Product type inequalities for the geometric category

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Abstract. The geometric category of a space was defined by R. H. Fox and it is a generalization of the Lusternik-Schnirelmann category. In the first part of this paper we give a characterization theorem of geometric category. Then we prove the product inequality and give some examples. In the last part of the paper we introduce the relative geometric category and we study some of its properties.

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1 Introduction

The Lusternik-Schnirelmann category of a topological space X is the minimal number n with the property that there exists a covering $U_1, U_2, \ldots, U_n$ of X such that U_i, $i = \overline{1,n}$, are open and contractible in X. Denote $cat(X) = n$. If such a covering does not exist, define $cat(X) = \infty$. If Y is a subspace of X, then the Lusternik-Schnirelmann category of Y in X is the minimal n with the property that there exists a covering $U_1, U_2, \ldots, U_n$ of Y in X such that all U_i are open and contractible in X. Denote $cat(Y, X) = n$. If such a covering does not exist, define $cat(Y, X) = \infty$. For $Y = \varnothing$, take $cat(Y, X) = 0$. Remark that $cat(X, X) = cat(X)$.

This homotopy invariant was defined by L. Lusternik and L. Schnirelmann in [17] over 70's years ago in order to provide a lower bound on the number of critical points for any smooth function on a manifold and used to prove the famous theorem concerning the existence of at least three closed geodesics on a sphere.

It is important to emphasize that Lusternik-Schnirelmann category, together with its generalizations, is still intensively studied and applied in modern domains, such as topology of manifolds, topological robotics or complexity of algorithms, see [1]-[4], [12], [13], [20] and [23].

A natural extension of Lusternik-Schnirelmann category is the geometric category; instead of subsets contractible in X, we take selfcontractible subsets. The origin of this definition can be found in the paper [15] of R. H. Fox. The author gave a characterization theorem of Lusternik-Schnirelmann category by using the notion of categorical sequence: if Y is a subspace of X, then a categorical sequence for Y in

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X is a family of subsets $Y_1, Y_2, \ldots, Y_k$ of X such that $Y_1 \subset Y_2 \subset \ldots \subset Y_k = Y$ and $Y_1, Y_2 \setminus Y_1, \ldots, Y_k \setminus Y_{k-1}$ are contained in open subsets of X which are contractible in X. k is called the length of the categorical sequence. Let X be a path-connected separable metric space and Y a subspace of X such that $cat(Y, X) < \infty$. Then $cat(Y, X)$ is the minimum length of all categorical sequences of Y in X.

In the first part of this paper we define the categorical sequence corresponding to geometric category and we prove a Fox type theorem. Then we obtain the product type inequality for geometric category. In the last part we define the relative geometric category and establish its basic properties.

2 g-categorical sequences

For a topological space X it is natural to consider covers with subsets contractible in themselves. We call such subsets *selfcontractible*. What we obtain is called geometric category. This variant of category has the origin in the above cited paper of Fox [15].

Definition 2.1. Let X be a topological space. The geometric category of X, $gcat(X)$, is the minimal number n such that there exists a covering of X with n open and selfcontractible subsets.

For any space X we have $cat(X) \leq gcat(X)$, any selfcontractible subset of X being contractible in X. An example of space whose geometric category does not coincide with its Lusternik-Schnirelmann category is $S^2/\{\text{three points}\}$ and it is given in [15]. It is proved that $cat(S^2/\{\text{three points}\}) = 2$ and $gcat(S^2/\{\text{three points}\}) = 3$. In fact, in [7] M. Clapp and L. Montejano were able to construct for every integer n a polyhedron K_n with the property that $gcat(K_n) - cat(K_n) \geq n$. See also [18], [19] and [6]. On the other hand, $S^2 \vee S^1 \vee S^1$ and $S^2/\{\text{three points}\}$ are homotopically equivalent, but $gcat(S^2 \vee S^1 \vee S^1) = 2$. See [15] and [9]. Moreover, the poliedra K_n have the additional property that $gcat(K_n) - gcat(K_n \times I) \geq n$. This fact prove that the geometric category is not a homotopy invariant and make the geometric category difficult to use. A possibility to obtain a homotopy invariant in this case is to consider the minimal value of the geometric category for all spaces Y which are homotopically equivalent with X. Then we get the strong category, $Cat(X)$, introduced by T. Ganea in [16].

Our first result gives a particular covering for any topological space X with finite geometric category. We need the following lemma, which follows from Definition 2.1.

Lemma 2.2. *For any subspaces X and Y of a topological space, the inequality*

$$gcat(X \cup Y) \leq gcat(X) + gcat(Y)$$

is satisfied.

Proposition 2.3. *Let X be a topological space. If $gcat(X) = n$, there exists a covering of X by n pairwise disjoint subsets of X such that any subset is contained in some selfcontractible subset of X.*

Proof. We prove the assertion by induction relative to n. The case $n = 1$ is obvious. Assume $gcat(X) = 2$. Then $X = U_1 \cup U_2$, where U_1 and U_2 are selfcontractible.

Take $\widetilde{U_1} = U_1$ and $\widetilde{U_2} = U_2 \setminus U_1$. Now let the assertion be true for $gcat(X) \leq k-1$ and prove it for $gcat(X) = k$. Take $X = U_1 \cup U_2 \cup \ldots \cup U_k$, where $U_1, U_2, \ldots, U_k$ are selfcontractible. Define $\widetilde{X} = U_2 \cup \ldots \cup U_k$. By Lemma 2.2, we have $gcat(X) \leq gcat(U_1) + gcat(\widetilde{X}) = 1 + gcat(\widetilde{X})$ and $gcat(\widetilde{X}) \geq k-1$. Assume now that we have $gcat(\widetilde{X}) < k-1$; we obtain $gcat(X) \leq gcat(U_1) + gcat(\widetilde{X}) < k$, contradiction. It follows that $gcat(\widetilde{X}) = k$. By the induction hypothesis, there exist $\widetilde{U_2}, \ldots, \widetilde{U_k}$ pairwise disjoint contained in selfcontractible subsets of X such that $\widetilde{X} = \widetilde{U_2} \cup \ldots \cup \widetilde{U_k}$. Then take $\widetilde{U_1} = U_1 \setminus \widetilde{X}$. □

Definition 2.4. We call a g-categorical sequence of length k for X a family $X_1, X_2, \ldots, X_k$ of subsets of X having the following properties:
(i) $X_1 \subset X_2 \subset \ldots \subset X_k = X$;
(ii) $X_1, X_2 \setminus X_1, \ldots, X_k \setminus X_{k-1}$ are contained in open selfcontractible subsets of X.

We give now a characterization of geometric category by using g-categorical sequences. More precisely, we prove a generalization of [15, Theorem 5.1].

Theorem 2.5. *Let X be a topological space such that $gcat(X) < \infty$. Then $gcat(X)$ is the minimum length of all g-categorical sequences of X.*

Proof. Let $X_1, X_2, \ldots, X_k$ be a g-categorical sequence of X. We prove that $gcat(X) \leq k$ by induction. For $k = 1$ it is true. Assume the assertion true for $gcat(X) \leq k-1$. Because the family $X_2 \setminus X_1, \ldots, X_k \setminus X_1$ satisfy the Definition 2.4, it is a g-categorical sequence of $X \setminus X_1$. From the induction hypothesis we have $gcat(X \setminus X_1) \leq k-1$. We apply the subadditivity property for geometric category and the inequality $gcat(X) \leq gcat(X \setminus X_1) + gcat(X_1) \leq k$ follows. Conversely, assume that $gcat(X) = k$ and prove the existence of a g-categorical sequence of X with length k. Let $U_1, \ldots, U_k$ open and selfcontractible subsets of X such that $X = U_1 \cup \ldots \cup U_k$. We can take $X_1 = U_1, X_2 = U_1 \cup U_2, \ldots, X_k = U_1 \cup \ldots \cup U_k$. Remark that $X_1, X_2, \ldots, X_k$ is a g-categorical sequence of X. □

3 The product inequality for geometric category

We give a method to compute the geometric category for product spaces. For this, we follow the standard construction introduced in [15] for Lusternik-Schnirelmann category. See [22] and [8] for some generalizations.

Recall that a space is completely normal if any subsets A, B such that $\overline{A} \cap B = \varnothing$ and $A \cap \overline{B} = \varnothing$ have open and disjoint neighborhoods, which means that every subspace of the space is normal. Metric spaces and CW-complexes are completely normal. See [10].

The main result of this section uses the following lemma, which can be proved as in [15].

Lemma 3.1. *Let X be a completely normal space and $A, B \subset X$. If A and B are open and disjoint in their union, then*

$$gcat(A \cup B) = \max\{gcat(A), gcat(B)\}.$$

Theorem 3.2. *For any two path-connected spaces X and Y such that $X \times Y$ is completely normal, the following inequality is satisfied:*

$$gcat(X \times Y) \leq gcat(X) + gcat(Y) - 1.$$

Proof. Denote $gcat(X) = m$ and $gcat(Y) = n$. Then there exist g-categorical sequences of X respectively Y, say $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$. Define $A_k = \bigcup_{i+j=k+1} X_i \times Y_j$ and remark that $A_{k+1} \setminus A_k = \bigcup_{i+j=k+2} (X_i \setminus X_{i-1}) \times (Y_j \setminus Y_{j-1})$. Any $X_i \setminus X_{i-1}, Y_j \setminus Y_{j-1}$ is contained in a selfcontractible subset of X respectively Y and any two sets of the above union are disjoint. By applying Lemma 3.1, it follows that any $A_{k+1} \setminus A_k$ is contained in a selfcontractible subset of $X \times Y$. Then $A_1, A_2, \ldots, A_{m+n-1}$ is a g-categorical sequence for $X \times Y$ and $gcat(X \times Y) \leq m + n - 1$. □

Remark 3.3. For $X_1, \ldots, X_n$ path-connected spaces, Theorem 3.2 implies

$$gcat(X_1 \times \ldots \times X_n) \leq \sum_{i=1}^{n} gcat(X_i) - n + 1.$$

Remark 3.4. We have the following inequality:

$$gcat(X \times Y) \geq \max\{gcat(X), gcat(Y)\}.$$

The proof is similar to its classical Lusternik-Schnirelmann counterpart, established in [15].

Denote the cohomology ring of a space X with some chosen coefficients R by $H^*(X, R)$ and the reduced cohomology by $\widetilde{H}^*(X, R)$.

Definition 3.5. The cuplength of X with coefficients in R is the least integer k (or ∞) such that all $(k+1)$-fold cup products vanish in the reduced cohomology. Denote this number by $cup\text{-}length_R(X)$ or, simply, by $cup\text{-}length(X)$.

The cuplength of a space is useful in order to estimate from below the Lusternik-Schnirelmann category of the space:

Theorem 3.6. ([5]) *The following inequality holds for any R:*

$$1 + cup\text{-}length_R(X) \leq cat(X).$$

Remark 3.7. Theorem 3.6 gives a lower bound for geometric category:

$$1 + cup\text{-}length_R(X) \leq cat(X) \leq gcat(X).$$

We need the following property of cuplength:

Proposition 3.8. ([14]) *If X and Y are topological spaces, then*

$$cup\text{-}length(X \times Y) \geq cup\text{-}length(X) + cup\text{-}length(Y).$$

Example 3.9. The sphere S^n can be covered by two hemispheres extended to open sets. Any hemisphere being selfcontractible, it follows that $gcat(S^n) \leq 2$. On the other hand, it is known that $cup\text{-}length_Z(S^n) = 1$. Theorem 3.6 implies $gcat(S^n) \geq cat(S^n) \geq 2$. Then $gcat(S^n) = cat(S^n) = 2$.

Example 3.10. The torus T^n has the cohomology ring $H^*(T^n, Q)$ an exterior algebra with n generators; then $cup\text{-}length_Q(T^n) = n$. The Remarks 3.3 and 3.7 give: $n+1 \leq cat(T^n) \leq gcat(T^n) \leq ngcat(S^1) - n + 1 = n + 1$ and $gcat(T^n) = n + 1$.

Example 3.11. We prove that $gcat(S^n \times T^k) = 2 + k$. Theorem 3.2 implies $gcat(S^n \times T^k) \leq gcat(S^n) + gcat(T^k) - 1 = 2 + k$. By Proposition 3.8 and Remark 3.7 we obtain $gcat(S^n \times T^k) \geq cat(S^n \times T^k) \geq 1 + cup\text{-}length(S^n) + cup\text{-}length(T^k) = 2 + k$.

Example 3.12. The above arguments give $gcat(S^n \times S^k) \leq gcat(S^n) + gcat(S^k) - 1 = 3$ and $gcat(S^n \times S^k) \geq cat(S^n \times S^k) \geq 1 + cup\text{-}length(S^n) + cup\text{-}length(S^k) = 3$. Then $gcat(S^n \times S^k) = 3$.

4 Relative geometric category

Let X be a topological space and Y a subspace of X. Let $A \neq \varnothing$ be closed in X such that $A \subset Y$. We say that a subset U of X containing A is *categorical relative to A in X* if $j \circ \rho = i$ up to a homotopy for some $\rho : (U, A) \to (A, A)$, where $i : (U, A) \hookrightarrow (X, A)$ and $j : (A, A) \hookrightarrow (X, A)$ are inclusions, i.e. U is deformable in X into A relative to A.

In accordance with E. Fadell [11], the *category of Y in X relative to A*, denoted $cat_A(Y, X)$, is the minimal number n with the property that there exists a covering $U_0, U_1, \ldots, U_n$ of Y in X such that U_0 is open and categorical relative to A in X and $U_i, i = \overline{1, n}$, are open and contractible in X. If such a covering does not exist, define $cat_A(Y, X) = \infty$. If $A = \varnothing$ we take $U_0 = \varnothing$ and $cat_A(Y, X) = cat(Y, X)$. The category of X relative to A is defined by $cat_A(X, X) = cat_A(X)$. For any pair (X, A) we have $cat(X) \geq cat_A(X)$.

We mention that if (X, A) is an ANR pair, then this is a minor variation of the definition given by M. Reeken in [21].

It is natural to define the relative geometric category. For this purpose, we fix a topological space X and a closed nonempty subset A of X.

Definition 4.1. The geometric category of X relative to A is the minimal number n such that $X = U_0 \cup U_1 \cup \ldots \cup U_n$, where U_0 is categorical in X relative to A and $U_i, i = \overline{1, n}$, are open in X and selfcontractible. Denote it by $gcat_A(X)$. If such a covering does not exist, define $gcat_A(X) = \infty$.

Remark 4.2. If $A = \varnothing$, take $U_0 = \varnothing$ and $gcat_{\varnothing}(X) = gcat(X)$.

Remark 4.3. In fact, E. Fadell [11] gave two definitions of relative category, but for relative geometric category it is not possible to do this. By denoting $cat^*_A(X) = n$ if $X = U_1 \cup \ldots \cup U_n$, each U_i being categorical relative to A in X and n minimal with this property, it is obvious that the corresponding relative geometric category, say $gcat^*_A(X)$, coincides with $cat^*_A(X)$.

We give now some basic properties of relative geometric category.

Proposition 4.4. *Let X and Y be subspaces of a topological space and $\varnothing \neq A \subset X$ be closed.*
(i) The following subadditivity property holds: $gcat_A(X \cup Y) \leq gcat_A(X) + gcat(Y)$.
(ii) (Monotonicity property) If $X \subset Y$, then $gcat_A(X) \leq gcat_A(Y)$.
(iii) If $X \subset Y$, then $gcat(Y \setminus X) \geq gcat_A(Y) - gcat_A(X)$.
(iv) The following inequalities hold: $cat_A(X) \leq gcat_A(X) \leq gcat(X)$.

Proof. (i) Assume $gcat_A(X) = m$ and $gcat(Y) = n$. Then we can write $X = U_0 \cup U_1 \cup \ldots \cup U_m$, $Y = V_1 \cup \ldots \cup V_n$, U_0 categorical relative to A in X and U_i, $i = \overline{1,m}$, V_j, $j = \overline{1,n}$ open in X respectively Y and selfcontractible. The covering $\{U_0, U_1, \ldots, U_m, V_1, \ldots, V_n\}$ of $X \cup Y$ satisfy the Definition 4.1 and we deduce that $gcat_A(X \cup Y) \leq m + n$. (ii) and (iii) are obvious. (iv) Because any selfcontractible subset of X is contractible in X, it follows the first inequality. By applying (i) the second part is obtained. □

Proposition 4.5. *If $h : X \to Y$ is a homeomorphism and $\varnothing \neq A \subset X$ is closed, then $gcat_A(X) = gcat_{h(A)}(Y)$.*

Proof. Consider $\{U_i\}_{i=\overline{0,n}}$ a covering of X as in Definition 4.1. Then $\{h(U_i)\}_{i=\overline{0,n}}$ is a corresponding covering of Y and $gcat_{h(A)}(Y) \leq gcat_A(X)$. Conversely, let $\{V_i\}_{i=\overline{0,n}}$ be a covering of Y as in Definition 4.1 and consider $\{h^{-1}(V_i)\}_{i=\overline{0,n}}$ the corresponding covering of X. □

Remark 4.6. Examples given by R. H. Fox in [15] and M. Clapp, L. Montejano in [7] together with Remark 4.2 prove that the relative geometric category cannot be a homotopy invariant.

We define now the relative g-categorical sequences:

Definition 4.7. We call a g-categorical sequence of length k for X, relative to a closed subset A, a family $X_0, X_1, X_2, \ldots, X_k$ of subsets of X having the following properties:
(i) $X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_k = X$;
(ii) X_0 is categorical relative to A in X;
(iii) $X_1 \setminus X_0, \ldots, X_k \setminus X_{k-1}$ are contained in open selfcontractible subsets of X.

We prove now the result which corresponds to Proposition 2.3:

Proposition 4.8. *Let X be a topological space and A a closed subset of X such that $gcat_A(X) = n$. Then there exists a covering of X by $n + 1$ pairwise disjoint subsets of X, one of them being categorical relative to A in X and n being contained in selfcontractible subsets of X.*

Proof. For $n = 1$ there exist U_0, U_1 such that $X = U_0 \cup U_1$, where U_0 is categorical relative to A in X and U_1 is selfcontractible. Take $\widetilde{U_0} = U_0$ and $\widetilde{U_1} = U_1 \setminus U_0$. Assume the assertion true for $gcat_A(X) \leq k - 1$ and prove it for $gcat_A(X) = k$. Write $X = U_0 \cup U_1 \cup \ldots \cup U_k$, where U_0 is categorical relative to A in X and $U_1, U_2, \ldots, U_k$ are contained in selfcontractible subsets of X. Take $\widetilde{X} = U_0 \cup U_2 \cup \ldots \cup U_k$; then $gcat_A(\widetilde{X}) = k - 1$. The induction hypothesis ensures the existence of a covering $\widetilde{U_0}, \widetilde{U_2}, \ldots, \widetilde{U_k}$ of $\widetilde{X}$ which contains only pairwise disjoint subsets of X, with the property that $\widetilde{U_0}$ is categorical relative to A in X and $\widetilde{U_2}, \ldots, \widetilde{U_k}$ are contained in selfcontractible subsets of X. Take $\widetilde{U_1} = U_1 \setminus \widetilde{X}$. □

We establish now the characterization of relative geometric category by using relative g-categorical sequences:

Theorem 4.9. *Let X be a topological space and let A be a closed subset of X. Assume $gcat_A(X) < \infty$. Then $gcat_A(X)$ is the minimum length of all g-categorical sequences of X relative to A.*

Proof. Let $X_0, X_1, \ldots, X_k$ be a g-categorical sequence of X relative to A. We prove that $gcat_A(X) \leq k$ by induction. For $k = 0$ it is true. Assume the assertion true for $gcat_A(X) \leq k-1$. Remark that $X_1 \backslash X_0, X_2 \backslash X_0, \ldots, X_k \backslash X_0$ is a g-categorical sequence of $X \backslash X_0$. Then, by using Theorem 2.5, we obtain $gcat(X \backslash X_0) \leq k$. Proposition 4.4 ($i$) implies $gcat_A(X) \leq gcat(X \setminus X_0) + gcat_A(X_0) \leq k$. Assume now that $gcat_A(X) = k$ and justify the existence of a g-categorical sequence of X relative to A, such that its length equals k. Let $U_0, U_1, \ldots, U_k$ such that $X = U_0 \cup U_1 \cup \ldots \cup U_k$, where U_0 is categorical in X relative to A and $U_i, i = \overline{1,k}$ are open and selfcontractible subsets of X. Define $X_0 = U_0, X_1 = U_0 \cup U_1, \ldots, X_k = U_0 \cup \ldots U_k$. Then $X_0, X_1, \ldots, X_k$ is a g-categorical sequence of X relative to A. □

Example 4.10. Let X be the 2-torrus T^2 and $A = S^1 \subset T^2$ be the inner meridian of X. There is a covering U_0, U_1, U_2 of X such that U_0 is categorical relative to S^1 and U_1, U_2 are selfcontractible. Then $gcat_{S^1}(T^2) = 2$. Take $X_0 = U_0, X_1 = U_0 \cup U_1$ and $X_2 = U_0 \cup U_1 \cup U_2 = T^2$. Then X_0, X_1, X_2 is a g-categorical sequence of T^2 relative to S^1 of length 2.

Finally, remark that a product type inequality for relative geometric category can be proved by using the Theorem 4.9 in the same way as in Theorem 3.2.

Theorem 4.11. *For any two path-connected spaces X and Y such that $X \times Y$ is completely normal, for any $A \neq \varnothing$ closed in X and $B \neq \varnothing$ closed in Y, the following inequality is satisfied:*

$$gcat_{A\times B}(X \times Y) \leq gcat_A(X) + gcat_B(Y) - 1.$$

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