for eukaryotic cells. A novel physical mechanism is proposed, based on the two-dimensional curvature of the membrane, for spontaneous lipid targeting to the poles and division site of rod-shaped bacterial cells. The energy of the membrane is expressed in terms of curvature and one remarks that some geometrical constraints of the plasma membrane by a more rigid bacterial cell wall leads to lipid microphase separation.

(iv) In [10], one presents a boundary effect detection method of pinpointing structural damage locations, using operational deflection shapes measured by a scanning laser vibrometer. The numerical and experimental studies rest on the fitting of four coefficients C_1, C_2, C_3 and C_4, from a linear equation. The quantity $C_1 \cdot C_3$ is proportional to the difference of kinetic and elastic energy densities, $C_3 - C_1$ is proportional to the curvature and $C_4 - C_2$ is proportional to the spatial derivative of the curvature. Even if the dependency is not directly proportional, variations of energy correspond to variations of curvature, and conversely.

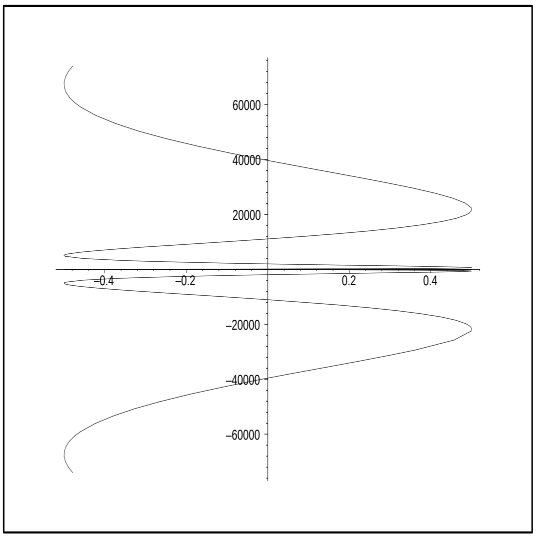

Figure 4: Definition interval [-16;16]

5 Conclusions

Given a left invariant vector field ("field of forces") ξ on a 2-dimensional Lie group G, we determined *all* the left invariant Riemannian metrics g, such that ξ becomes geodesic in (G, g); moreover, we characterized *all* the geodesics (i.e. the possible trajectories of free falling particles) moving in a "Universe" (G, g).

Two types of dynamics arise: a regular one (as we had expected), but also a "chaotic"-like one (which is quite unusual). This may explain why (and how) "symmetric" fields of forces in "symmetric" environments may produce "free falling" trajectories with broken symmetry. Examples in the previous section, imported from real life via experimental sciences, provide support for our speculative remarks.

Recent interesting results concerning the interplay between vector fields and Riemannian metrics were obtained in [9] and [18], where applications and interpretations in Convex optimization theory may be found.

References

[1] V. Arnold, *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits*, Ann. Inst. Fourier, 16 (1966), 316-361.

[2] K. Casey Huang, R. Mukhopadhyay and N. S. Wingreen, *A Curvature-Mediated Mechanism for Localization of Lipids to Bacterial Poles*, PLoS Comput Biol 2 (11), (2006), 1357-1364.

[3] B. Khesin, *Topological Fluid Dynamics*, Notices Amer. Math. Soc., 52, 1,(2005) 9-19.

[4] L. N. Mazzoni, Lapo Casetti, *Curvature of the Energy Landscape and Folding of Model Proteins*, Phys. Rev. Lett., 97 (2006), 218104- 218107.

[5] J. Milnor, *Curvatures of Left Invariant Metrics on Lie Groups*, Advances in Math., 21 (1976), 293-329.

[6] V. Obadeanu, *Structures géométriques associées à certaines systèmes dynamiques*, Balkan J. Geom. Appl., 5, 2, (2000), 81-89.

[7] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, N.Y. , 1983.

[8] A.L. Onishchik and E.B. Vinberg, *Lie Groups and Lie Algebras*, III, Encycl. Math. Sci. 41, Springer, 1994.

[9] T. Oprea, *2-Killing vector fields on Riemannian manifolds*, Balkan J. Geom. Appl. 13, 1 (2008), 87-92.

[10] P.F. Pai, S. Jin, *Locating structural damage by detecting boundary effects*, J.Sound and Vibration, 231, 4, (2000), 1079-1110.

[11] M. Postnikov, *Lie Groups and Lie Algebras*, Mir Publ. (Engl. ed.), 1986.

[12] G.T. Pripoae, *Vector fields dynamics as geodesic motion on Lie groups*, C.R.A.S. Paris, I, 342 (2006), 865-868.

[13] G.T. Pripoae, *Geodesic left invariant vector fields on 2 and 3-dimensional Lie Groups*, Tensor N.S., 68 (2007), 82-88.

[14] G.T. Pripoae, C.L. Pripoae, *Geodesible vector fields and adapted invariant Riemannian metrics*, to be published in BSGP (Proceedings), (2009).

[15] J.M. Selig, *Geometrics Fundamentals of Robotics*, (2-nd ed.), Springer Verlag, 2005.

[16] D.H. Sattinger,O.L.Weaver, *Lie Groups and Algebras with Applications to Physics, Geometry, and Mechanics*, Apl. Math. Sci. 61, Springer, 1986.

[17] E. Sharon, M. Marder, H. L. Swinney, *Leaves, Flowers and Garbage Bags: Making Waves*, American Scientist, 92 (2004), 254- 261.

[18] C. Udriste, T. Oprea, *H-convex Riemannian submanifolds*, Balkan J. Geom. Appl. 13, 2 (2008), 112-119.

[19] C. Udriste, A. Udriste, *Electromagnetic dynamical systems*, Balkan J. Geom. Appl. 2, 1, (1997), 129-140.

[20] C. Udriste, *Geometric dynamics*, Kluwer Acad. Publ., 2000.

Authors' addresses:

Cristina-Liliana Pripoae
Academy of Economic Studies, Department of Mathematics,
Piata Romana nr. 6, RO- 010374, Bucharest, Romania.
E-mail adress: cristinapripoae@yahoo.com

Gabriel-Teodor Pripoae
University of Bucharest, Faculty of Mathematics and Informatics,
Str. Academiei nr.14, RO- 010014, Bucharest, Romania.
E-mail adress: gpripoae@fmi.unibuc.ro