Quadratic and homogeneous Hamilton-Poisson systems on $A^*_{3,6,-1}$

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Dedicated to the 70-th anniversary of Professor Constantin Udriste

Abstract. Let $A_{3,6,-1}$ be the real Lie algebra of type (VI), the real parameter being equal to -1, in the Bianchi classification of the 3-dimensional real Lie algebras, and $A^*_{3,6,-1}$ the dual of the real vector space $A_{3,6,-1}$. The dynamics associated to a quadratic and homogeneous Hamilton-Poisson system on $A^*_{3,6,-1}$ is considered. The spectral and nonlinear stability of its equilibrium states and the existence of its periodic orbits are analyzed. The numerical integration of this dynamical system via Kahan's integrator is also discussed. Similar problems have been studied in [2] and [3]. An open problem is the extension of such problems to Poisson-Lie algebroids (see [9]).

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Key words: Hamilton-Poisson system; Kahan integrator; Runge-Kutta integrator; spectral stability; nonlinear stability.

1 The geometrical picture of the problem

Let (e_1, e_2, e_3) be the canonical basis for $\mathbb{R}^3$, i.e.

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Definition 1.1. *Let $A_{3,6,-1}$ be the Lie algebra $\mathbb{R}^3$ with the bracket operation given by:*

$[.,.]$	e_1	e_2	e_3
e_1	0	0	e_1
e_2	0	0	$-e_2$
e_3	$-e_1$	e_2	0

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This is a real Lie algebra of type (VI), the real parameter being equal to -1, in the Bianchi classification of the 3-dimensional real Lie algebras (see for details [5]).

Then the minus Lie-Poisson structure on $A^*_{3,6,-1} \cong \mathbb{R}^3$ is generated by the matrix:

$$\Pi_-(x_1, x_2, x_3) := \begin{bmatrix} 0 & 0 & -x_1 \\ 0 & 0 & x_2 \\ x_1 & -x_2 & 0 \end{bmatrix}.$$

Definition 1.2. *A quadratic and homogeneous Hamilton-Poisson system on* $A^*_{3,6,-1} \cong \mathbb{R}^3$ *is the triple*

$$\left(\mathbb{R}^3, \Pi_-, H\right),$$

where

$$H(x_1, x_2, x_3) := \frac{1}{2}(b_1 x_1^2 + b_2 x_2^2 + b_3 x_3^2)$$
$$+ d_1 x_2 x_3 + d_2 x_1 x_3 + d_3 x_1 x_2,$$
$$b_1, b_2, b_3, d_1, d_2, d_3 \in \mathbb{R}, b_1^2 + b_2^2 + b_3^2 + d_1^2 + d_2^2 + d_3^2 \neq 0.$$

Its dynamics is described by the following set of differential equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \Pi_- \cdot \nabla H,$$

or equivalently:

$$\left\{ \begin{array}{l} \dot{x_1} = -x_1 \left(b_3 x_3 + d_1 x_2 + d_2 x_1\right), \\ \dot{x_2} = x_2 \left(b_3 x_3 + d_1 x_2 + d_2 x_1\right), \\ \dot{x_3} = x_1 \left(b_1 x_1 + d_2 x_3\right) - x_2 \left(b_2 x_2 + d_1 x_3\right). \end{array} \right. \tag{1.1}$$

Using Bermejo-Fairén technique (see [4]) we are immediately lead to:

Proposition 1.1. *There exists only one functionally independent Casimir of our Poisson configuration* $\left(\mathbb{R}^3, \Pi_-\right)$ *given by*

$$C\left(x_1, x_2, x_3\right) := x_1 x_2.$$

The phase curves of the dynamics (1.1) are intersections of

$$H\left(x_1, x_2, x_3\right) = constant$$

and

$$C\left(x_1, x_2, x_3\right) = constant.$$

2 Spectral stability

It is easy to see that the dynamics (1.1) has the following equilibrium states:

$$e_1^M := (0, 0, M), M \in \mathbb{R}.$$

If

$$\begin{cases} b_1b_3 - d_2^2 > 0, \\ b_2b_3 - d_1^2 \geq 0, \\ b_3 \in \mathbb{R}^*, \end{cases}$$

then we have other two families of equilibrium states, namely

$$e_2^M := \left(\frac{\sqrt{b_2b_3 - d_1^2}}{\sqrt{b_1b_3 - d_2^2}} M, M, -\frac{d_1 + \frac{\sqrt{b_2b_3 - d_1^2}}{\sqrt{b_1b_3 - d_2^2}} d_2}{b_3} M \right), M \in \mathbb{R},$$

and

$$e_3^M := \left(-\frac{\sqrt{b_2b_3 - d_1^2}}{\sqrt{b_1b_3 - d_2^2}} M, M, \frac{-d_1 + \frac{\sqrt{b_2b_3 - d_1^2}}{\sqrt{b_1b_3 - d_2^2}} d_2}{b_3} M \right), M \in \mathbb{R}.$$

Proposition 2.1. *The equilibrium states $e_1^M, M \in \mathbb{R}^*$, are unstable.*

Proof. It is easy to see that the characteristic polynomial of the matrix corresponding to the linear part of our system (1.1) at the equilibrium state e_1^M, where $M \in \mathbb{R}^*$, has the following roots: $\lambda_1 = 0, \lambda_{2,3} = \pm b_3 M$, and then our assertion immediately follows. □

Proposition 2.2. *The equilibrium states $e_2^M, M \in \mathbb{R}$, are spectrally stable.*

Proof. Using MATHEMATICA 7 we obtain that the characteristic polynomial of the matrix corresponding to the linear part of our system (1.1) at the equilibrium state e_2^M, where $M \in \mathbb{R}$, has the following roots: $\lambda_1 = 0, \lambda_{2,3} = \pm Mi\sqrt{b_2b_3 - d_1^2}$, and then our assertion follows. □

Proposition 2.3. *The equilibrium states $e_3^M, M \in \mathbb{R}$, are spectrally stable.*

Proof. Using MATHEMATICA 7 we obtain that the characteristic polynomial of the matrix corresponding to the linear part of our system (1.1) at the equilibrium state e_3^M, where $M \in \mathbb{R}$, has the following roots: $\lambda_1 = 0, \lambda_{2,3} = \pm Mi\sqrt{b_2b_3 - d_1^2}$, and then our assertion follows. □

3 Nonlinear stability

We shall discuss now the nonlinear stability of the equilibrium states e_2^M and $e_3^M, M \in \mathbb{R}^*$.

Proposition 3.1. *If $b_1 > 0, b_2 > 0, b_3 > 0, d_1d_2 > 0, b_1b_3 - d_2^2 > 0$, then the equilibrium states $e_2^M, M \in \mathbb{R}^*$, are nonlinearly stable.*

Proof. We shall prove the claim using Arnold's method [1] (see also [6]). Let $F_\lambda \in C^\infty(\mathbb{R}^3, \mathbb{R})$ be the smooth real function given by

$$F_\lambda(x_1, x_2, x_3) := \frac{1}{2}(b_1x_1^2 + b_2x_2^2 + b_3x_3^2) + d_1x_2x_3$$

$$+d_2x_1x_3 + d_3x_1x_2 + \lambda x_1x_2,$$

where λ is a real parameter. Then we successively have:
(i) $\nabla F_\lambda(e_2^M) = 0$ if and only if

$$\lambda = -\frac{-d_1d_2\sqrt{b_2b_3-d_1^2}+b_2b_3\sqrt{b_1b_3-d_2^2}-d_1^2\sqrt{b_1b_3-d_2^2}+b_3d_3\sqrt{b_2b_3-d_1^2}}{b_3\sqrt{b_2b_3-d_1^2}} =: \lambda_0;$$

(ii) $W := \ker dC(e_2^M) = span\left(\begin{bmatrix} -\frac{\sqrt{b_2b_3-d_1^2}}{\sqrt{b_1b_3-d_2^2}} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right);$

(iii) $\nabla^2 F_{\lambda_0}(e_2^M)|_{W\times W}$ induces a positive definite quadratic form on W.

Therefore, via Arnold's method, the equilibrium states $e_2^M, M \in \mathbb{R}$, are nonlinearly stable. □

Proposition 3.2. *If $b_1 > 0, b_2 > 0, b_3 > 0, b_1b_3 - d_2^2 > 0, b_2b_3 - d_1^2 > 0$, then the equilibrium states $e_3^M, M \in \mathbb{R}^*$, are nonlinearly stable.*

Proof. Let $F_\lambda \in C^\infty(\mathbb{R}^3, \mathbb{R})$ be the smooth real function given by

$$F_\lambda(x_1, x_2, x_3) := \frac{1}{2}(b_1x_1^2 + b_2x_2^2 + b_3x_3^2) + d_1x_2x_3$$

$$+d_2x_1x_3 + d_3x_1x_2 + \lambda x_1x_2.$$

Then we successively have:
(i) $\nabla F_\lambda(e_3^M) = 0$ if and only if

$$\lambda = -\frac{b_1\sqrt{b_2b_3-d_1^2}}{\sqrt{b_1b_3-d_2^2}} + d_3 + \frac{d_2\left(-d_1+d_2\frac{\sqrt{b_2b_3-d_1^2}}{\sqrt{b_1b_3-d_2^2}}\right)}{b_3} =: \lambda_0^*;$$

(ii) $W := \ker dC(e_3^M) = span\left(\begin{bmatrix} -\frac{\sqrt{b_2b_3-d_1^2}}{\sqrt{b_1b_3-d_2^2}} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right);$

(iii) $\nabla^2 F_{\lambda_0^*}(e_3^M)|_{W\times W}$ induces a positive definite quadratic form on W.

Therefore, via Arnold's method, the equilibrium states $e_3^M, M \in \mathbb{R}^*$, are nonlinearly stable. □

4 Periodic orbits

Let us assume that

$$b_1, b_2, b_3 > 0, \quad d_1d_2 > 0, \quad b_1b_3 - d_2^2 > 0, \quad b_2b_3 - d_1^2 > 0.$$

Then we can prove the following two results:

Proposition 4.1. *Near to*

$$e_2^M := \left(\frac{\sqrt{b_2b_3-d_1^2}}{\sqrt{b_1b_3-d_2^2}}M, M, -\frac{d_1+\frac{\sqrt{b_2b_3-d_1^2}}{\sqrt{b_1b_3-d_2^2}}d_2}{b_3}M \right), M \in \mathbb{R}^*,$$

the reduced dynamics has, for each sufficiently small value of the reduced energy, at least 1-periodic solution whose period is close to $\dfrac{2\pi}{M\sqrt{b_2b_3-d_1^2}}$.

Proof. Indeed, we have:

(i) The matrix of the linear part of the reduced dynamics has purely imaginary roots. More precisely, $\lambda_{2,3} = \pm Mi\sqrt{b_2b_3-d_1^2}$.

(ii) $span(\nabla C(e_2^M)) = V_0$, where $V_0 := \ker(A(e_2^M))$, and $A(e_2^M)$ is the linear operator corresponding to the matrix of the linear part of the dynamics (1.1) at the equilibrium e_2^M.

(iii) The reduced Hamiltonian has a local minimum at the equilibrium state e_2^M (see the proof of Proposition 3.1).

Then our assertion follows via the Moser-Weinstein theorem with zero eigenvalue (see for details [7]). □

Proposition 4.2. *Near to*

$$e_3^M := \left(-\frac{\sqrt{b_2b_3-d_1^2}}{\sqrt{b_1b_3-d_2^2}}M, M, \frac{-d_1+\frac{\sqrt{b_2b_3-d_1^2}}{\sqrt{b_1b_3-d_2^2}}d_2}{b_3}M \right), M \in \mathbb{R}^*,$$

the reduced dynamics has, for each sufficiently small value of the reduced energy, at least 1-periodic solution whose period is close to $\dfrac{2\pi}{M\sqrt{b_2b_3-d_1^2}}$.

Proof. Indeed, we have:

(i) The matrix of the linear part of the reduced dynamics has purely imaginary roots. More precisely, $\lambda_{2,3} = \pm Mi\sqrt{b_2b_3-d_1^2}$.

(ii) $\mathrm{span}(\nabla C(e_3^M)) = V_0$, where $V_0 := \ker(A(e_3^M))$.

(iii) The reduced Hamiltonian has a local minimum at the equilibrium state e_3^M (see the proof of Proposition 3.2).

Then our assertion follows via the Moser-Weinstein theorem with zero eigenvalue (see for details [7]). □

5 Numerical integration of the dynamics (1.1) via Kahan's integrator

It is well-known that Kahan's integrator (see [8]) for the dynamics (1.1) can be written in the following form:

$$(5.1)\qquad \begin{cases} x_1^{n+1} - x_1^n = -\frac{h}{2}\left[x_1^n\left(b_3x_3^{n+1} + d_1x_2^{n+1} + d_2x_1^{n+1}\right) + \right. \\ \qquad \left. +x_1^{n+1}\left(b_3x_3^n + d_1x_2^n + d_2x_1^n\right)\right], \\ x_2^{n+1} - x_2^n = \frac{h}{2}\left[x_2^n\left(b_3x_3^{n+1} + d_1x_2^{n+1} + d_2x_1^{n+1}\right) + \right. \\ \qquad \left. +x_2^{n+1}\left(b_3x_3^n + d_1x_2^n + d_2x_1^n\right)\right], \\ x_3^{n+1} - x_3^n = \frac{h}{2}\left[x_1^n\left(b_1x_1^{n+1} + d_2x_3^{n+1}\right) - \right. \\ \qquad -x_2^n\left(b_2x_2^{n+1} + d_1x_3^{n+1}\right) + x_1^{n+1}\left(b_1x_1^n + d_2x_3^n\right) \\ \qquad \left. -x_2^{n+1}\left(b_2x_2^n + d_1x_3^n\right)\right]. \end{cases}$$

Using MATHEMATICA 7, it follows that:

Proposition 5.1. *Kahan's integrator* (5.1) *is Poisson* (*resp. energy, resp. Casimir*) *preserving if and only if one of the following conditions hold:*

(i) $b_1 = \frac{d_2^2}{b_3}, b_2 = \frac{d_1^2}{b_3}, d_1, d_2, d_3 \in \mathbb{R}, b_3 \in \mathbb{R}^*$;

(ii) $d_1 = 0, d_2 = 0, b_3 = 0, d_3, b_1, b_2 \in \mathbb{R}$.

If we make a comparison with the 4th Runge-Kutta integrator, we obtain almost the same results, see Figures 5.1, 5.2 and 5.3, respectively, Figure 5.4. However Kahan's integrator has the advantage to be easier implemented.

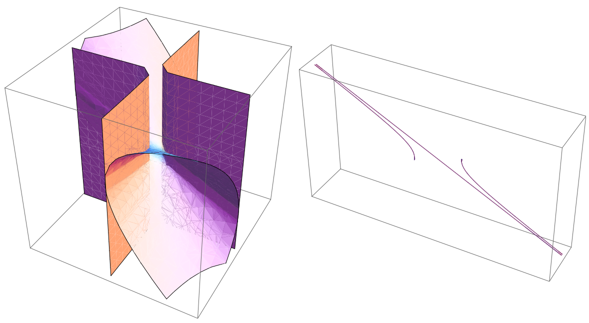
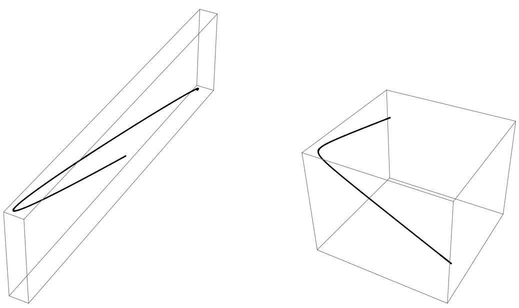


Fig. 5.1 The phase curves of the dynamics (1.1) for case (i)



Integrators for case (i). Fig. 5.2: Kahan; Fig. 5.3: 4th order Runge-Kutta.

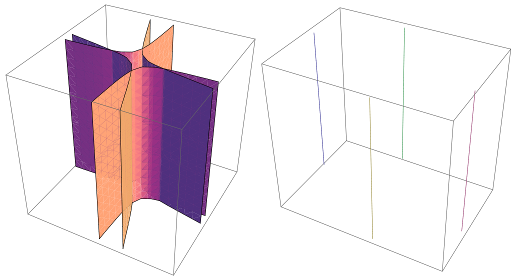


Fig. 5.4 The phase curves of the dynamics (1.1) for case (*ii*)

In this case Kahan's integrator does not provide any relevant results.

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