Optimization problems via second order Lagrangians

Constantin Udrişte and Ariana Pitea

Abstract. The aim of this paper is to study several optimality properties in relation with a second order Lagrangian and its associated Hamiltonian. Section 1 includes original results on Lagrangian and Hamiltonian dynamics based on second order Lagrangians, Riemannian metrics determined by second order Lagrangians, second order Lagrangians linear affine in acceleration, and the pull-back of Lagrange single-time 1-form on the first order jet bundle. Two examples, the first coming from Economics (the problem of optimal growth) and the second coming from Physics (the motion of a spinning particle), illuminate the theoretical aspects. Section 2 justifies a new version of Hamilton-Jacobi PDE. Section 3 analyzes the constrained optimization problems based on second order Lagrangians. Section 4 proves Theorems regarding the dynamics induced by second-order forms.

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1 Lagrangian or Hamiltonian dynamics based on second order Lagrangians

The Analytical Mechanics based on second order Lagrangians has been studied, with remarkable results, by many researchers (see [5], [6], [7]). Here we develop our viewpoint by introducing some new results.

Let $\mathbb{R}$ and M be manifolds of dimensions 1 and n, with the local coordinates t and $x = (x^i)$. Consider $J^1(\mathbb{R}, M)$ and $J^2(\mathbb{R}, M)$ the first, respectively the second order jet bundle associated to $\mathbb{R}$ and M, [14]. In order to develop our theory, we need the following background about jet bundles, [27].

Definition 1.1. A mapping $\phi\colon [t_0, t_1] \subset \mathbb{R} \to \mathbb{R} \times M$ is called *local section* of $(\mathbb{R} \times M, \pi_1, \mathbb{R})$ if it satisfies the condition $\pi_1 \circ \phi = \mathrm{id}_{[t_0,t_1]}$. If $t \in \mathbb{R}$, then the set of all local sections of π_1, whose domains contain the point t, will be denoted $\Gamma_t(\pi_1)$.

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If $\phi \in \Gamma_t(\pi_1)$ and (t, x^i) are coordinate functions around $\phi(t) \in \mathbb{R} \times M$, then $x^i\big(\phi(t)\big) = \phi^i(t)$, $i = \overline{1, n}$.

Definition 1.2. Two local sections $\phi, \psi \in \Gamma_t(\pi_1)$ are called 2-*equivalent* at the point t if

$$\phi(t) = \psi(t), \quad \frac{\mathrm{d}\phi^i}{\mathrm{d}t}(t) = \frac{\mathrm{d}\psi^i}{\mathrm{d}t}(t), \quad \frac{\mathrm{d}^2\phi^i}{\mathrm{d}t^2}(t) = \frac{\mathrm{d}^2\psi^i}{\mathrm{d}t^2}(t), \quad i = \overline{1, n}.$$

The equivalence class containing ϕ is called the 2-jet of ϕ at the point t and is denoted by $j^2_t(\phi)$.

Definition 1.3. The set $J^2(\mathbb{R}, M) = \{j^2_t\phi \,|\, t \in T,\ \phi \in \Gamma_t(\pi_1)\}$ is called *the second order jet bundle.*

A smooth function of the form $L(t, x, \dot{x}, \ddot{x})$ is called *second order Lagrangian on* $J^2(\mathbb{R}, M)$. A solution of the unconstrained optimization problem

$$\min I(x(\cdot)) = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t), \ddot{x}(t))\mathrm{d}t,\ x(t_\alpha) = x_\alpha,\ \dot{x}(t_\alpha) = \dot{x}_\alpha,\ \alpha = 0, 1,$$

satisfies the Euler-Lagrange ODEs

$$\frac{\partial L}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}^i} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L}{\partial \ddot{x}^i} = 0,\ x(t_\alpha) = x_\alpha,\ \dot{x}(t_\alpha) = \dot{x}_\alpha,\ \alpha = 0, 1.$$

These ODEs can be written in the *canonical (normal) form* as equations of the fourth order if and only if

$$\det\left(\frac{\partial^2 L}{\partial \ddot{x}^i \partial \ddot{x}^j}\right) \neq 0.$$

In this case, the second order Lagrangian is called a *regular Lagrangian.*

Now, for a fixed function $x(\cdot)$, we define *the generalized momenta* $p = (p_i), q = (q_i)$ by the algebraic system

$$p_i(t) = \frac{\partial L}{\partial \dot{x}^i}(t, x(t), \dot{x}(t), \ddot{x}(t)),\ q_i(t) = \frac{\partial L}{\partial \ddot{x}^i}(t, x(t), \dot{x}(t), \ddot{x}(t)).$$

Suppose this system defines the functions $\dot{x} = \dot{x}(t, x, p, q)$, $\ddot{x} = \ddot{x}(t, x, p, q)$. Locally, a necessary and sufficient condition is

$$\det\begin{pmatrix} \dfrac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} & \dfrac{\partial^2 L}{\partial \dot{x}^i \partial \ddot{x}^k} \\[2ex] \dfrac{\partial^2 L}{\partial \ddot{x}^k \partial \dot{x}^i} & \dfrac{\partial^2 L}{\partial \ddot{x}^k \partial \ddot{x}^\ell} \end{pmatrix} \neq 0.$$

In this case, the Lagrangian is called *super-regular* and enters in duality with the function of Hamiltonian type

$$H(t, x, p, q) = \dot{x}^i(t, x, p, q)\frac{\partial L}{\partial \dot{x}^i}(t, x, \dot{x}^i(t, x, p, q), \ddot{x}^i(t, x, p, q))$$

$$+\ddot{x}^i(t,x,p,q)\frac{\partial L}{\partial \ddot{x}^i}(t,x,\dot{x}^i(t,x,p,q),\ddot{x}^i(t,x,p,q)) - L(t,x,\dot{x}^i(t,x,p,q),\ddot{x}^i(t,x,p,q))$$

(*second order non-standard Legendrian duality*) or, shortly,

$$H = \dot{x}^i p_i + \ddot{x}^i q_i - L.$$

Although in books [5], [6], [7] is underlined that an autonomous function like H is not conserved along the trajectories of the previous Euler-Lagrange dynamics, that is it is *not a classical Hamiltonian*, we prefer to use it as a *Hamiltonian which produces a conservation law.*

Theorem 1.1. *If $x(\cdot)$ is a solution of the Euler-Lagrange ODEs and the momenta $p(\cdot)$, $q(\cdot)$ are defined as in the previous, then the triple $(x(\cdot), p(\cdot), q(\cdot))$ is a solution of the Hamilton ODEs*

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} - \frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} = -\frac{\partial H}{\partial x^i}, \quad \frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \quad \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = \frac{\partial H}{\partial q_i}.$$

If the Lagrangian L is autonomous, then $H - \dot{q}_i \dfrac{\partial H}{\partial p_i}$ represents a conservation law.

Proof. By computation, we find

$$\frac{\partial H}{\partial x^j} = p_i \frac{\partial \dot{x}^i}{\partial x^j} + q_i \frac{\partial \ddot{x}^i}{\partial x^j} - \frac{\partial L}{\partial x^j} - \frac{\partial L}{\partial \dot{x}^i}\frac{\partial \dot{x}^i}{\partial x^j} - \frac{\partial L}{\partial \ddot{x}^i}\frac{\partial \ddot{x}^i}{\partial x^j} = -\frac{\partial L}{\partial x^j}.$$

The Euler-Lagrange ODEs produce

$$\frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} = \frac{\mathrm{d}p_i}{\mathrm{d}t} + \frac{\partial H}{\partial x^i}.$$

On the other hand,

$$\frac{\partial H}{\partial p_j} = \dot{x}^j, \quad \frac{\partial H}{\partial q_j} = \ddot{x}^j.$$

Finally,

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial x^i}\dot{x}^i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{q}_i \dot{x}^i) + \frac{\partial H}{\partial t}.$$

For other different but connected viewpoints to this subject, the reader is addressed to [9], [16], [21], [26].

1.1 Riemannian metrics associated to second order Lagrangians

If L is a regular second order Lagrangian, then, traditionally, the associated *metric* on the manifold M is defined by

$$g_{ij} = \frac{\partial^2 L}{\partial \ddot{x}^i \partial \ddot{x}^j}.$$

Denoting

$$a_{ij} = \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}, \ b_{ij} = \frac{\partial^2 L}{\partial \ddot{x}^i \partial \dot{x}^j}$$

and adding the condition for super-regular Lagrangian, we obtain the *augmented metric* (non-degenerate $(0,2)$-tensor field)

$$G = \begin{pmatrix} a & b \\ {}^t b & g \end{pmatrix}.$$

This tensor field has the following properties:

1) If the tensor field a is nonsingular, then

$$\det G = (\det a) \det(g - {}^t b a^{-1} b).$$

2) There exists the inverse G^{-1} and

$$G^{-1} = \begin{pmatrix} (a - b g^{-1}\, {}^t b)^{-1} & a^{-1} b ({}^t b a^{-1} b - g)^{-1} \\ ({}^t b a^{-1} b - g)^{-1}\, {}^t b a^{-1} & (g - {}^t b a^{-1} b)^{-1} \end{pmatrix},$$

if the inverses used here exist.

3) If the tensor field G is positive definite, then:

(i) its inverse is positive definite; (ii) the tensor fields $(a - b g^{-1}\, {}^t b)^{-1}$, $a - b g^{-1}\, {}^t b$ are positive definite; the tensor fields $g - {}^t b a^{-1} b$, a, g are positive definite; (iii) $(\det a)(\det g) \geq (\det b)^2$; $\det G \leq (\det a)(\det g)$.

1.2 Second order Lagrangians linear affine in acceleration

A second order Lagrangian is called *linear affine in acceleration* if

$$\frac{\partial^2 L}{\partial \ddot{x}^i \partial \ddot{x}^j} = 0.$$

Since the general form of a second order Lagrangian linear affine in acceleration is

$$L(t, x, \dot{x}, \ddot{x}) = A(t, x, \dot{x}) + B_i(t, x, \dot{x}) \ddot{x}^i,$$

the associated Euler-Lagrange equations

$$\frac{\partial L}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{x}^i} + \frac{\mathrm{d}^2}{\mathrm{d}t^2} B_i = 0$$

have *at most order three.* If $B_i(t, x, \dot{x}) = B_i(t, x)$, then the associated Euler-Lagrange equations have *at most order two.*

1.3 The pull-back of Lagrange 1-form on first order jet bundle

The general form of a single-time Lagrange 1-form on the first order jet bundle $J^1(\mathbb{R}, M)$ is

$$\omega = L(t, x, \dot{x}) \mathrm{d}t + M_i(t, x, \dot{x}) \mathrm{d}x^i + N_i(t, x, \dot{x}) \mathrm{d}\dot{x}^i,$$

where L, M_i and N_i, $i = \overline{1, n}$ are first order smooth Lagrangians. Let us consider the pullback

$$x^* \omega = \big(L(t, x(t), \dot{x}(t)) + M_i(t, x(t), \dot{x}(t)) \dot{x}^i(t) + N_i(t, x(t), \dot{x}(t)) \ddot{x}^i(t) \big)\, \mathrm{d}t,$$

whose coefficient, $\mathcal{L} = L + M_i\dot{x}^i + N_i\ddot{x}^i$, is a smooth *second order Lagrangian on* $J^2(\mathbb{R}, M)$*, linear in acceleration.*

The Euler-Lagrange ODEs associated to the second order Lagrangian $\mathcal{L}$ are

$$\frac{\partial N_j}{\partial \dot{x}^i}\dddot{x}^j = \frac{\partial L}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^i}\right)$$
$$+\frac{\partial M_j}{\partial x^i}\dot{x}^j - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial M_j}{\partial \dot{x}^i}\right)\dot{x}^j - \frac{\partial M_j}{\partial \dot{x}^i}\ddot{x}^j - \frac{\partial M_i}{\partial t}$$
$$+\frac{\partial N_j}{\partial x^i}\ddot{x}^j - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial N_j}{\partial \dot{x}^i}\right)\ddot{x}^j + \frac{\partial^2 N_i}{\partial t^2}, \quad i = \overline{1, n}.$$

Of course, instead of a system of at most fourth order, we obtained a system of the third order. This fact is implied by the special form of the foregoing considered second order Lagrangian $\mathcal{L}$, which is linear with respect to the acceleration. Regarding the ODEs system given above, we remark that it is a normal one, if $\det\left(\frac{\partial N_j}{\partial \dot{x}^i}\right) \neq 0$.

We continue with two examples whose development needs the results we have obtained.

Example 1 [15]. Let us consider a practical example which comes from Economics, and regards dynamic utility and capital accumulation.

Our *problem of optimal growth* deals with the *consumption level function* C and its *growth rate* $\dot{C}$. We consider the *utility* $U(C, \dot{C})$ as determined by the consumption level function

$$C = Y(K) - \dot{K},$$

where Y is the *Gross national income*. Therefore, C is the Gross national product left over after capital accumulation $\dot{K}$ is met.

In order to transform $U(C, \dot{C})$ into a Lagrangian of second order, linear in acceleration, it is appropriate to consider $Y(K) = bK$, $b = const$ and

$$U(C, \dot{C}) = C^a + \gamma\dot{C},$$

where $0 \leq a,\ \gamma \leq 1$. In these conditions, our study refers to *maximizing the functional*

$$J(K(\cdot)) = \int_0^\infty U(K(t), \dot{K}(t), \ddot{K}(t))\mathrm{d}t.$$

We get the necessary conditions of optimality

$$ab(bK - \dot{K})^{a-1} - a(1-a)(bK - \dot{K})^{a-2}(b\dot{K} - \ddot{K}) = 0.$$

After that, we divide by $a(bK - \dot{K})^{a-2})$, which is not null. We are led to the following second order differential equation

$$(1-a)\ddot{K} + b(a-2)\dot{K} + b^2K = 0,$$

having the solution

$$K(t) = A_1\exp(bt) + A_2\exp\left(\frac{bt}{1-a}\right),$$

with A_1 and A_2 constants determined by the boundary conditions $K(0) = K_0$ and $K(T) = K_T$.

Example 2 [16]. We end this section by presenting a second order Lagrangian directly connected to the *motion of a spinning particle*.

Every kinematic system of the first order can be prolonged by certain methods to appropriate dynamical systems of order two [19], [18], whose trajectories are geodesics of a Lagrangian defined by the velocity vector field and the Riemannian metric. In a similar manner, every dynamical system of order two (or three) can be prolonged by differentiation and other methods to a corresponding dynamical system of order four, whose trajectories are geodesics of a Lagrangian defined by the velocity, acceleration and metric. The foregoing remarks allow us to create examples for higher order Lagrangians spaces, see [7].

For example, we consider the Riemannian manifold $(\mathbb{R}^3, \delta_{ij})$ and a point $x = (x^1, x^2, x^3) \in \mathbb{R}^3$, whose motion is described by the differential system

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + x = at + b,$$

where a and b are constant vectors. Differentiating two times, this differential system is prolonged to the following differential system of order four

$$\frac{\mathrm{d}^4 x}{\mathrm{d}t^2} + \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = 0,$$

representing the *motion of a spinning particle* (the motion of a particle rotating around its translating center). This differential system comes from the second order Lagrangian (Euler-Lagrange ODEs)

$$\mathcal{L} = \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j - \frac{1}{2}\delta_{ij}\ddot{x}^i\ddot{x}^j, \quad i,j = \overline{1,3}$$

and admits the first integral

$$\mathcal{H} = \frac{1}{2}\delta^{ij}p_i p_j - \frac{1}{2}\delta^{ij}q_i q_j + \delta^{ij}p_i\,\dot{q}_j, \quad i,j = \overline{1,3}.$$

2 Hamilton-Jacobi PDE

We consider the C^2-clsss function $S\colon J^1(\mathbb{R}, \mathbb{R}^n) \equiv \mathbb{R}^{2n+1} \to \mathbb{R}$ and the constant level sets $\Sigma_c\colon S(t, x, \dot{x}) = c$. Suppose that these sets are hypersurfaces in $\mathbb{R}^{2n+1}$, that is the normal vector field $\left(\frac{\partial S}{\partial t}, \frac{\partial S}{\partial x^i}, \frac{\partial S}{\partial \dot{x}^i}\right)$ is nowhere zero. Let $\Gamma\colon (t, x(t), \dot{x}(t))$, $t \in \mathbb{R}$, be a transversal C^1-class curve to the hypersurfaces Σ_c. Then the function $c(t) = S(t, x(t), \dot{x}(t))$ has a derivative, which is non null, namely

$$\frac{\mathrm{d}c}{\mathrm{d}t}(t) = \frac{\partial S}{\partial t}(t, x(t), \dot{x}(t)) + \frac{\partial S}{\partial x^i}(t, x(t), \dot{x}(t))\dot{x}^i(t)$$

$$+\frac{\partial S}{\partial \dot{x}^i}(t, x(t), \dot{x}(t))\ddot{x}^i(t) = L(t, x(t), \dot{x}(t)) \neq 0.$$

Using the second order Lagrangian of L, it follows the momenta

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = \frac{\partial S}{\partial x^i},\ q_i = \frac{\partial L}{\partial \ddot{x}^i} = \frac{\partial S}{\partial \dot{x}^i}.$$

On one hand, the equalities

$$\dot{x}(t) = \dot{x}(t, x(t), p(t), q(t)),\ \ddot{x}(t) = \ddot{x}(t, x(t), p(t), q(t))$$

become

$$\dot{x}(t) = \dot{x}\left(t, x(t), \frac{\partial S}{\partial x}(t, x(t), \dot{x}(t)), \frac{\partial S}{\partial \dot{x}}(t, x(t), \dot{x}(t))\right)$$

$$\ddot{x}(t) = \ddot{x}\left(t, x(t), \frac{\partial S}{\partial x}(t, x(t), \dot{x}(t)), \frac{\partial S}{\partial \dot{x}}(t, x(t), \dot{x}(t))\right).$$

On the other hand, the definition of L implies

$$-\frac{\partial S}{\partial t} = \frac{\partial S}{\partial x^i}\left(t, x(t), \dot{x}\left(t, x(t), \frac{\partial S}{\partial x}(t, x(t), \dot{x}(t)), \frac{\partial S}{\partial \dot{x}}(t, x(t), \dot{x}(t))\right)\right)$$

$$\dot{x}^i\left(t, x(t), \frac{\partial S}{\partial x}(t, x(t), \dot{x}(t)), \frac{\partial S}{\partial \dot{x}}(t, x(t), \dot{x}(t))\right)$$

$$+\frac{\partial S}{\partial \dot{x}^i}\left(t, x(t), \dot{x}\left(t, x(t), \frac{\partial S}{\partial x}(t, x(t), \dot{x}(t)), \frac{\partial S}{\partial \dot{x}}(t, x(t), \dot{x}(t))\right)\right)$$

$$\ddot{x}^i\left(t, x(t), \frac{\partial S}{\partial x}(t, x(t), \dot{x}(t)), \frac{\partial S}{\partial \dot{x}}(t, x(t), \dot{x}(t))\right) - L(t, x(t), \dot{x}(t)).$$

This relation reveals a new Hamilton-Jacobi PDE

$$\frac{\partial S}{\partial t} + H\left(t, x, \dot{x}, \frac{\partial S}{\partial x}, \frac{\partial S}{\partial \dot{x}}\right) = 0.$$

As a rule, this PDE is endowed with the initial condition $S(0, x, \dot{x}) = S_0(x, \dot{x})$. The solution $S(t, x, \dot{x})$ is called the *generating function* of the canonical momenta.

Conversely, if $S(t, x, \dot{x})$ is a solution of the Hamilton-Jacobi PDE, we define

$$p_i(t) = \frac{\partial S}{\partial x^i}(t, x(t), \dot{x}(t)),\ q_i(t) = \frac{\partial S}{\partial \dot{x}^i}(t, x(t), \dot{x}(t)),$$

and then

$$\int_{t_0}^{t_1} L(t, x(t), \dot{x}(t), \ddot{x}(t))\mathrm{d}t = \int_{t_0}^{t_1} (p_i \dot{x}^i + q_i \ddot{x}^i - H)dt = \int_\Gamma dS.$$

The last formula shows that the action integral can be written as a path independent curvilinear integral.

Theorem 2.1. *The generating function of the canonical momenta is a solution of the Cauchy problem*

$$\frac{\partial S}{\partial t} + H\left(t, x, \dot{x}, \frac{\partial S}{\partial x}, \frac{\partial S}{\partial \dot{x}}\right) = 0, \quad S(0, x, \dot{x}) = S_0(x, \dot{x}).$$

3 Constrained optimization problems based on second order Lagrangians

Let $L\colon J^2(T,M) \to \mathbb{R}$, $g\colon J^2(T,M) \to \mathbb{R}^a$ and $h\colon J^2(T,M) \to \mathbb{R}^b$ be functions of C^2-class. The aim of this section is to study the *constrained optimization problems* determined by these functions in the sense of "optimal functional constrained by PDIs and PDEs" or "optimal functional with isoperimetric constraints".

The first model is the *optimization constrained by PDIs and PDEs*:

$$\min_{x(\cdot)} I(x(\cdot)) = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t), \ddot{x}(t))\mathrm{d}t,$$

subject to

$$g(t, x(t), \dot{x}(t), \ddot{x}(t)) \leqq 0,\ h(t, x(t), \dot{x}(t), \ddot{x}(t)) = 0, \qquad t \in [t_0, t_1],$$

$$x(t_\alpha) = x_\alpha,\ \dot{x}(t_\alpha) = \dot{x}_\alpha,\ \alpha = 0, 1.$$

We transform this constrained optimization problem into a free one, using the Lagrangian

$$\bar{L} = L + < \mu, g >_{\mathbb{R}^a} + < \nu, h >_{\mathbb{R}^b},$$

where $\mu(t)$ and $\nu(t)$ are *vector multipliers*. The necessary conditions for optimality are contained in the following

Theorem 3.1. *If $x^\circ(\cdot)$ is an optimal solution of the previous program, then there are two smooth vector functions, $\mu\colon \mathbb{R} \to \mathbb{R}^a$ and $\nu\colon \mathbb{R} \to \mathbb{R}^b$, such that the following conditions are satisfied at $x^\circ(\cdot)$:*

$$\frac{\partial \bar{L}}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \bar{L}}{\partial \dot{x}^i} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial \bar{L}}{\partial \ddot{x}^i} = 0,$$

$$< \mu(t), g(t, x^\circ(t), \dot{x}^\circ(t), \ddot{x}^\circ(t)) >_{\mathbb{R}^a} = 0, \quad \mu(t) \leqq 0, \quad t \in [t_0, t_1],$$

$$h(t, x(t), \dot{x}(t), \ddot{x}(t)) = 0, \quad t \in [t_0, t_1].$$

The second model is the *optimization with isoperimetric constraints*:

$$\min_{x(\cdot)} I(x(\cdot)) = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t), \ddot{x}(t))\mathrm{d}t,$$

subject to

$$\int_{t_0}^{t_1} g(t, x(t), \dot{x}(t), \ddot{x}(t))dt \leqq 0, \qquad \int_{t_0}^{t_1} h(t, x(t), \dot{x}(t), \ddot{x}(t))dt = 0.$$

In this case, we use a similar Lagrangian

$$\bar{L} = L + < \mu, g > + < \nu, h >,$$

but now the multipliers μ and ν are constant vectors. They are well determined only if the extremals depending on them are not extremals for at list one of the functionals

$$\int_{t_0}^{t_1} g(t, x(t), \dot{x}(t), \ddot{x}(t))dt, \quad \int_{t_0}^{t_1} h(t, x(t), \dot{x}(t), \ddot{x}(t))dt.$$

For all the rest, we have similar necessary optimality conditions.

For other ideas connected to this subject, we address the reader to works [10], [12], [11] and [20]-[30].

4 Dynamics induced by a second-order form

Now we want to extend our explanations to *second-order forms*

$$\theta = \theta_i(x)\mathrm{d}^2x^i + \theta_{ij}(x)\mathrm{d}x^i \otimes \mathrm{d}x^j, \quad \theta_{ij} = \theta_{ji},$$

since they can reflect some dynamical systems coming from Biomathematics, Economical Mathematics, Industrial Mathematics etc.. A second-order form is denoted by (θ_i, θ_{ij}).

Let $\omega_i(x)$ be given potentials (given form) on the Riemannian manifold $(\mathbb{R}^n, \delta_{ij})$. The metric δ_{ij} determines the Christoffel symbols $\Gamma^i_{jk} = 0$. The usual covariant (partial) derivative $\omega_{i,j}$ may be decomposed as $\omega_{i,j} = \frac{1}{2}(\omega_{i,j} - \omega_{j,i}) + \frac{1}{2}(\omega_{i,j} + \omega_{j,i})$, where the anti-symmetric part $\mathcal{M}_{ij} = \frac{1}{2}(\omega_{i,j} - \omega_{j,i})$ is the *Maxwell tensor field* (vortex) and the symmetric part $\mathcal{N}_{ij} = \frac{1}{2}(\omega_{i,j} + \omega_{j,i})$ is the *deformation rate tensor field.*

The pair $(\omega_i, \mathcal{N}_{ij})$ is a second-order form. If (ω_i, ω_{ij}) is a general second-order form, then we suppose that the difference $g_{ij} = \omega_{ij} - \mathcal{N}_{ij}$ represents the components of a new metric, that is g_{ij} is a (0,2) tensor field and $\det(g_{ij}) \neq 0$.

The foregoing ingredients produce the following energy Lagrangians:

1) *second order potential-produced energy Lagrangian*,

$$L_{pp} = \omega_i(x(t))\,\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}(t) + \mathcal{N}_{ij}(x(t))\frac{\mathrm{d}x^i}{\mathrm{d}t}(t)\frac{\mathrm{d}x^j}{\mathrm{d}t}(t),$$

2) *first order gravitational energy Lagrangian*,

$$L_g = g_{ij}(x(t))\frac{\mathrm{d}x^i}{\mathrm{d}t}(t)\frac{\mathrm{d}x^j}{\mathrm{d}t}(t),$$

3) *second order general energy Lagrangian*,

$$L_{ge} = \omega_i(x(t))\,\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + \omega_{i,j}(x(t))\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}.$$

All three energy Lagrangians are related by

$$L_{ge} = L_{pp} + L_g$$

and to each energy Lagrangian there may correspond a *field theory.*

The *Pfaff equation* $\omega_i(x)dx^i = 0$, $i = \overline{1,n}$, defines an $(n-1)$-dimensional *distribution* on M. The symmetric part

$$\frac{1}{2}(\omega_{i,j} + \omega_{j,i})$$

is the *second fundamental form* of this distribution [8]. The potential-produced energy Lagrangian is zero along the integral curves of the distribution generated by the given 1-form $\omega = (\omega_i(x))$.

In the autonomous case, the second order general energy Lagrangian produces the *energy functional*

$$\int_{t_0}^{t_1} \left(\omega_i(x(t)) \, \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2}(t) + \omega_{ij}(x(t)) \frac{\mathrm{d}x^i}{\mathrm{d}t}(t) \frac{\mathrm{d}x^j}{\mathrm{d}t}(t) \right) \mathrm{d}t.$$

By applying the foregoing theory, the Euler-Lagrange ODEs are

$$\omega_{lj,i}\dot{x}^l\dot{x}^j - \omega_{ij,l}\dot{x}^j\dot{x}^l - \omega_{ij}\ddot{x}^j - \omega_{ij,l}\dot{x}^l\dot{x}^j - \omega_{ij}\ddot{x}^j + \omega_{i,jk}\dot{x}^k\dot{x}^l + \omega_{i,l}\ddot{x}^l = 0.$$

If we denote the Christoffel symbols associated to ω_{ij} by

$$\omega_{ijk} = \frac{1}{2}(\omega_{kj,i} + \omega_{ki,j} - \omega_{ij,k}),$$

and

$$\Omega_{ijk} = \frac{1}{2}\left(\omega_{k,ij} + \omega_{j,ik} - \omega_{i,jk}\right),$$

we obtain the following

Theorem 4.1. *The extremals of the energy functional are solutions of the Euler-Lagrange ODEs*

$$g_{ki}\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + (\omega_{ijk} - \Omega_{ijk})\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t} = 0, \; x(t_0) = x_0, \; x(t_1) = x_1$$

(geodesics with respect to an Otsuki type connection [8]*).*

To involve the other elements, we introduce $\Gamma_{ijk} = \omega_{ijk} - \Omega_{ijk}$. After calculations, we find $\Gamma_{ijk} = g_{ijk} + \mathrm{m}_{ijk}$, where g_{ijk} are the Christoffel symbols of g_{ij}, and $\mathrm{m}_{ijk} = \mathcal{M}_{ij,k} + \mathcal{M}_{ik,j}$ is the symmetrized derivative of the Maxwell tensor field $\mathcal{M}$.

Corollary 4.1. *The extremals of the energy functional are solutions of the Euler-Lagrange ODEs*

$$g_{ki}\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + (g_{kji} + \mathrm{m}_{kji})\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{dx^j}{\mathrm{d}t} = 0, \; x(t_0) = x_0, \; x(t_1) = x_1$$

(geodesics with respect to an Otsuki type connection [8]*).*

Regarding this theory of dynamics induced by a second-order general form, there are several open problems, as follows [1], [3], [13], [31]: (1) Find the linear connections in the sense of Crampin [2] associated to the foregoing second-order ODEs; (2) Analyze the second variations of the preceding energy functionals and the symmetries of the foregoing second order differential systems (see also [32]); (3) Find practical interpretations for the motions known as *geometric dynamics, gravi-tovortex motion and second-order force motion* (see also [4]).

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Authors' addresses:

Constantin Udrişte, Ariana Pitea,
University "Politehnica" of Bucharest,
Faculty of Applied Sciences, Department of Mathematics-Informatics I,
Splaiul Independentei 313, Bucharest 060042, Romania;
E-mail: udriste@mathem.pub.ro , anet.udri@yahoo.com ; apitea@mathem.pub.ro