A new Tzitzeica hypersurface and cubic Finslerian metrics of Berwald type

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Abstract. A new hypersurface of Tzitzeica type is obtained in all three forms: parametric, implicit and explicit. To a two-parameters family of cubic Tzitzeica surfaces we associate a cubic Finsler function for which the regularity is expressed as the non-flatness of the Tzitzeica indicatrix. A natural relationship is obtained between cubic Tzitzeica surfaces and three-dimensional Berwald spaces with cubic fundamental Finsler function.

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1 Tzitzeica hypersurfaces and a new example

The first centroaffine invariant was introduced in 1907 by G. Tzitzeica in the classical theory of surfaces, [14]. Namely, if M^2 is such a (non-flat) surface embedded in $\mathbb{R}^3$ the Tzitzeica invariant is $Tzitzeica(M^2) = \frac{K}{d^4}$, where K is the Gaussian curvature and d is the distance from the origin of $\mathbb{R}^3$ to the tangent plane in an arbitrary point of M^2, see also [15].

Since then, the class of surfaces with a constant $Tzitzeica$ was the subject of several fruitful research, see for example [2], [17] and the surveys [5], [13] and [16]. For example, they are called *Tzitzeica surfaces* by Romanian geometers, *affine spheres* by Blaschke and *projective spheres* by Wilczynski. One interesting direction of study is the generalization to higher dimension: a hypersurface M^n of $\mathbb{R}^{n+1}$ was called *Tzitzeica hypersurface* if $Tzitzeica(M^n) = \frac{K}{d^{n+2}}$ is a real constant.

The simplest examples of Tzitzeica surfaces are the quadrics with center, particularly spheres. Tzitzeica himself obtained the surface $M_1^2 : xyz = 1$ which was generalized by Calabi to $M_1^n : x^1 \cdot ... \cdot x^{n+1} = 1$ in [4].

The aim of this section is to derive the $n(\geq 3)$-dimensional version of another well-known Tzitzeica surface, [14]:

$$M_2^2 : z(x^2 + y^2) = 1. \tag{1.1}$$

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Inspired by [3] we search for M_2^n being a hypersurface of rotation in the parametrical form:

$$M^n : x^1 = \frac{2u^n u^1}{\Delta}, ..., x^{n-1} = \frac{2u^n u^{n-1}}{\Delta}, x^n = \frac{u^n(\Delta - 2)}{\Delta}, x^{n+1} = f(u^n)$$

with $f : (0, +\infty) \to \mathbb{R}$ and $\Delta = (u^1)^2 + ... + (u^{n-1})^2 + 1$. We consider the parameters $u^1, ..., u^{n-1} \in \mathbb{R}$ and $u^n > 0$.

For easy computations we derive an implicit equation of M^n. More precisely, from the above equations we have: $(u^n)^2 = (x^1)^2 + ... + (x^n)^2$ and then $M^n : x^{n+1} = f(\sqrt{(x^1)^2 + ... + (x^n)^2})$ i.e.:

$$M^n : F(x^1, ..., x^{n+1}) := f(\sqrt{(x^1)^2 + ... + (x^n)^2}) - x^{n+1} = 0. \tag{1.2}$$

Recall that choosing the normal $N = -\frac{\nabla F}{\|\nabla F\|}$, the Gaussian curvature of M^n is:

$$K = -\frac{\begin{vmatrix} F_{ij} & F_i \\ F_j & 0 \end{vmatrix}}{\| \nabla F \|^{n+2}},$$

where F_i denotes the partial derivative of F with respect to x^i.

For $p = (x^i) \in M^n$ the tangent hyperplane T_pM has the equation $F_i(p)(X^i - x^i) = 0$ and then:

$$d = \frac{|F_i x^i|}{\|\nabla F\|}.$$

From the last two relations the Tzitzeica condition reads:

$$\begin{vmatrix} F_{ij} & F_i \\ F_j & 0 \end{vmatrix} = -Tzitzeica(M^n)|F_i x^i|^{n+2}. \tag{1.3}$$

With F of (1.2) we obtain that the LHS of (1.3) is $\frac{f''(f')^{n-1}}{\delta^{\frac{n-1}{2}}}$ while the RHS of (1.3) is $-Tzitzeica(M^n)|f'\sqrt{\delta} - f|^{n+2}$, where $\delta = (x^1)^2 + ... + (x^n)^2$. Therefore, denoting $\sqrt{\delta} = t > 0$ and considering $f = f(t)$ we have:

$$f''(f')^{n-1} = -Tzitzeica(M^n)t^{n-1}|tf' - f|^{n+2}$$

for which we search $f(t) = t^a$ with $a \in \mathbb{R}$. The comparison of degrees of t gives $a = -n$. Hence:

Theorem 1.1. *The hypersurface of* $\mathbb{R}^{n+1}$*:*

$$M_2^n : x^{n+1}[(x^1)^2 + ... + (x^n)^2]^{\frac{n}{2}} = 1 \tag{1.4}$$

is a Tzitzeica one with:

$$Tzitzeica(M_2^n) = \frac{(-n)^n}{(n+1)^{n+1}}.$$

One can remark that if in computing K we choose the opposite normal $-N$ then we obtain $-Tzitzeica(M^n)$. For example in [18, p. 137] M_2^2 has $Tzitzeica(M_2^2) = -\frac{4}{27}$. Another observation is that the main theorem of [6] gives the 4-dimensional *affine locally strongly convex* hypersurface: $(y^2 - z^2 - w^2)^3 x^2 = 1$ as generalization of (1.1). The authors continue their study in [7] where Theorem 1 states the 5-dimensional version $(y^2 - z^2 - w^2 - t^2)^2 x = 1$. It results that Tzitzeica hypersurfaces are not affine hyperspheres.

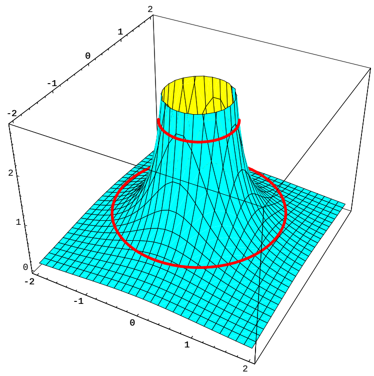

Figure 1: $z(x^2 + y^2) = 1$.

The Calabi-Tzitzeica hypersurface $M_1^n : x^1 ... x^{n+1} = 1$ has:

$$Tzitzeica(M_1^n) = \frac{1}{(n+1)^{n+1}}.$$

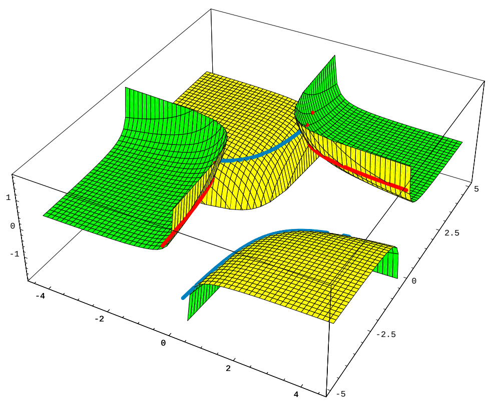

Figure 2: $xyz = 1$.

2 Cubic Finsler functions for a class of cubic Tzitzeica surfaces

The aim of this section is to provide a class of Finsler cubic-Minkowski metrics, connected to a family of 2-parameters Tzizteica surfaces. The geometric meaning of the regularity of these cubic metrics is expressed by the Tzitzeica condition $Tzitzeica(M^2) \neq 0$.

Firstly, we can give an unified formula for M_1^2 and M_2^2. With the transformation:

$$\begin{cases} x = \frac{1}{\sqrt{2}}(\tilde{x} + \tilde{y}) \\ y = \frac{1}{\sqrt{2}}(\tilde{x} - \tilde{y}) \\ z = \tilde{z} \end{cases}$$

which is a rotation of angle $\frac{\pi}{4}$ in the $(\tilde{x}, \tilde{y})$-plane, we have: $M_1^2 : (\tilde{x}^2 - \tilde{y}^2)\tilde{z} = 2$ and $M_2^2 : (\tilde{x}^2 + \tilde{y}^2)\tilde{z} = 1$. Let $\varepsilon \in \{1, 2\}$, then:

$$M_\varepsilon^2 : [\tilde{x}^2 + (-1)^\varepsilon \tilde{y}^2]\tilde{z} = \frac{2}{\varepsilon} \tag{2.1}$$

with:

$$Tzitzeica(M_\varepsilon^2) = \frac{(-1)^{\varepsilon+1}\varepsilon^2}{27}. \tag{2.2}$$

The above considerations leads to:

Definition 2.1. The Tzizteica surface $xyz = 1$ can be called *vertical hyperbolic-cubic Tzitzeica surface* while the surface $z(x^2 + y^2) = 1$ can be called *vertical elliptic-cubic Tzitzeica surface.*

A first natural problem is to search about a parabolic version of these surfaces:

Proposition 2.1. *In the class of surfaces* $M_{p,\rho} : z(y^2 - 2px) = \rho$ *with* $\rho \neq 0$ *there are no Tzitzeica surfaces.*

Proof. The Tzitzeica condition for $M_{p,\rho}$ reads:

$$8p\rho(2p + \rho) = -Tzitzeica(M_{p,\rho})[\rho + z(y^2 + 1)]$$

which is impossible due to the presence of z and y. □

A second natural problem is about the general class:

$$M_{\alpha,\beta,\gamma}^2 : z(\alpha x^2 + \beta y^2 + \gamma z^2) = 1 \tag{2.3}$$

which is solved by:

Proposition 2.2. *The Tzitzeica surfaces of the class* $M_{\alpha,\beta,\gamma}^2$ *are characterized by* $\gamma = 0$.

Proof. The equation (1.3) for $M^2_{\alpha,\beta,\gamma}$ is:

$$4\alpha\beta(3-12\gamma z^3) = -3^4 Tzitzeica(M^2_{\alpha,\beta,\gamma}).$$

which yields $\gamma = 0$ and:

$$Tzitzeica(M^2_{\alpha,\beta,0}) = -\frac{4\alpha\beta}{3^3}. \tag{2.4}$$

For $\alpha = \frac{1}{2}, \beta = -\frac{1}{2}$ we recover M^2_1 while for $\alpha = \beta = 1$ it results M^2_2. □

Inspired by [1, p. 23] who associates to M^n_1 the Berwald-Moór Finslerian function $F(y) = \sqrt[n]{y^1...y^n}$ we study in this section a Finsler metric naturally associated to $M^2_{\alpha,\beta,0}$. Let us consider the 3-dimensional manifold $\mathbb{R}^3$ with global coordinates $x = (x^1, x^2, x^3)$ and for $x \in \mathbb{R}^3$ we denote on the tangent space $T_x\mathbb{R}^3$ the coordinates by (y^1, y^2, y^3).

In the tangent bundle $T\mathbb{R}^3$ let us fix the domain $D = \{(y^1, y^2, y^3); [\alpha(y^1)^2 + \beta(y^2)^2]y^3 > 0\}$ and the function $F : D \to \mathbb{R}^*_+$:

$$F(y^1, y^2, y^3) = \sqrt[3]{[\alpha(y^1)^2 + \beta(y^2)^2]y^3}. \tag{2.5}$$

F is a *Finsler fundamental function* of *cubic-Minkowski type* having as indicatrix $\{y \in TM/F(y) = 1\}$ the Tzitzeica surface provided by Proposition 2.2. For F we use the theory from [10] and [11], so we put the expression $F = \sqrt[3]{a_{i_1...i_3}y^{i_1}y^{i_2}y^{i_3}}$ with:

$$\begin{cases} a_{113} = a_{131} = a_{311} = \frac{\alpha}{3} \\ a_{223} = a_{232} = a_{322} = \frac{\beta}{3}. \end{cases} \tag{2.6}$$

The Matsumoto-Numata tensors:

$$F^2 a_i = a_{ijk}y^j y^k, \quad F a_{ij} = a_{ijk}y^k$$

are for our example (2.5):

$$a_1 = \frac{2\alpha y^1 y^3}{3F^2}, a_2 = \frac{2\beta y^2 y^3}{3F^2}, a_3 = \frac{\alpha(y^1)^2 + \beta(y^2)^2}{3F^2} = \frac{F}{3y^3} \tag{2.7}$$

$$a_{11} = \frac{\alpha y^3}{3F}, a_{22} = \frac{\beta y^3}{3F}, a_{33} = a_{12} = 0, a_{13} = \frac{\alpha y^1}{3F}, a_{23} = \frac{\beta y^2}{3F}. \tag{2.8}$$

From [11, p. 94] the *Finsler metric* $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is:

$$g_{ij} = 2a_{ij} - a_i a_j \tag{2.9}$$

which yields:

$$\begin{cases} g_{11} = \frac{2\alpha(y^3)^2}{9F^4}[\alpha(y^1)^2 + 3\beta(y^2)^2] \\ g_{22} = \frac{2\beta(y^3)^2}{9F^4}[3\alpha(y^1)^2 + \beta(y^2)^2] \\ g_{33} = -\frac{[(y^1)^2+(y^2)^2]^2}{9F^4} = -(\frac{F}{3y^3})^2 \end{cases} \tag{2.10}$$

$$g_{12} = \frac{-4\alpha\beta y^1 y^2 (y^3)^2}{9F^4}, g_{13} = \frac{4\alpha y^1}{9F}, g_{23} = \frac{4\beta y^2}{9F}. \tag{2.11}$$

The determinant of the *basic tensor* (a_{ij}) is:

$$\det(a_{ij}) = -\frac{\alpha\beta}{27} = \frac{1}{4} Tzitzeica(M^2_{\alpha,\beta,0}) \tag{2.12}$$

and then, after [11, p. 94], the function F of (2.5) is *regular* if and only if $\alpha\beta \neq 0$. In other words, the regularity of F is in correspondence with the non-flatness of the Tzitzeica surface $M^2_{\alpha,\beta} : (\alpha x^2 + \beta y^2)z = 1$.

3 A relationship between cubic Tzitzeica surfaces and cubic three-dimensional Berwald metrics

Recall that one of the most important geometrical object in Finsler geometry is the *Berwald connection* who lives on the tangent bundle. A simplified condition in that connection is to depend only on base coordinates, [12, p. 723], which yields:

Definition 3.1. A Finsler manifold is a *Berwald space* if there exists a symmetric affine connection Γ such that the parallel transport with respect to this connection preserves the function F.

For example, any Riemannian metric is Berwald and the associated connection is the Levi-Civita connection. Therefore Berwald spaces are close to Riemannian ones. The aim of this section is to show that M^2_1 and M^2_2 correspond to Berwald metrics.

Let a three-dimensional manifold M with coordinates $(x^i) = (x, y, z)$ and tangent bundle coordinates $(y^i) = (p, q, r)$. In [12, p. 886] it is proved that three-dimensional Berwald spaces with cubic metric are conformally with $F_1 = (pqr)^{1/3}$ and $F_2 = (p^3 + q^3 + r^3 - 3pqr)^{1/3}$, the conformal factor being an arbitrary function of (x, y, z). Obviously, F_1 corresponds to the Tzitzeica surface M^2_1 from the indicatrices point of view and we perform a change of coordinates in M in order to associate F_2 to M^2_2.

This coordinates change is inspired by [12, p. 887], $(x, y, z) \to (u, v, w)$:

$$\begin{cases} u = z - \frac{x}{2} - \frac{y}{2} \\ v = \frac{\sqrt{3}}{2}(x - y) \\ w = x + y + z. \end{cases} \tag{3.1}$$

In these new coordinates we have: $F_2(\dot{u}, \dot{v}, \dot{w}) = (\dot{u}^2 + \dot{v}^2)\dot{w}$ and we obtain the Finslerian function corresponding to M^2_2. Let us remark that a picture of M^2_2 in the coordinates (p, q, r) appears in [9, p. 6] where it is called *Appell sphere* or *ternary unit sphere*. As is pointed out in [8, p. 168], this surface is the ternary analogue of the circle S^1. In conclusion, there is a natural relationship between cubic three-dimensional Berwald spaces and cubic Tzitzeica surfaces.

We close with another natural problem, namely to find cubic Tzitzeica surfaces inspired by the expression of F_2:

$$M^2_{c_1,c_2,c_3,b} : c_1 x^3 + c_2 y^3 + c_3 z^3 - 6bxyz = 1. \tag{3.2}$$

Proposition 3.1. *The Tzitzeica surfaces of the class* $M^2_{c_1,c_2,c_3,b}$ *are characterized by* $c_1c_2c_3 = 8b^3$.

Proof. A straightforward computation gives:

$$4[b^2 + (8b^3 - c_1c_2c_3)xyz] = -Tzitzeica(M^2_{c_1,c_2,c_3,b})$$

which gives the conclusion and:

$$Tzitzeica(M^2_{c_1,c_2,c_3,b}) = -(2b)^2. \tag{3.3}$$

We get $M^2_{2b,2b,2b,b} : 2b(x^3 + y^3 + z^3 - 3xyz) = 1$ and then for $b = \frac{1}{2}$ we recover M^2_2 in the (p, q, r)-expression above. □

If the transformation (3.1) is performed for F_1 we get $F_1(\dot{u}, \dot{v}, \dot{w}) = \frac{1}{27}(2\dot{u}^3 + \dot{w}^3 - 3\dot{u}^2\dot{w} - 6\dot{u}\dot{v}^2 - 3\dot{v}^2\dot{w})$. Then, with a factor of $\frac{1}{3}$ we obtain the Tzitzeica surface:

$$\widehat{M}^2_1 : 2x^3 + z^3 - 3x^2z - 6xy^2 - 3y^2z = 1 \tag{3.3}$$

with:

$$Tzitzeica(\widehat{M}^2_1) = 4. \tag{3.4}$$

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