Fractional calculus of variations for double integrals

Tatiana Odzijewicz and Delfim F.M. Torres

Abstract. We consider fractional isoperimetric problems of calculus of variations with double integrals via the recent modified Riemann–Liouville approach. A necessary optimality condition of Euler–Lagrange type, in the form of a multitime fractional PDE, is proved, as well as a sufficient condition and fractional natural boundary conditions.

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1 Introduction

The calculus of variations was born in 1697 with the solution to the brachistochrone problem (see, e.g., [40]). It is a very active research area in the XXI century (see, e.g., [7, 13, 21–23]). Motivated by the study of several natural phenomena in such areas as aerodynamics, economics, medicine, environmental engineering, and biology, there has been a recent increase of interest in the study of problems of the calculus of variations and optimal control where the cost is a multiple integral functional with several independent time variables. The reader interested in the area of multitime calculus of variations and multitime optimal control is referred to [24,27,31–35,37–39] and references therein.

Fractional calculus, i.e., the calculus of non-integer order derivatives, has its origin also in the 1600s. During three centuries the theory of fractional derivatives of real or complex order developed as a pure theoretical field of mathematics, useful only for mathematicians. In the last few decades, however, fractional differentiation proved very useful in various fields of applied sciences and engineering: physics (classic and quantum mechanics, thermodynamics, etc.), chemistry, biology, economics, engineering, signal and image processing, and control theory [8, 14, 18, 25, 26, 28].

The calculus of variations and the fractional calculus are connected since the XIX century. Indeed, in 1823 Niels Henrik Abel applied the fractional calculus in the solution of an integral equation that arises in the formulation of the tautochrone problem. This problem, sometimes also called the isochrone problem, is that of finding the shape of a frictionless wire lying in a vertical plane such that the time of a bead

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placed on the wire slides to the lowest point of the wire in the same time regardless of where the bead is placed. It turns out that the cycloid is the isochrone as well as the brachistochrone curve, solving simultaneously the brachistochrone problem of the calculus of variations and Abel's fractional problem [1]. It is however in the XX century that both areas are joined in a unique research field: the fractional calculus of variations.

The Fractional Calculus of Variations (FCV) was born in 1996-97 with the proof, by Riewe, of the Euler-Lagrange fractional differential equations [29, 30]. Nowadays, FCV is subject of strong current research – see, e.g., [2–6, 11, 20]. The first works on FCV were developed using fractional derivatives in the sense of Riemann–Liouville [2]. Later, problems of FCV with Grunwald–Letnikow, Caputo, Riesz and Jumarie fractional operators, among others, were considered [3, 9, 12, 20]. The literature on FCV is now vast. However, most results refer to the single time case. Results for multitime FCV are scarce, and reduce to those in [3, 10, 36]. Here we develop further the theory of multitime fractional calculus of variations, by considering fractional isoperimetric problems with two independent time variables. Previous results on fractional isoperimetric problems are for the single time case only [4, 5]. In our paper we study isoperimetric problems for variational functionals with double integrals involving fractional partial derivatives.

The paper is organized as follows. In Section 2 we recall some basic definitions of multidimensional fractional calculus. Our results are stated and proved in Section 3. The main results of the paper include natural boundary conditions (Theorem 3.5) and a necessary optimality condition (Theorem 3.4) that becomes sufficient under appropriate convexity assumptions (Theorem 3.6).

2 Preliminaries

In this section we fix notations by collecting the definitions of fractional derivatives and integrals in the modified Riemann–Liouville sense. For more information on the subject we refer the reader to [3, 15–17, 19].

Definition 2.1 (The Jumarie fractional derivative [17]). Let f be a continuous function in the interval $[a, b]$ and $\alpha \in (0, 1)$. The operator defined by

$$f^{(\alpha)}(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha}(f(t) - f(a))dt \tag{2.1}$$

is called the Jumarie fractional derivative of order α.

Let us consider continuous functions $f = f(x_1, \ldots, x_n)$ defined on

$$R = \prod_{i=1}^{n} [a_i, b_i] \subset \mathbb{R}^n.$$

Definition 2.2 (The fractional volume integral [3]). For $\alpha \in (0, 1)$ the fractional volume integral of f over the whole domain R is given by

$$I_R^\alpha f = \alpha^n \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} f(t_1,\dots,t_n)(b_1-t_1)^{\alpha-1}\dots(b_n-t_n)^{\alpha-1} dt_n \dots dt_1.$$

Definition 2.3 (Fractional partial derivatives [3]). Let $x_i \in [a_i,b_i]$, $i=1,\dots,n$, and $\alpha \in (0,1)$. The operator ${}_{a_i}D_{x_i}^\alpha[i]$ defined by

$$ {}_{a_i}D_{x_i}^\alpha[i] f(x_1,\dots,x_n) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial x_i}\int_{a_i}^{x_i}(x_i-t)^{-\alpha}\Big[f(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_n) - f(x_1,\dots,x_{i-1},a_i,x_{i+1},\dots,x_n)\Big]dt$$

is called the ith fractional partial derivative of order α, $i=1,\dots,n$.

Remark 2.1. *The Jumarie fractional derivative [15,17] given by* (2.1) *can be obtained by putting $n=1$ in Definition 2.3:*

$${}_aD_x^\alpha[1]f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (x-t)^{-\alpha}(f(t)-f(a))dt = f^{(\alpha)}(x).$$

Definition 2.4 (The fractional line integral [3]). Let $R=[a,b]\times[c,d]$. The fractional line integral on ∂R is defined by

$$I_{\partial R}^\alpha f = I_{\partial R}^\alpha[1]f + I_{\partial R}^\alpha[2]f,$$

where

$$I_{\partial R}^\alpha[1]f = \alpha\int_a^b [f(t,c)-f(t,d)]\,(b-t)^{\alpha-1}dt$$

and

$$I_{\partial R}^\alpha[2]f = \alpha\int_c^d [f(b,t)-f(a,t)]\,(d-t)^{\alpha-1}dt.$$

3 Main Results

Let us consider functions $u=u(x,y)$. We assume that the domain of functions u contain the rectangle $R=[a,b]\times[c,d]$ and are continuous on R. Moreover, functions u under our consideration are such that the fractional partial derivatives ${}_aD_x^\alpha[1]u$ and ${}_cD_y^\alpha[2]u$ are continuous on R, $\alpha\in(0,1)$. We investigate the following fractional problem of the calculus of variations: to minimize a given functional

$$J[u(\cdot,\cdot)] = \alpha^2\int_a^b\int_c^d f\left(x,y,u,{}_aD_x^\alpha[1]u,{}_cD_y^\alpha[2]u\right)(b-x)^{\alpha-1}(d-y)^{\alpha-1}dydx \tag{3.1}$$

when subject to an isoperimetric constraint

$$\alpha^2 \int_a^b \int_c^d g\left(x, y, u, {}_a D_x^\alpha[1]u, {}_c D_y^\alpha[2]u\right)(b-x)^{\alpha-1}(d-y)^{\alpha-1} dy dx = K \tag{3.2}$$

and a boundary condition

$$u(x,y)|_{\partial R} = \psi(x,y). \tag{3.3}$$

We are assuming that ψ is some given function, K is a constant, and f and g are at least of class of C^1. Moreover, we assume that $\partial_4 f$ and $\partial_4 g$ have continuous fractional partial derivatives ${}_a D_x^\alpha[1]$; and $\partial_5 f$ and $\partial_5 g$ have continuous fractional partial derivatives ${}_c D_y^\alpha[2]$. Along the work, we denote by $\partial_i f$ and $\partial_i g$ the standard partial derivatives of f and g with respect to their ith argument, $i = 1, \ldots, 5$.

Definition 3.1. A continuous function $u = u(x,y)$ that satisfies the given isoperimetric constraint (3.2) and boundary condition (3.3), is said to be admissible for problem (3.1)-(3.3).

Remark 3.1. *Contrary to the classical setting of the calculus of variations, where admissible functions are necessarily differentiable, here we are considering our variational problem* (3.1)-(3.3) *on the set of continuous curves u (without assuming differentiability of u). Indeed, the modified Riemann–Liouville derivatives have the advantage of both the standard Riemann–Liouville and Caputo fractional derivatives: they are defined for arbitrarily continuous (not necessarily differentiable) functions, like the standard Riemann–Liouville ones, and the fractional derivative of a constant is equal to zero, as it happens with the Caputo derivatives.*

Definition 3.2 (Local minimizer to (3.1)-(3.3)). An admissible function $u = u(x,y)$ is said to be a local minimizer to problem (3.1)-(3.3) if there exists some $\gamma > 0$ such that for all admissible functions $\hat{u}$ with $\|\hat{u} - u\|_{1,\infty} < \gamma$ one has $J[\hat{u}] - J[u] \geq 0$, where

$$\|u\|_{1,\infty} := \max_{(x,y)\in R} |u(x,y)| + \max_{(x,y)\in R} \left|{}_a D_x^\alpha[1]u(x,y)\right| + \max_{(x,y)\in R} \left|{}_c D_y^\alpha[2]u(x,y)\right|.$$

We make use of the following result proved in [3]:

Lemma 3.2 (Green's fractional formula [3]). *Let $h, k,$ and η be continuous functions whose domains contain R. Then,*

$$\begin{aligned}
&\int_a^b \int_c^d \left[h(x,y)\, {}_a D_x^\alpha[1]\eta(x,y) - k(x,y)\, {}_c D_y^\alpha[2]\eta(x,y)\right](b-x)^{\alpha-1}(d-y)^{\alpha-1} dy dx \\
&= -\int_a^b \int_c^d \left[{}_a D_x^\alpha[1]h(x,y) - {}_c D_y^\alpha[2]k(x,y)\right]\eta(x,y)(b-x)^{\alpha-1}(d-y)^{\alpha-1} dy dx \\
&\quad + \alpha! \left[I_{\partial R}^\alpha[1](h\eta) + I_{\partial R}^\alpha[2](k\eta)\right].
\end{aligned}$$

Remark 3.3. *If $\eta \equiv 0$ on ∂R in Lemma 3.2, then*

$$\int_a^b \int_c^d \left[h(x,y)_a D_x^\alpha[1]\eta(x,y) - k(x,y)_c D_y^\alpha[2]\eta(x,y)\right](b-x)^{\alpha-1}(d-y)^{\alpha-1}dydx \tag{3.4}$$
$$= -\int_a^b \int_c^d \left[{}_a D_x^\alpha[1]h(x,y) -_c D_y^\alpha[2]k(x,y)\right]\eta(x,y)(b-x)^{\alpha-1}(d-y)^{\alpha-1}dydx.$$

3.1 Necessary Optimality Condition

The next theorem gives a necessary optimality condition for u to be a solution of the fractional isoperimetric problem defined by (3.1)-(3.3).

Theorem 3.4 (Euler–Lagrange fractional optimality condition to (3.1)-(3.3)). *If u is a local minimizer to problem (3.1)-(3.3), then there exists a nonzero pair of constants (λ_0, λ) such that u satisfies the fractional PDE*

$$\partial_3 H\{u\}(x,y) -_a D_x^\alpha[1]\partial_4 H\{u\}(x,y) -_c D_y^\alpha[2]\partial_5 H\{u\}(x,y) = 0 \tag{3.5}$$

for all $(x,y) \in R$, where

$$H(x,y,u,v,w,\lambda_0,\lambda) := \lambda_0 f(x,y,u,v,w) + \lambda g(x,y,u,v,w)$$

and, for simplicity of notation, we use the operator $\{\cdot\}$ defined by

$$\{u\}(x,y) := \left(x,y,u(x,y),_a D_x^\alpha[1]u(x,y),_c D_y^\alpha[2]u(x,y),\lambda_0,\lambda\right).$$

Proof. Let us define the function

$$\hat{u}_\varepsilon(x,y) = u(x,y) + \varepsilon\eta(x,y), \tag{3.6}$$

where η is such that $\eta \in C^1(R)$,

$$\eta(x,y)|_{\partial R} = 0,$$

and $\varepsilon \in \mathbb{R}$. If ε take values sufficiently close to zero, then (3.6) is included into the first order neighborhood of u, i.e., there exists $\delta > 0$ such that $\hat{u}_\varepsilon \in U_1(u,\delta)$, where

$$U_1(u,\delta) := \left\{\hat{u}(x,y) : \|u - \hat{u}\|_{1,\infty} < \delta\right\}.$$

On the other hand,

$$\hat{u}_0(x,y) = u, \frac{\partial \hat{u}_\varepsilon(x,y)}{\partial \varepsilon} = \eta, \frac{\partial_a D_x^\alpha[1]\hat{u}_\varepsilon(x,y)}{\partial \varepsilon} =_a D_x^\alpha[1]\eta, \frac{\partial_c D_y^\alpha[2]\hat{u}_\varepsilon(x,y)}{\partial \varepsilon} =_c D_y^\alpha[2]\eta.$$

Let

$$F(\varepsilon) = \alpha^2 \int_a^b \int_c^d f(x,y,\hat{u}_\varepsilon(x,y),_a D_x^\alpha[1]\hat{u}_\varepsilon(x,y),_c D_y^\alpha[2]\hat{u}_\varepsilon(x,y))(b-x)^{\alpha-1}(d-y)^{\alpha-1}dydx,$$

and

$$G(\varepsilon) = \alpha^2 \int_a^b \int_c^d g(x, y, \hat{u}_\varepsilon(x,y), {}_a D_x^\alpha[1]\hat{u}_\varepsilon(x,y), {}_c D_y^\alpha[2]\hat{u}_\varepsilon(x,y))(b-x)^{\alpha-1}(d-y)^{\alpha-1}dydx.$$

Define the Lagrange function by

$$L(\varepsilon, \lambda_0, \lambda) = \lambda_0 F(\varepsilon) + \lambda \left(G(\varepsilon) - K\right).$$

Then, by the extended Lagrange multiplier rule (see, e.g., [40]), we can choose multipliers λ_0 and λ, not both zero, such that

$$\frac{\partial L(0, \lambda_0, \lambda)}{\partial \varepsilon} = \lambda_0 \frac{\partial F}{\partial \varepsilon}\bigg|_{\varepsilon=0} + \lambda \frac{\partial G}{\partial \varepsilon}\bigg|_{\varepsilon=0} = 0. \tag{3.7}$$

The term $\frac{\partial F}{\partial \varepsilon}\big|_{\varepsilon=0}$ is equal to

$$\begin{aligned}
&\alpha^2 \int_a^b \int_c^d \left\{ \frac{\partial}{\partial \varepsilon} \left[f(x, y, \hat{u}_\varepsilon, {}_a D_x^\alpha[1]\hat{u}_\varepsilon, {}_c D_y^\alpha[2]\hat{u}_\varepsilon)(b-x)^{\alpha-1}(d-y)^{\alpha-1} \right] \right\}_{\varepsilon=0} dydx \\
&= \alpha^2 \int_a^b \int_c^d \partial_3 f (b-x)^{\alpha-1}(d-y)^{\alpha-1} dydx \qquad (3.8)\\
&\quad + \alpha^2 \int_a^b \int_c^d \left[\partial_4 f_a D_x^\alpha[1]\eta + \partial_5 f_c D_y^\alpha[2]\eta \right] (b-x)^{\alpha-1}(d-y)^{\alpha-1} dydx.
\end{aligned}$$

By (3.4) the last double integral in (3.8) may be transformed as follows:

$$\begin{aligned}
&\alpha^2 \int_a^b \int_c^d \left[\partial_4 f_a D_x^\alpha[1]\eta + \partial_5 f_c D_y^\alpha[2]\eta \right] (b-x)^{\alpha-1}(d-y)^{\alpha-1} dydx \\
&= -\alpha^2 \int_a^b \int_c^d \left[{}_a D_x^\alpha[1]\partial_4 f + {}_c D_y^\alpha[2]\partial_5 f \right] \eta (b-x)^{\alpha-1}(d-y)^{\alpha-1} dydx.
\end{aligned}$$

Hence,

$$\frac{\partial F}{\partial \varepsilon}\bigg|_{\varepsilon=0} = \alpha^2 \int_a^b \int_c^d \left[\partial_3 f - {}_a D_x^\alpha[1]\partial_4 f - {}_c D_y^\alpha[2]\partial_5 f \right] \eta (b-x)^{\alpha-1}(d-y)^{\alpha-1} dydx. \tag{3.9}$$

Similarly,

$$\frac{\partial G}{\partial \varepsilon}\bigg|_{\varepsilon=0} = \alpha^2 \int_a^b \int_c^d \left[\partial_3 g - {}_a D_x^\alpha[1]\partial_4 g - {}_c D_y^\alpha[2]\partial_5 g \right] \eta (b-x)^{\alpha-1}(d-y)^{\alpha-1} dydx. \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.7), it results that

$$\frac{\partial L(\varepsilon,\lambda_0,\lambda)}{\partial\varepsilon}=\alpha^2\int\limits_a^b\int\limits_c^d\Big[\lambda_0\left(\partial_3 f -_a D_x^\alpha[1]\partial_4 f -_c D_y^\alpha[2]\partial_5 f\right)$$

$$+\lambda\left(\partial_3 g -_a D_x^\alpha[1]\partial_4 g -_c D_y^\alpha[2]\partial_5 g\right)\Big]\eta(b-x)^{\alpha-1}(d-y)^{\alpha-1}dydx=0.$$

Finally, since $\eta\equiv 0$ on ∂R, the fundamental lemma of the calculus of variations (see, e.g., [24]) implies that

$$\partial_3 H\left\{u\right\}(x,y) -_a D_x^\alpha[1]\partial_4 H\left\{u\right\}(x,y) -_c D_y^\alpha[2]\partial_5 H\left\{u\right\}(x,y)=0.$$

□

3.2 Natural Boundary Conditions

In this section we consider problem (3.1)-(3.2), i.e., we consider the case when the value of function $u=u(x,y)$ is not preassigned on ∂R.

Theorem 3.5 (Fractional natural boundary conditions to (3.1)-(3.2)**).** *If u is a local minimizer to problem* (3.1)-(3.2)*, then u is a solution of the fractional differential equation* (3.5)*. Moreover, it satisfies the following conditions:*

1. $\partial_4 H\left\{u\right\}(a,y)=0$ *for all* $y\in[c,d]$*;*
2. $\partial_4 H\left\{u\right\}(b,y)=0$ *for all* $y\in[c,d]$*;*
3. $\partial_5 H\left\{u\right\}(x,c)=0$ *for all* $x\in[a,b]$*;*
4. $\partial_5 H\left\{u\right\}(x,d)=0$ *for all* $x\in[a,b]$*.*

Proof. Since in problem (3.1)-(3.2) no boundary condition is imposed, we do not require η in the proof o Theorem 3.4 to vanish on ∂R. Therefore, following the proof of Theorem 3.4, we obtain

$$\alpha^2\int\limits_a^b\int\limits_c^d\left(\partial_3 H\left\{u\right\}(x,y) +_a D_x^\alpha[1]\partial_4 H\left\{u\right\}(x,y)\right.$$
$$\left. +_c D_y^\alpha[2]\partial_5 H\left\{u\right\}(x,y)\right)\eta(b-x)^{\alpha-1}(d-y)^{\alpha-1}dydx$$
$$+\alpha!\left[I_{\partial R}^\alpha[1](\partial_4 H\left\{u\right\}(x,y)\eta)+I_{\partial R}^\alpha[2](\partial_5 H\left\{u\right\}(x,y)\eta)\right]=0, \tag{3.11}$$

where η is an arbitrary continuous function. In particular, the above equation holds for $\eta\equiv 0$ on ∂R. If $\eta(x,y)|_{\partial R}=0$, the second member of the sum in (3.11) vanishes and the fundamental lemma of the calculus of variations (see, e.g., [24]) implies (3.5). With this result equation (3.11) takes the form

$$\int\limits_c^d \partial_4 H\left\{u\right\}(b,y)\eta(b,y)(d-y)^{\alpha-1}dy-\int\limits_c^d \partial_4 H\left\{u\right\}(a,y)\eta(a,y)(d-y)^{\alpha-1}dy \tag{3.12}$$

$$-\int\limits_a^b \partial_5 H\left\{u\right\}(x,c)\eta(x,c)(b-x)^{\alpha-1}dx-\int\limits_a^b \partial_5 H\left\{u\right\}(x,d)\eta(x,d)(b-x)^{\alpha-1}dx=0.$$

Let $S_1 = ([a,b] \times c) \cup ([a,b] \times d) \cup (b \times [c,d])$. Since η is an arbitrary function, we can consider the subclass of functions for which $\eta(x,y)|_{S_1} = 0$. For such η, equation (3.12) reduces to

$$0 = \int_c^d \partial_4 H \{u\}(a,y)\eta(a,y)(d-y)^{\alpha-1} dy.$$

By the fundamental lemma of calculus of variations, we obtain that

$$\partial_4 H \{u\}(a,y) = 0$$

for all $y \in [c,d]$. We prove the other natural boundary conditions in a similar way. □

3.3 Sufficient Condition

We now prove a sufficient condition that ensures existence of global minimum under appropriate convexity assumptions.

Theorem 3.6. *Let $H(x,y,u,v,w,\lambda_0,\lambda) = \lambda_0 f(x,y,u,v,w) + \lambda g(x,y,u,v,w)$ be a convex function of u, v and w. If $u(x,y)$ satisfies* (3.5)*, then for an arbitrary admissible function $\hat{u}(\cdot,\cdot)$ the following holds:*

$$J[\hat{u}(\cdot,\cdot)] \geq J[u(\cdot,\cdot)],$$

i.e., $u(\cdot,\cdot)$ minimizes (3.1).

Proof. Define the following function:

$$\mu(x,y) := \hat{u}(x,y) - u(x,y).$$

Obviously,

$$\mu(x,y)|_{\partial R} = 0.$$

Since $H\{\hat{u}\}(x,y)$ is convex and ${}_aD_x^\alpha[1]$, ${}_cD_y^\alpha[2]$ are linear operators, we obtain that

$$\begin{aligned}
&H\{\hat{u}\}(x,y) - H\{u\}(x,y) \\
&\geq (\hat{u}(x,y) - u(x,y))\partial_3 H\{u\}(x,y) + \left({}_aD_a^\alpha[1]\hat{u}(x,y) - {}_aD_x^\alpha[1]u(x,y)\right)\partial_4 H\{u\}(x,y) \\
&\quad + \left({}_cD_y^\alpha[2]\hat{u}(x,y) - {}_cD_y^\alpha[2]u(x,y)\right)\partial_5 H\{u\}(x,y) \\
&= (\hat{u}(x,y) - u(x,y))\partial_3 H\{u\}(x,y) + {}_aD_x^\alpha[1]\left(\hat{u}(x,y) - u(x,y)\right)\partial_4 H\{u\}(x,y) \\
&\quad + {}_cD_y^\alpha[2]\left(\hat{u}(x,y) - u(x,y)\right)\partial_5 H\{u\}(x,y) \\
&= \mu(x,y)\partial_3 H\{u\}(x,y) + {}_aD_x^\alpha[1]\mu(x,y)\partial_4 H\{u\}(x,y) + {}_cD_y^\alpha[2]\mu(x,y)\partial_5 H\{u\}(x,y),
\end{aligned} \tag{3.13}$$

where the λ_0 and λ that appear in $\{u\}(x,y)$ are constants whose existence is assured by Theorem 3.4. Therefore,[1]

$$
\begin{aligned}
&J[\hat{u}(\cdot,\cdot)] - J[u(\cdot,\cdot)]\\
&= \alpha^2 \int_a^b \int_c^d f(x,y,\hat{u},{}_aD_x^\alpha[1]\hat{u},{}_cD_y^\alpha[2]\hat{u})(b-x)^{\alpha-1}(d-y)^{\alpha-1}dydx\\
&\quad - \alpha^2 \int_a^b \int_c^d f(x,y,u,{}_aD_x^\alpha[1]u,{}_cD_y^\alpha[2]u)(b-x)^{\alpha-1}(d-y)^{\alpha-1}dydx\\
&\quad + \lambda_0 \left(\alpha^2 \int_a^b \int_c^d g(x,y,\hat{u},{}_aD_x^\alpha[1]\hat{u},{}_cD_y^\alpha[2]\hat{u})(b-x)^{\alpha-1}(d-y)^{\alpha-1}dydx - K\right)\\
&\quad - \lambda_0 \left(\alpha^2 \int_a^b \int_c^d g(x,y,\hat{u},{}_aD_x^\alpha[1]\hat{u},{}_cD_y^\alpha[2]u)(b-x)^{\alpha-1}(d-y)^{\alpha-1}dydx - K\right)\\
&= \alpha^2 \int_a^b \int_c^d \left(H\{\hat{u}\} - H\{u\}\right)(b-x)^{\alpha-1}(d-y)^{\alpha-1}dydx.
\end{aligned}
$$

Using (3.13) and (3.4), we get

$$
\begin{aligned}
&\alpha^2 \int_a^b \int_c^d \left(H\{\hat{u}\} - H\{u\}\right)(b-x)^{\alpha-1}(d-y)^{\alpha-1}dydx\\
&\geq \alpha^2 \int_a^b \int_c^d \mu\partial_3 H\{u\}(b-x)^{\alpha-1}(d-y)^{\alpha-1}dydx\\
&\quad + \alpha^2 \int_a^b \int_c^d \left({}_aD_x^\alpha[1]\mu\partial_4 H\{u\} +{}_c D_y^\alpha[2]\mu\partial_5 H\{u\}\right)(b-x)^{\alpha-1}(d-y)^{\alpha-1}dydx\\
&= \alpha^2 \int_a^b \int_c^d \mu\partial_3 H\{u\}(b-x)^{\alpha-1}(d-y)^{\alpha-1}dydx\\
&\quad + \alpha^2 \int_a^b \int_c^d \left({}_aD_x^\alpha[1]\partial_4 H\{u\} +{}_c D_y^\alpha[2]\partial_5 H\{u\}\right)\mu(b-x)^{\alpha-1}(d-y)^{\alpha-1}dydx\\
&= \alpha^2 \int_a^b \int_c^d \left(\partial_3 H\{u\} +{}_a D_x^\alpha[1]\partial_4 H\{u\}\right.\\
&\quad \left. +{}_cD_y^\alpha[2]\partial_5 H\{u\}\right)\mu(b-x)^{\alpha-1}(d-y)^{\alpha-1}dydx\\
&= 0.
\end{aligned}
$$

Thus, $J[\hat{u}(\cdot,\cdot)] \geq J[u(\cdot,\cdot)]$. □

[1] From now on we omit, for brevity, the arguments (x,y).

4 Conclusion

The fractional calculus provides a very useful framework to deal with nonlocal dynamics: if one wants to include memory effects, i.e., the influence of the past on the behavior of the system at present time, then one may use fractional derivatives. The proof of fractional Euler–Lagrange equations is a subject of strong current study because of its numerous applications. However, while the single time case is well developed, the multitime fractional variational theory is in its childhood, and much remains to be done. In this work we consider a new class of multitime fractional functionals of the calculus of variations subject to isoperimetric constraints. We prove both necessary and sufficient optimality conditions via the modified Riemann–Liouville approach.

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References

[1] N.H. Abel, *Euvres completes de Niels Henrik Abel*, Christiana: Imprimerie de Grondahl and Son; New York and London: Johnson Reprint Corporation. VIII, 621 pp., 1965.

[2] O.P. Agrawal, *Formulation of Euler-Lagrange equations for fractional variational problems*, J. Math. Anal. Appl. 272, 1 (2002), 368-379.

[3] R. Almeida, A.B. Malinowska and D.F.M. Torres, *A fractional calculus of variations for multiple integrals with application to vibrating string*, J. Math. Phys. 51, 3 (2010), 033503, 12 pp.

[4] R. Almeida and D.F.M. Torres, *Calculus of variations with fractional derivatives and fractional integrals*, Appl. Math. Lett. 22, 12 (2009), 1816-1820.

[5] R. Almeida and D.F.M. Torres, *Necessary and sufficient conditions for the fractional calculus of variations with Caputo derivatives*, Commun. Nonlinear Sci. Numer. Simul. 16, 3 (2011), 1490-1500.

[6] D. Baleanu, A.K. Golmankhaneh, R. Nigmatullin and A.K. Golmankhaneh, *Fractional Newtonian mechanics*, Cent. Eur. J. Phys. 8, 1 (2010), 120-125.

[7] N.R.O. Bastos, R.A.C. Ferreira and D.F.M. Torres, *Necessary optimality conditions for fractional difference problems of the calculus of variations*, Discrete Contin. Dyn. Syst. 29, 2 (2011), 417-437.

[8] N.R.O. Bastos, R.A.C. Ferreira and D.F.M. Torres, *Discrete-time fractional variational problems*, Signal Process. 91, 3 (2011), 513-524.

[9] R.A. El-Nabulsi and D.F.M. Torres, *Necessary optimality conditions for fractional action-like integrals of variational calculus with Riemann-Liouville derivatives of order* (α, β), Math. Methods Appl. Sci. 30, 15 (2007), 1931-1939.

[10] R.A. El-Nabulsi and D.F.M. Torres, *Fractional actionlike variational problems*, J. Math. Phys. 49, 5 (2008), 053521, 7 pp.

[11] G.S.F. Frederico and D.F.M. Torres, *Fractional conservation laws in optimal control theory*, Nonlinear Dynam. 53, 3 (2008), 215-222.

[12] G.S.F. Frederico and D.F.M. Torres, *Fractional Noether's theorem in the Riesz-Caputo sense*, Appl. Math. Comput. 217, 3 (2010), 1023-1033.

[13] E. Girejko, A.B. Malinowska and D.F.M. Torres, *The contingent epiderivative and the calculus of variations on time scales*, Optimization (2010), in press. DOI: 10.1080/02331934.2010.506615

[14] R. Hilfer, *Applications of fractional calculus in physics*, World Sci. Publishing, River Edge, NJ, 2000.

[15] G. Jumarie, *Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results*, Comput. Math. Appl. 51, 9-10 (2006), 1367-1376.

[16] G. Jumarie, *Fractional Hamilton-Jacobi equation for the optimal control of nonrandom fractional dynamics with fractional cost function*, J. Appl. Math. Comput. 23, 1-2 (2007), 215-228.

[17] G. Jumarie, *Table of some basic fractional calculus formulae derived from a modified Riemann-Liouville derivative for non-differentiable functions*, Appl. Math. Lett. 22, 3 (2009), 378-385.

[18] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[19] A.B. Malinowska, M.R. Sidi Ammi and D.F.M. Torres, *Composition functionals in fractional calculus of variations*, Commun. Frac. Calc. 1, 1 (2010), 32-40.

[20] A.B. Malinowska and D.F.M. Torres, *Generalized natural boundary conditions for fractional variational problems in terms of the Caputo derivative*, Comput. Math. Appl. 59, 9 (2010), 3110-3116.

[21] A.B. Malinowska and D.F.M. Torres, *Natural boundary conditions in the calculus of variations*, Math. Methods Appl. Sci. 33, 14 (2010), 1712-1722.

[22] A.B. Malinowska and D.F.M. Torres, *A general backwards calculus of variations via duality*, Optim. Lett. (2010), in press. DOI: 10.1007/s11590-010-0222-x

[23] N. Martins and D.F.M. Torres, *Calculus of variations on time scales with nabla derivatives*, Nonlinear Anal. 71, 12 (2009), e763-e773.

[24] T. Marutani, *Canonical forms of Euler's equation and natural boundary condition—2 dimensional case*, The Kwansei Gakuin Economic Review 34 (2003), 21-28.

[25] D. Mozyrska and D.F.M. Torres, *Minimal modified energy control for fractional linear control systems with the Caputo derivative*, Carpathian J. Math. 26, 2 (2010), 210–221.

[26] D. Mozyrska and D.F.M. Torres, *Modified optimal energy and initial memory of fractional continuous-time linear systems*, Signal Process. 91, 3 (2011), 379-385.

[27] M. Pirvan and C. Udrişte, *Optimal control of electromagnetic energy*, Balkan J. Geom. Appl. 15, 1 (2010), 131-141.

[28] I. Podlubny, *Fractional Differential Equations*, Mathematics in Sciences and Engineering, 198, Academic Press, San Diego, 1999.

[29] F. Riewe, *Nonconservative Lagrangian and Hamiltonian mechanics*, Phys. Rev. E (3) 53, 2 (1996), 1890-1899.

[30] F. Riewe, *Mechanics with fractional derivatives*, Phys. Rev. E (3) 55, 3-B (1997), 3581-3592.

[31] C. Udrişte, *Multitime controllability, observability and bang-bang principle*, J. Optim. Theory Appl. 139, 1 (2008), 141-157.

[32] C. Udrişte, *Simplified multitime maximum principle*, Balkan J. Geom. Appl. 14, 1 (2009), 102-119.

[33] C. Udrişte, *Nonholonomic approach of multitime maximum principle*, Balkan J. Geom. Appl. 14, 2 (2009), 101-116.

[34] C. Udrişte, *Equivalence of multitime optimal control problems*, Balkan J. Geom. Appl. 15, 1 (2010), 155-162.

[35] C. Udrişte, O. Dogaru and I. Ţevy, *Null Lagrangian forms and Euler-Lagrange PDEs*, J. Adv. Math. Stud. 1, 1-2 (2008), 143-156.

[36] C. Udrişte and D. Opris, *Euler-Lagrange-Hamilton dynamics with fractional action*, WSEAS Trans. Math. 7, 1 (2008), 19-30.

[37] C. Udrişte, P. Popescu and M. Popescu, *Generalized multitime Lagrangians and Hamiltonians*, WSEAS Trans. Math. 7, 1 (2008), 66-72.

[38] C. Udrişte and A.-M. Teleman, *Hamiltonian approaches of field theory*, Int. J. Math. Math. Sci. 2004, 57-60 (2004), 3045-3056.

[39] C. Udrişte and I. Ţevy, *Multi-time Euler-Lagrange-Hamilton theory*, WSEAS Trans. Math. 6, 6 (2007), 701-709.

[40] B. van Brunt, *The Calculus of Variations*, Springer, New York, 2004.

Author's address:

Tatiana Odzijewicz and Delfim F. M. Torres
Department of Mathematics,
University of Aveiro,
3810-193 Aveiro, Portugal.
E-mail: tatianao@ua.pt; delfim@ua.pt