On some pseudo-symmetric Riemann spaces

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Abstract. Let (M, g) be a Riemannian manifold. It is called pseudo-symmetric if at every point of M the tensor $R \cdot R$ and the Tachibana tensor $Q(g, R)$ are linearly dependent. Any semi-symmetric manifold $(R \cdot R = 0)$ is pseudo-symmetric. This general notion arose during the study of totally umbilical submanifolds of semi-symmetric spaces, as well as during the consideration of geodesic mappings.

We continue the study in this direction, considering subgeodesic mappings, which are a natural generalization of geodesic mappings on Riemannian manifolds. We study ξ-subgeodesically related spaces, extending some known results concerning pseudo-symmetric spaces admitting geodesic mappings. Conharmonic semi-symmetric spaces geodesically related are also characterized.

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1 Classes of Riemannian manifolds

Let (M, g) be a Riemann manifold. The notion of pseudo-symmetry [10] is a natural generalization of semi-symmetry [14], [2] along the line of spaces of constant sectional curvature and locally symmetric spaces.

$$\mathcal{R}_0 \quad \subset \quad \mathcal{R}_1 \quad \subset \quad \mathcal{R}_2 \quad \subset \quad \mathcal{R}_3,$$

where $\mathcal{R}_0$ is the class of constant sectional curvature Riemann spaces,
$\mathcal{R}_1$ is the class of locally symmetric Riemann spaces (i.e. $\nabla R = 0$),
$\mathcal{R}_2$ is the class of semi-symmetric Riemann spaces (i.e. $R \cdot R = 0$),
$\mathcal{R}_3$ is the class of pseudo-symmetric Riemann spaces (i.e. $R \cdot R = LQ(g, R)$).

Remark A. Let $T \in \mathcal{T}^{0,k}M$. We define $R \cdot T, Q(g, T) \in \mathcal{T}^{0,k+2}M$, by

$$(R \cdot T)(X_1, \ldots, X_k; X, Y) = (R(X, Y) \cdot T)(X_1, \ldots, X_k) =$$
$$= -T(R(X, Y)X_1, \ldots, X_k) - \cdots - T(X_1, \ldots, R(X, Y)X_k).$$

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$$Q(g,T)(X_1,\dots,X_k;X,Y) = -((X\wedge Y)\cdot T)(X_1,\dots,X_k) = \\ = T((X\wedge Y)X_1,\dots,X_k)+\dots+T(X_1,\dots,(X\wedge Y)X_k),$$

where $(X\wedge_g Y)U = g(U,Y)X - g(U,X)Y$.

Remark B. The class $\mathcal{R}_2$ of semi-symmetric spaces was introduced by E. Cartan. These spaces were classified by Z.I. Szabo [11] and semi-symmetric hypersurfaces in E^{n+1} were studied by K.Nomizu.

a) It is clear that any semi-symmetric manifold ($R\cdot R = 0$) is Ricci semi-symmetric ($R\cdot S = 0$).

b)(Open Problem) *It is a long standing question whether these notions are equivalent for hypersurfaces of Euclidean spaces.*

c) *Ricci semi-symmetric hypersurfaces of Euclidean spaces ($n > 3$), with positive scalar curvature are semi-symmetric.*

d) *Both properties are equivalent for hypersurfaces of Euclidean space $E^{n+1}(n > 3)$, under the additional global condition of completness.*

The class $\mathcal{R}_3$ of pseudo-symmetric manifolds (i.e. $R\cdot R$ and $Q(g,R)$ are linearly dependent) arose:

I) during the study of totally umbilical submanifolds in semi-symmetric manifolds [4], [5], [6]:

Theorem A. *Let $M^n \subset \overline{M}^{n+1}$ be a totally umbilical hypersurface. If $\overline{M}^{n+1}$ is semi-symmetric then M is conformally flat or is a pseudo-symmetric space.*

Theorem B. *The hypersurface $M \subset E^{n+1}, n \geq 3$, is pseudo-symmetric if and only if the shape operator has one of the following forms:*

1) 0_n;

2) $\lambda I_n, \lambda \neq 0$;

3) $\lambda I_1 \oplus 0_{n-1}, \lambda \neq 0$;

4) $\lambda I_k \oplus 0_{n-k}, \lambda \neq 0, k > 1$;

5) $\lambda I_1 \oplus \mu I_1 \oplus 0_{n-2}, \lambda\mu \neq 0$;

6) $\lambda I_1 \oplus \mu I_{n-1}, \lambda\mu \neq 0$;

7) $\lambda I_k \oplus \mu I_{n-k}, \lambda\mu \neq 0, k > 1$.

II) during the study of geodesic and subgeodesic mappings:

Remark C.

a) Let $\xi \in \mathcal{X}(M)$. A diffeomorphism $f : V_n = (M,g) \mapsto \overline{V}_n = (M,\bar{g})$ is called $\xi-$ subgeodesic mapping if maps $\xi-$ subgeodesics into $\xi-$ subgeodesics, where $\xi-$ subgeodesics on M are given by the following equations:

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^k}{dt}\frac{dx^j}{dt} = a\frac{dx^i}{dt} + b\xi^i, \xi^i = g^{ij}\xi_j, a, b \in \mathcal{F}(M).$$

b) There exists a $\xi-$ subgeodesic mapping f if and only if the Yano formulae are satisfied

$$\overline{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X - g(X,Y)\xi, \psi \in \wedge^1(M).$$

c) f is called nontrivial if $\psi_i - \xi_i \neq 0, \forall i \in \{1, \dots n\}$.

d) There exists f geodesic mapping (i.e. $\xi = 0$) if and only if the Weyl formulae are satisfied

$$\overline{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X.$$

e) The geodesic correspondence is special if $\psi_{ij} = f g_{ij}$, where

$$\psi_{ij} = \psi_{i,j} - \psi_i \psi_j, f \in \mathcal{F}(M).$$

Example.

Let $V_n = (M,g), \overline{V}_n = (M,\bar{g})$ be geodesically related Riemann spaces, where one considers the warped product [12] $M = (a,b) \times {}_F \tilde{M}$ of an open interval (a,b) of R^n and of a Riemann space of constant sectional curvature $(\tilde{M}^{n-1}, \tilde{g})$. Let $F : (a,b) \mapsto R$ be a positive differentiable function. The geodesically related metrics are defined in the following manner [5]

$$\begin{cases} g_{11} = \epsilon \in \{-1,1\} \\ g_{\alpha\beta} = F\tilde{g}_{\alpha\beta} \\ g_{1\alpha} = 0. \end{cases}$$

$$\begin{cases} \bar{g}_{11} = \dfrac{c}{(F+d)^2} \\ \bar{g}_{\alpha\beta} = \epsilon F \dfrac{cF}{d(f+d)} \tilde{g}_{\alpha\beta} \\ \bar{g}_{1\alpha} = 0. \end{cases}$$

Also one has

$$\begin{cases} \psi_1 = \dfrac{-1}{2} \dfrac{F'}{F+d} \quad c, d \in R^*, \alpha, \beta = \overline{2,n}. \\ \psi_\alpha = 0, \end{cases}$$

$$\begin{cases} L = \dfrac{\epsilon}{2F}(F" - \dfrac{(F')^2}{2F}) \\ \bar{L} = \dfrac{d^2}{2cF}(F" - \dfrac{(F')^2}{2F}) + \dfrac{d}{2c}(F" - \dfrac{(F')^2}{F}). \end{cases}$$

One can take, for example, $F(x^1) = (kx^1 + d)^2$. So, $L = 0, \bar{L} = -\dfrac{dk^2}{c}$.

Theorem C. [4] *Let (M,g) be a pseudo-symmetric manifold admitting a nontrivial geodesic mapping f on $(M,\bar{g})$. Then $(M,\bar{g})$ is also a pseudo-symmetric space.*

Remark D. We should point out that one can consider the general context of pseudo-Riemannian case.

Many spacetimes (Robertson-Walker, Schwarzchild, Einstein-de Sitter etc) are pseudo-symmetric and those which are not pseudo-symmetric verify certain conditions of pseudo-symmetric type [6].

Extensive literature concerning similar problems for Einstein equations, PDE's and integral equations can be mentioned from different perspectives [1], [3], [7], [13].

2 Conharmonic semi-symmetric spaces

Let $V_n = (M, g)$ be a Riemann space, $n \geq 3$. The conharmonic curvature tensor C is defined by

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}\{(AX \wedge Y)Z + (X \wedge AY)Z\},$$

where A is the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor S, i.e. $g(AX, Y) = S(X, Y)$.

Let $\xi \in \mathcal{X}(M)$. The conformal transformation

$$g \mapsto \tilde{g} = e^{2u}g, u \in \mathcal{F}(M), \frac{\partial u}{\partial x^i} = \xi_i = g_{ij}\xi^j$$

is called a conharmonic transformation if $\xi_{hk} = 0$, where

$\xi_{hk} = \xi_{h,k} - \xi_h\xi_k + \frac{1}{2}\xi_i\xi^i g_{hk}$.

The conharmonic curvature tensor is invariant under these transformations.

The conharmonic curvature tensor has been introduced by Y. Ishii and characterizes conformally flat spaces with vanishing scalar curvature, if it vanishes identically.

The space V_n is called conharmonic semi-symmetric if $R \cdot C = 0$.

Our aim is to characterize conharmonic semi-symmetric spaces geodesically related.

Theorem 2.1. *Let $V_n = (M, g)$ and $\overline{V}_n = (M, \overline{g}), n \geq 3$, be two nontrivial geodesically related Riemann spaces.*

If $\overline{V}_n$ is $\overline{C}$-semi-symmetric, then V_n and $\overline{V}_n$ are spaces with constant sectional curvature or are special geodesically related.

Proof. $\overline{V_n}$ is $\overline{C}$-semi-symmetric.

Then $(\overline{R} \cdot \overline{C})^h_{ijkrm} = \overline{C}^i_{jkh;sm} - \overline{C}^i_{jkh;ms} = 0$.

Contracting this relation with g^{kr} one gets

$$\begin{gathered} g^{kr}(R^s_{ikj}R_{hsmr} + R^s_{imr}R_{hsjk} + R^s_{jmr}R_{hisk} + \\ + R^s_{kmr}R_{hijs}) + R^s_{ihj}\Psi_{sm} - g_{hm}g^{kr}R^s_{ikj}\Psi_{sr} + \\ + \Psi_{im}S_{jh} - \Psi_{is}R^s_{jmh} + \Psi_{js}R^s_{imh} - g_{jh}g^{kr}\Psi_{sk}R^s_{imr} - \\ - \Psi_{js}R^s_{mih} + \Psi_{is}R^s_{jmh} + \Psi_{ms}R^s_{jih} - fR_{hijm} - g_{jh}\Psi_{is}g^{sr}S_{rm} = 0, \end{gathered} \tag{2.1}$$

where $f = g^{ij}\Psi_{ij}$.

Summing the above equation with the same obtained interchanging the indices h and i, we obtain

$$\begin{gathered} \Psi_{sm}R^s_{ihj} + \Psi_{sm}R^s_{hij} - g_{hm}g^{kr}\Psi_{sr}R^s_{ikj} - \psi_{sr}g_{im}g^{kr}R^s_{hkj} + \\ + S_{jh}\Psi_{im} + S_{ij}\Psi_{mh} + \Psi_{js}R^s_{imh} + \Psi_{js}R^s_{hmi} - \\ - g_{jh}g^{kr}\Psi_{ks}R^s_{imr} - g_{ij}g^{kr}\Psi_{ks}R^s_{hmr} - g_{jh}\Psi_{is}g^{sr}S_{rm} - \\ - g_{ij}\Psi_{hs}S_{rm} = 0. \end{gathered} \tag{2.2}$$

Summing the relation (2.2) with the same equation obtained permuting the indices j with m, we have

$$\begin{gathered} S_{jh}\Psi_{im} + S_{ij}\Psi_{hm} - g_{jh}\Psi_{is}A^s_m + S_{hm}\Psi_{ij} - g_{ij}\Psi_{sh}A^s_m + \\ + S_{im}\Psi_{hj} - g_{mh}\Psi_{is}A^s_j - g_{im}\Psi_{sh}A^s_j = 0. \end{gathered} \tag{2.3}$$

After a contraction of (2.3) with g^{ij}, we get the equation

$$(n+1)\Psi_{hs}A^s_m - \rho\Psi_{hm} - fS_{hm} + g_{hm}\Psi_{sr}S^{sr} - \Psi_{sm}A^s_h = 0, \tag{2.4}$$

where $S^{ij} = g^{ir}A^j_r$, $\rho = g^{ij}S_{ij}$. From (2.4) we obtain

$$\rho f = nS^{ij}\Psi_{ij}. \tag{2.5}$$

The relations (2.5) and (2.3) lead to

$$\Psi_{sh}A^s_m = \frac{f}{n}S_{mh} - \frac{f\rho}{n^2}g_{mh} + \frac{\rho}{n}\Psi_{mh} = \Psi_{sm}A^s_h. \tag{2.6}$$

Using (2.6), the relation (2.3) becomes

$$(fg_{hm} - n\Psi_{hm})(nS_{ij} - \rho g_{ij}) + (fg_{ij} - n\Psi_{ij})(nS_{hm} - \rho g_{hm}) +$$

$$+(fg_{jm} - n\Psi_{jm})(nS_{ih} - \rho g_{ih}) + (fg_{ih} - n\Psi_{ih})(nS_{jm} - \rho g_{jm}) = 0.$$

We obtain $(\Psi_{ij} - \frac{f}{n}g_{ij})(S_{hm} - \frac{\rho}{n}g_{hm}) = 0$. Hence the correspondence is special or the space V_n is Einstein. In the second case one has

$$\Psi_{ir} - \frac{f}{n}g_{ir} = 0 \qquad \text{or} \qquad P_{ijkh} = 0,$$

where P is the projective Weyl curvature tensor [9], [8]. V_n being an Einstein space, if $P = 0$ then V_n becomes a space with constant curvature. Hence, V_n and $\overline{V_n}$ are spaces with constant curvature, using the Beltrami theorem. □

Theorem 2.2. *Let $V_n = (M, g)$ and $\overline{V}_n = (M, \overline{g}), n \geq 3$, be two nontrivial geodesically related Riemann spaces. If $\overline{V}_n$ is $\overline{C}$-semi-symmetric, with irreducible curvature tensor, then V_n and $\overline{V}_n$ are spaces with constant sectional curvature.*

Proof. If V_n and $\overline{V_n}$ are two special geodesically related Riemannian spaces then

$$\overline{R}^i_{jkh} = R^i_{jkh} + f(\delta^i_h g_{jk} - \delta^i_k g_{jh}), \text{ where } \Psi_{ij} = fg_{ij}.$$

The above relation leads to

$$g_{is}\overline{R}^s_{jkh} + g_{js}\overline{R}^s_{ikh} = 0$$

The space $\overline{V_n}$ being with irreducible curvature tensor, then the system

$$x_{is}\overline{R}^s_{jkh} + x_{js}\overline{R}^s_{ikh} = 0 \tag{2.7}$$

has an unique solution, abstraction a factor. Because g_{ij} and $\overline{g}_{ij}$ are solutions of the system (2.7) we obtain $\overline{g}_{ij} = e^{2u}g_{ij}$, where u is a function with variables $x^1, ..., x^n$. V_n and $\overline{V_n}$ being geodesically related, we have $u =$ ct. and we obtain $\overline{\left| \begin{matrix} i \\ j\ k \end{matrix} \right|} = \left| \begin{matrix} i \\ j\ k \end{matrix} \right|$. Then $\delta^i_j\Psi_k + \delta^i_k\Psi_j = 0$ and $\Psi_k = 0$. Using the previous result, the theorem is proved. □

The relation between the subgeodesic correspondence and the conformal related spaces leads to the

Theorem 2.3. *Let $V_n = (M, g)$ and $\overline{V}_n = (M, \overline{g}), n \geq 3$, be two nontrivial ξ-subgeodesically related Riemann spaces. If $\overline{V}_n$ is $\overline{C}$-semi-symmetric, with irreducible curvature tensor, then $\overline{V}_n$ and $\tilde{V}_n = (M, \tilde{g} = e^{2u} g)$ are spaces with constant sectional curvature.*

Proof. V_n and $\overline{V_n}$ being subgeodesically related, we have

$$\overline{\left|\begin{matrix} i \\ j\ k \end{matrix}\right|} = \left|\begin{matrix} i \\ j\ k \end{matrix}\right| + \delta^i_j \Psi_k + \delta^i_k \Psi_j - g_{jk}\xi^i.$$

Because V_n and $\widetilde{V_n}$ are conformally related, the Christoffel symbols are transformed by

$$\widetilde{\left|\begin{matrix} i \\ j\ k \end{matrix}\right|} = \left|\begin{matrix} i \\ j\ k \end{matrix}\right| + \delta^i_j \xi_k + \delta^i_k \xi_j - g_{jk}\xi^i.$$

Then we have $\overline{\left|\begin{matrix} i \\ j\ k \end{matrix}\right|} = \widetilde{\left|\begin{matrix} i \\ j\ k \end{matrix}\right|} + \delta^i_j \omega_k + \delta^i_k \omega_j$, where $\omega_k = \Psi_k - \xi_k$.

So, $\overline{V_n}$ and $\widetilde{V_n}$ are non-trivial geodesically related. Applying the previous theorem for spaces $\overline{V_n}$ and $\widetilde{V_n}$, we obtain the conclusion. □

3 Pseudo-symmetric subgeodesically related Riemann spaces

One can obtain certain conditions of pseudo-symmetric type for $\xi-$subgeodesically related spaces:

Theorem 3.1. *Let $V_n = (M, g)$ and $\overline{V}_n = (M, \overline{g}), n \geq 3$, be nontrivial ξ-subgeodesically related Riemann spaces.*

Then

$$\overline{R} \cdot g = Q(g, F),$$

where

$$F_{ij} = \xi_{i;j} - \psi_{i;j} - (\xi_i - \psi_i)(\xi_j - \psi_j).$$

Proof. Using the Yano formulae, we get

$$g_{jk;ir} = -2\Psi_{i;r} g_{jk} - (\Psi_{j;r} - \xi_{j;r}) g_{ik} - (\Psi_{k;r} - \xi_{k;r}) g_{ij} -$$
$$-2\Psi_i \left[-2\Psi_r g_{jk} - (\Psi_j - \xi_j) g_{rk} - (\Psi_k - \xi_k) g_{rj} \right] -$$
$$-(\Psi_j - \xi_j) \left[-2\Psi_r g_{ik} - (\Psi_i - \xi_i) g_{rk} - (\Psi_k - \xi_k) g_{ir} \right] -$$
$$-(\Psi_k - \xi_k) \left[-2\Psi_r g_{ij} - (\Psi_j - \xi_j) g_{ri} - (\Psi_i - \xi_i) g_{rj} \right].$$

Hence

$$(\overline{R} \cdot g)_{jkri} = g_{jk;ir} - g_{jk;ri} = Q(g, F)_{jkri},$$

where $F_{ij} = \xi_{i;j} - \Psi_{i;j} - (\xi_i - \Psi_i)(\xi_j - \Psi_j)$. □

Theorem 3.2. *Let $V_n = (M,g)$ and $\overline{V}_n = (M,\bar{g}), n \geq 3$, be nontrivial ξ-subgeodesically related Riemann spaces.*

Let $\overline{V}_n = (M,\bar{g})$ be a pseudo-symmetric space such that

$$\overline{R}\cdot\overline{R} = \overline{L}Q(\overline{g},\overline{R}),$$

where $\overline{L}$ is constant on the set $\overline{U} = \{x \in M \mid \overline{Z} \neq 0 \text{ at } x\}$,$\overline{Z}$ being the concircular curvature tensor.

If $F = fg + h\overline{g}, f,h \in \mathcal{F}(M)$, then spaces are conformally related or $\overline{L} = h$ on $\overline{U}$.

Proof. Because $F = fg + h\overline{g}$, using the previous theorem, we have $\overline{R}\cdot g = Q(\overline{g},\ -hg)$.

The tensor $E = -hg - \overline{L}g$ satisfies on $\overline{U}$ the relation

$$E - \frac{1}{n}(\overline{g}^{ij}E_{ij})\overline{g} = 0.$$

This condition is equivalent [5] with

$$(\overline{L}+h)\left[g - \frac{1}{n}(\overline{g}^{ij}g_{ij})\overline{g}\right] = 0$$

on $\overline{U}$. □

Conjectures:

Let $V_n = (M,g)$ and $\overline{V}_n = (M,\overline{g}), n \geq 3$, be nontrivial geodesically or ξ-subgeodesically related Riemann spaces.

If $\overline{V}_n$ is conharmonic pseudo-symmetric (i.e. $\overline{R}\cdot\overline{C} = \overline{L}Q(\overline{g},\overline{C})$) then

a) V_n is conharmonic pseudo-symmetric (i.e. $R\cdot C = LQ(g,C)$);

b) both spaces have constant sectional curvature.

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