Minimal tensor product immersions

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Abstract. The differential geometric study of the tensor product immersion of two Riemannian immersions was initiated by Chen in [2]. In this paper we consider the tensor product immersion of an arbitrary curve in $\mathbb{R}^m$ and an arbitrary curve in $\mathbb{R}^n$. We are particularly interested when such a tensor product immersion produces a minimal surface in Euclidean space. In case that $m = 2$ and $n = 2$ or $n = 3$ this question was previously studied in [7] and [1]. Here in the present paper we obtain the general classification result, and at the same time correct some small errors in the results of [1].

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1 Introduction

Among all submanifolds, minimal submanifolds and in particular minimal surfaces in euclidean space 3-space are the most widely studied. As minimal surfaces in the three dimensional euclidean space are by now well understood, two possible generalisations are nowadays widely studied, namely the study of minimal surfaces in more general 3-dimensional spaces like the Heisenberg group see a.o. [5] or [8] or the study of minimal surfaces in higher dimensional euclidean spaces. An easy way of constructing examples of surfaces is using the notion of tensor product immersions. The systematic study of the tensor product immersion of two Riemmannian immersions was initiated by Chen in [2]. Let $f : M \to \mathbb{R}^n$ and $g : N \to \mathbb{R}^m$ be two isometric immersions of Riemannian manifolds M and N respectively, then the tensor product immersion $f \otimes g : M \times N \to \mathbb{R}^{nm}$ is defined by

$$(f \otimes g)(p, q) = f(p) \otimes g(q).$$

Here we represent an element of $\mathbb{R}^{nm}$ as a matrix with n rows and m columns. Hence $f(p) \otimes g(q) = A$, where $A = [a_{ij}] = [(f(p))_i g(q)_j] = f(p)\, {}^t g(q)$.

Regarding elements of $\mathbb{R}^{nm}$ as matrices, we see that the natural metric on $\mathbb{R}^{nm}$ can be expressed using matrix multiplication by

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$$\langle A,B\rangle = \mathrm{trace}\,(A\,{}^tB) = \mathrm{trace}\,({}^tAB) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}b_{ij}.$$

Necessary and sufficient conditions for $(f\otimes g)$ to be an immersion were derived in [2].

In this paper we are in particular interested in the tensor product immersion of an arbitrary curve $\alpha : I \to \mathbb{R}^n$ with an arbitrary curve $\beta : J \to \mathbb{R}^m$. We want to investigate when the tensor product surface of these two curves defines a minimal surface. We will prove the following theorem:

Theorem 1.1. *Let $\alpha : I \to \mathbb{R}^n$ and $\beta : J \to \mathbb{R}^m$ be curves such that the tensor product $\alpha\otimes\beta$ is a regular surface. Then the tensor product $\alpha\otimes\beta$ is a minimal surface if and only if, if necessary after interchanging α and β, one of the following conditions hold:*

(i) α is an open part of a straight line through the origin, β is contained in a plane through the origin and consequently $\alpha\otimes\beta$ is an open part of a plane,

(ii) β is congruent to an open part of a hyperbola centered at the origin and α is congruent to an open part of a circle centered at the origin.

For $n = 2$ and $m = 2$ this theorem was obtained in [7], whereas for $m = 2$ and $n = 3$ the above problem was investigated in [1]. In Theorem 2.1 of [1], 6 cases occurred. However, case (5) by an orthogonal transformation reduces to case (3), whereas case (6) (as is explained in a remark at the end of the paper) can only occur for a specific parameter in which case it reduces by an orthogonal transformation to (4). Furthermore also the extra condition on α was omitted in Theorem 2.1 of [1].

Finally we want to remark that properties of tensor products of spherical curves were investigated in [6] in order to obtain explicit examples of Willmore surfaces.

The paper is organised as follows. In the next section we recall some basic properties about the tensor product of vectors which will then be used in Section 3 in order to prove the main theorem.

2 Preliminaries

As explained in the introduction, we denote the tensor product of $u\otimes v$ of a vector $u\in\mathbb{R}^n$ and a vector $v\in\mathbb{R}^m$, as the matrix given by $u\otimes v = u\,{}^tv$. Then we have the following lemmas:

Lemma 2.1. *Let O_1 (resp. O_2) be an orthogonal transformation of $\mathbb{R}^n$ (resp. $\mathbb{R}^m$). Then the application $\mathcal{H} : \mathbb{R}^{n\times m} \to R^{n\times m} : A \mapsto O_1A\,{}^tO_2$ is an orthogonal transformation of $\mathbb{R}^{nm} = \mathbb{R}^{n\times m}$.*

Proof. As for a matrix $A\in\mathbb{R}^{n\times m}$ we have that

$$\langle O_1A\,{}^tO_2, O_1A\,{}^tO_2\rangle = \mathrm{trace}(O_1A\,{}^tO_2O_2\,{}^tA\,{}^tO_1) = \mathrm{trace}(O_1A\,{}^tA\,{}^tO_1) = \langle O_1A, O_1A\rangle$$

and

$$\langle O_1A, O_1A\rangle = \mathrm{trace}({}^tA\,{}^tO_1O_1A) = \langle A,A\rangle,$$

we conclude that $\mathcal{H}$ preserves the scalar product and hence $\mathcal{H}$ is an isometry. □

Lemma 2.2. *Let $u, \bar{u} \in \mathbb{R}^n$ and $v, \bar{v} \in \mathbb{R}^m$. Then $\langle u \otimes v, \bar{u} \otimes \bar{v} \rangle = \langle u, \bar{u} \rangle \langle v, \bar{v} \rangle$.*

Proof. We remark that the usual metric on $\mathbb{R}^n$ is given by

$$\langle u, \bar{u} \rangle = \sum_{i=1}^{n} u_i \bar{u}_i = {}^t u \bar{u} = \mathrm{trace}(u\, {}^t\bar{u}).$$

Therefore we find that

$$\begin{aligned}\langle u \otimes v, \bar{u} \otimes \bar{v} \rangle &= \mathrm{trace}(u \otimes v)\, {}^t(\bar{u} \otimes \bar{v}) = \mathrm{trace}(u\, {}^t v\, {}^t(\bar{u}\, {}^t\bar{v})) = \mathrm{trace}(u\, {}^t v \bar{v}\, {}^t\bar{u}) \\ &= \mathrm{trace}(u \langle v, \bar{v} \rangle\, {}^t\bar{u}) = \langle v, \bar{v} \rangle\, \mathrm{trace}(u\, {}^t\bar{u}) = \langle v, \bar{v} \rangle \langle u, \bar{u} \rangle.\end{aligned}$$

□

3 Minimal tensor product surfaces

From now on we will assume that $\alpha : I \to \mathbb{R}^n : t \mapsto \alpha(t)$, where $\alpha(t) = (\alpha_1, \ldots, \alpha_n)$ and $\beta : J \to \mathbb{R}^m : s \mapsto \beta(s) = (\beta_1(s), \ldots, \beta_m(s))$ are two curves in Euclidean space such that their tensor product defines an immersion of $I \times J$ into $\mathbb{R}^{nm} = \mathbb{R}^{n \times m}$.

Hence we have that $\alpha \otimes \beta(t, s) = {}^t(\alpha_1(t), \ldots, \alpha_n(t))(\beta_1, \ldots, \beta_m(s))$. Note now that

$$O_1\alpha \otimes O_2\beta = O_1\alpha\, {}^t\beta\, {}^tO_2 = \langle(\alpha \otimes \beta),$$

where O_1 and O_2 are respectively orthogonal transformations of $\mathbb{R}^n$ and $\mathbb{R}^m$. Consequently from Lemma 1 we have that $\alpha \otimes \beta$ and $O_1\alpha \otimes O_2\beta$ are related by an isometry. This shows that we still have the freedom to change the curves α and β by an orthogonal transformation.

Similarly from the expression of the tensor product, $(\alpha \otimes \beta) = (\lambda\alpha \otimes \frac{1}{\lambda}\beta)$, we see that we can also multiply the curve α by an arbitrary non zero constant, provided we divide the curve β by the same non zero constant.

We now write $f(t, s) = \alpha \otimes \beta(t, s)$. A straightforward computation then gives that

$$f_t = \frac{\partial f}{\partial t} = \alpha'(t) \otimes \beta(s) \qquad f_s = \frac{\partial f}{\partial s} = \alpha(t) \otimes \beta'(s).$$

It then follows that the components of the induced metric are respectively given by

$$\begin{aligned} g_{11} &= \langle f_t, f_t \rangle = \|\beta\|^2 \|\alpha'\|^2 \\ g_{12} &= \langle f_t, f_s \rangle = \langle \beta, \beta' \rangle \langle \alpha, \alpha' \rangle, \\ g_{22} &= \langle f_s, f_s \rangle = \|\beta'\|^2 \|\alpha\|^2. \end{aligned}$$

Note that f defines an immersion if and only if $g_{11}g_{22} - g_{12}^2 \neq 0$. Note that by the Cauchy-Schwartz inequality we have that $g_{11}g_{22} - g_{12}^2 \geq 0$. It also follows that in order to have an immersion, we have to exclude the points such that either

1. $\|\alpha(t_0)\| = 0$ or $\|\alpha'(t_0)\| = 0$,
2. $\|\beta(s_0)\| = 0$ or $\|\beta'(s_0)\| = 0$,
3. the vectors $\alpha'(t_0)$ and $\alpha(t_0)$ as well as the vectors $\beta'(s_0)$ and $\beta(s_0)$ are dependent.

Using the Gram-Schmidt orthonormalisation procedure we also find that $e_1 = \frac{1}{\sqrt{g_{11}}} f_t$ and $e_2 = \frac{1}{\sqrt{g_{11}(g_{11}g_{22}-g_{12}^2)}}(g_{11}f_s - g_{12}f_t)$ form an orthonormal basis of the tangent space.

We now introduce some more notation. Let $i, j \in \{1, \dots, n\}$, with $i \neq j$ and $a, b \in \mathbb{R}$. We denote by $v_{ij}(a, b)$ the vector $v = {}^t(v_1, \dots, v_n)$ of $\mathbb{R}^n$ given by

$$\begin{aligned} &v_k = 0, \quad k \neq i, k \neq j \\ &v_i = a \\ &v_j = b. \end{aligned}$$

Similarly for $p, q \in \{1, \dots, m\}$, $p \neq q$ and $a, b \in \mathbb{R}$ we define $w_{pq}(a, b) \in \mathbb{R}^m$.

We now define vectors of $\mathbb{R}^{nm}$ by

$$n^1_{ijpq} = v_{ij}(-\alpha_j, \alpha_i) \otimes w_{pq}(-\beta_q, \beta_p) \qquad n^2_{ijpq} = v_{ij}(-\alpha'_j, \alpha'_i) \otimes w_{pq}(-\beta'_q, \beta'_p).$$

Then we have

$$\begin{aligned} \langle n^1_{ijpq}, f_t\rangle &= (-\alpha_j\alpha'_i + \alpha_i\alpha'_j)(-\beta_p\beta_q + \beta_q\beta_p) = 0 \\ \langle n^1_{ijpq}, f_s\rangle &= (-\alpha_j\alpha_i + \alpha_i\alpha_j)(-\beta'_p\beta_q + \beta'_q\beta_p) = 0 \\ \langle n^2_{ijpq}, f_s\rangle &= (-\alpha'_j\alpha_i + \alpha'_i\alpha_j)(-\beta'_p\beta'_q + \beta'_q\beta'_p) = 0 \\ \langle n^2_{ijpq}, f_t\rangle &= (-\alpha'_j\alpha'_i + \alpha'_i\alpha'_j)(-\beta_p\beta'_q + \beta_q\beta'_p) = 0 \end{aligned}$$

Consequently we have that the vectors n^1_{ijpq} and n^2_{ijpq}, $i, j \in \{1, \dots, n\}$, with $i \neq j$, and $p, q \in \{1, \dots, m\}$, with $p \neq q$ are normal vectors.

We then have the following lemmas.

Lemma 3.1. *Let $\alpha : I \to \mathbb{R}^n$ and $\beta : J \to \mathbb{R}^m$ be curves such that the tensor product of α and β is a regular surface. Then the tensor product $f = \alpha \otimes \beta$ is a minimal surface if and only if $g_{22}f_{tt} + g_{11}f_{ss} - 2g_{12}f_{ts}$ is a tangent vector.*

Proof. We have that the surface is minimal if and only if, for any orthonormal basis $\{e_1, e_2\}$ we have that $h(e_1, e_1) + h(e_2, e_2) = 0$, where h denotes the second fundamental form of the immersion. Using the previously constructed orthonormal basis, we find that this is equivalent with $\langle g_{22}f_{tt} + g_{11}f_{ss} - 2g_{12}f_{st}, n\rangle = 0$, for any normal vector n. This concludes the proof. □

Lemma 3.2. *Let $\alpha : I \to \mathbb{R}^n$ and $\beta : J \to \mathbb{R}^m$ be curves. Suppose that the tensor product $f = \alpha \otimes \beta$ is a minimal surface then the components of α and β satisfy the following system of differential equations:*

$$\langle \alpha, \alpha'\rangle\langle \beta, \beta'\rangle(\beta'_p\beta_q - \beta'_q\beta_p)(-\alpha'_i\alpha_j + \alpha'_j\alpha_i) = 0, \tag{3.1}$$

$$(\beta''_p\beta'_q - \beta''_q\beta'_p)(\alpha_i\alpha'_j - \alpha_j\alpha'_i) + \frac{\langle \alpha, \alpha\rangle\langle \beta', \beta'\rangle}{\langle \alpha', \alpha'\rangle\langle \beta, \beta\rangle}(\beta_p\beta'_q - \beta_q\beta'_p)(\alpha''_i\alpha'_j - \alpha''_j\alpha'_i) = 0. \tag{3.2}$$

Proof. First we remark that

$$\begin{aligned} \langle f_{tt}, n^2_{ijpq}\rangle &= (-\alpha''_i\alpha'_j + \alpha''_j\alpha'_i)(-\beta'_q\beta_p + \beta'_p\beta_q), \qquad &\langle f_{ss}, n^1_{ijpq}\rangle = 0, \\ \langle f_{st}, n^1_{ijpq}\rangle &= (-\alpha'_i\alpha_j + \alpha'_j\alpha_i)(-\beta_q\beta'_p + \beta_p\beta'_q), \qquad &\langle f_{tt}, n^1_{ijpq}\rangle = 0, \\ \langle f_{ss}, n^2_{ij}\rangle &= (-\alpha_i\alpha'_j + \alpha_j\alpha'_i)(-\beta'_q\beta''_p + \beta''_q\beta'_p), \qquad &\langle f_{st}, n^2_{ij}\rangle = 0. \end{aligned}$$

Substituting the above expressions in the previous lemma then completes the proof. □

Now we will show how we can solve the above system of differential equations and determine the curves α and β explicitly. Remark that if $\beta'_p\beta_q - \beta_p\beta'_q = 0$, for all indices $p, q \in \{1, \ldots, m\}$, we have that $\beta'(s)$ and $\beta(s)$ are linearly dependent vectors. This implies that locally β is an open part of a straight line through the origin. Similarly if $\alpha_i\alpha'_j - \alpha_j\alpha'_i = 0$, for all indices i and j, we have that $\alpha'(t)$ and $\alpha(t)$ are linearly dependent vectors. Hence in that case α is an open part of a straight line through the origin. In order to complete the proof we now consider several cases.

3.1 Case 1

First we assume that neither α nor β are locally contained in a line through the origin. This implies that there exist indices p and q such that $\beta'_p\beta_q - \beta_p\beta'_q \neq 0$ and that there exist indices i and j such that $\alpha_i\alpha'_j - \alpha_j\alpha'_i \neq 0$. From (3.1) of Lemma 4 this implies that $\langle\alpha, \alpha'\rangle\langle\beta, \beta'\rangle = 0$. If necessary after interchanging α and β we may assume that $\langle\alpha, \alpha'\rangle = 0$. This implies that there exists a constant c such that $\langle\alpha, \alpha\rangle = c$. As we assumed that the surface is regular, we must have that $c \neq 0$ and therefore as mentioned before, we can still rescale the curve α (and correspondingly rescale the curve β). Hence we may assume that

$$\langle\alpha, \alpha\rangle = 1.$$

We also assume that the curve α is arclength parametrised. Taking now the indices p and q mentioned before, together with arbitrary indices i and j and substituting this into (3.2) we find that

$$\frac{\langle\beta,\beta\rangle(\beta''_p\beta'_q - \beta''_q\beta'_p)}{\langle\beta',\beta'\rangle(\beta_p\beta'_q - \beta_q\beta'_p)}(\alpha_i\alpha'_j - \alpha_j\alpha'_i) + (\alpha''_i\alpha'_j - \alpha''_j\alpha'_i) = 0 \tag{3.3}$$

Note that the above equation is valid for every value of s and t. So, if we take s_0 and put $k = \frac{\langle\beta,\beta\rangle(\beta''_p\beta'_q - \beta''_q\beta'_p)}{\langle\beta',\beta'\rangle(\beta_p\beta'_q - \beta_q\beta'_p)}(s_0)$, we get that

$$k(\alpha_i\alpha'_j - \alpha_j\alpha'_i) + (\alpha''_i\alpha'_j - \alpha''_j\alpha'_i) = 0.$$

As before the above equations, which are valid for all indices i and j, imply that the vectors $\alpha'' + k\alpha$ and α' are linearly dependent. On the other hand, deriving $\langle\alpha, \alpha\rangle = 1 = \langle\alpha', \alpha'\rangle$ we get that those vectors are also mutually orthogonal. Consequently we must have that

$$\alpha'' = -k\alpha. \tag{3.4}$$

However, as $\langle\alpha, \alpha\rangle = 1$, we have that $\langle\alpha'', \alpha\rangle = -\langle\alpha', \alpha'\rangle = -1$. Combining this with (3.4) we deduce that $k = 1$. Hence there exist constant vectors C_1 and C_2 such that $\alpha(t) = C_1\cos t + C_2\sin t$. As $\langle\alpha(t), \alpha(t)\rangle = 1$, we deduce that $\langle C_1, C_1\rangle = \langle C_2, C_2\rangle = 1$ and $\langle C_1, C_2\rangle = 0$. Hence by applying an orthogonal transformation, we may assume that

$$\alpha(t) = (\cos t, \sin t, 0, \ldots, 0).$$

This shows that α is a circle centered at the origin. Substituting these expressions into the first equation of Lemma 4 we deduce for arbitrary indices p and q that

$$\langle \beta, \beta \rangle (\beta_p'' \beta_q' - \beta_q'' \beta_p') = \langle \beta', \beta' \rangle (\beta_p \beta_q' - \beta_q \beta_p').$$

So far we have not yet chosen a parameter for the curve β. In order to simplify the above equation, we can take a parameter for the curve β such that $\langle \beta, \beta \rangle = \langle \beta', \beta' \rangle$. Rewriting now the above equation we find that $-\beta_q'(\beta_p'' - \beta_p) + \beta_p'(\beta_q'' - \beta_q) = 0$. This can be interpreted as the condition that the vectors $\beta'' - \beta$ and β' are linearly dependent. On the other hand deriving $\langle \beta, \beta \rangle = \langle \beta', \beta' \rangle$, we obtain that $\beta'' - \beta$ and β' are also mutually orthogonal. So we must have that $\beta'' = \beta$ and hence there exist constant vectors D_1 and D_2 such that $\beta(s) = D_1 e^s + D_2 e^{-s}$. ¿From $\langle \beta, \beta \rangle = \langle \beta', \beta' \rangle$, we then deduce that D_1 and D_2 are orthogonal vectors. Hence by an orthogonal transformation we may assume that $D_1 = (a, 0)$ and $D_2 = (0, b)$. Hence the curve α satisfies

$$\beta(s) = (ae^s, be^{-s}, 0, \ldots, 0),$$

which is the equation of an orthogonal hyperboloid centered at the origin. This completes the proof in this case.

3.2 Case 2

In this case we assume that at least one of the curves α or β is contained in a straight line through the origin. In view of the symmetry of the problem we may assume that β is an open part of a straight line. So by choosing an arc length parameter for the curve β and by applying an orthogonal transformation we may assume that $\beta(s) = (s, 0, \ldots, 0)$. So the tensor product immersion is actually contained in a totally geodesic $\mathbb{R}^n$ and given by $f(t, s) = s(\alpha_1(t), \ldots, \alpha_n(t))$. Note that the image of f corresponds with the image of g where $g(t, s) = s(\frac{\alpha_1(t)}{\|\alpha\|}, \ldots, \frac{\alpha_n(t)}{\|\alpha\|})$. Consequently without loss of generality we may assume that $\langle \alpha, \alpha \rangle = 1$ and $\langle \alpha', \alpha' \rangle = 1$. Note that now $f_{ss} = 0$ and $f_{ts} = \frac{1}{s} f_t$ is always a tangent vector. So in order for the immersion to be minimal, by Lemma 3.1, we must have that $\frac{1}{s} f_{tt} = (\alpha_1''(t), \ldots, \alpha_n''(t)) = \alpha''(t)$, is a tangent vector to the immersion. As the tangent space is spanned by α and α' and we also have that $\langle \alpha'', \alpha' \rangle = 0 = \langle \alpha, \alpha' \rangle$ this can only be the case if α'' and α are linearly dependent. Consequently there exists a function $\mu : I \to \mathbb{R}$ such that

$$\alpha''(t) = \mu(t)\alpha(t).$$

The function μ can be determined by

$$\mu(t) = \langle \alpha''(t), \alpha(t) \rangle = -\langle \alpha'(t), \alpha'(t) \rangle = -1.$$

Hence there exist constant vectors such that $\alpha(t) = C_1 \cos t + C_2 \sin t$. As before we get that $\langle C_1, C_1 \rangle = \langle C_2, C_2 \rangle = 1$ and $\langle C_1, C_2 \rangle = 0$. Consequently by an orthogonal transformation we may assume that $\alpha(t) = (\cos t, \sin t, 0, \ldots, 0)$ and we find that the image of the tensor product is again an open part of a plane.

Remark 3.1 We look at the complex immersion $\mathbb{C} \to \mathbb{C}^2 : z \mapsto (\cos(z), \sin(z))$. Writing $z = t + is$, we get that this corresponds to the map

$$(t, s) \mapsto (\cos(t)\cosh(s), \sin(t)\sinh(s), \cos(t)\sinh(s), \sin(t)\cosh(s)),$$

which is congruent to the tensor product of a circle and a hyperbola (both centered at the origin). As a complex immersion is always minimal we deduce that this tensor product is indeed a minimal immersion. Also a plane is trivially minimal. This shows the converse direction of the main theorem.

As none of the previous examples can be put together in a differentiable way we conclude the proof of the main theorem.

Remark 3.2 In the previously cited paper [1], one assumed that the normal space was spanned by the vectors n^1_{ijpq} and n^2_{ijpq}. Although that is true for a generic tensor product, it is no longer valid if one of the curves is a straight line through the origin. This explains the missing condition on the curve c_1 in their Theorem 2.1. In their final two cases, as here, they obtain that the curve α satisfies $\alpha''(t) = -k\alpha(t)$ together with $\langle \alpha, \alpha \rangle = \langle \alpha', \alpha' \rangle = 1$. However they fail to conclude that this is only possible if $k = 1$ leading in their case to additional (incorrect) solutions for the curve β (sinusoidal spiral).

Remark 3.3 Of course the same methods can also be applied for studying a tensor product of a Riemmannian and a Lorentzian curve, as was done in a special case in [4]. However, similar as in the Riemannian case, also in that case the additional spiral solutions are incorrect.

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