A class of almost contact metric manifolds and twisted products

Maria Falcitelli

Abstract. In the framework of Chinea-Gonzales we study the class of almost contact metric manifolds locally realized as twisted product manifolds $I \times_\lambda F$, I being an open interval, F an almost Hermitian manifold and $\lambda > 0$ a smooth function. Local classification theorems for the generalized Sasakian space-forms in the considered class are obtained as well.

M.S.C. 2010: 53C25, 53D15, 53C21.
Key words: twisted product manifold; generalized Sasakian space-form.

1 Introduction

Warped products play an interesting role in clarifying the interrelation between almost Hermitian (a.H.) and almost contact metric (a.c.m.) manifolds in a given class. The first result in this direction, due to Kenmotsu, states that any Kenmotsu manifold is, locally, isometric to a warped product manifold $I \times_\lambda F$, where F is a Kähler manifold, $I \subset \mathbb{R}$ an open interval and $\lambda : I \to \mathbb{R}$ the function defined by: $\lambda(t) = Ce^t$, $C > 0$ ([15]). In 2007 Dileo and Pastore extended this result, proving that any almost Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$ such that the tensor field $L_\xi\varphi$ vanishes is locally realized as a warped product manifold $I \times_\lambda F$, where F is an almost Kähler manifold and $\lambda(t) = Ce^t$, $C > 0$ ([7]).

On the other hand, suitable warped product manifolds are nice examples of generalized Sasakian space-forms (g.S. space-forms). In fact, given a smooth function $\lambda : \mathbb{R} \to \mathbb{R}$, $\lambda > 0$, and an a.H. manifold F, the warped product $\mathbb{R} \times_\lambda F$ is endowed with an a.c.m. structure naturally induced by the a.H. structure on F. If F is a generalized complex space-form, then $\mathbb{R} \times_\lambda F$ is a g.S. space-form ([1]).

As an extension of warped products, Bishop introduced the concept of umbilic products, also called twisted products ([4]). In [21] Ponge and Reckziegel stated a splitting theorem for a Riemannian manifold (M, g) that admits two complementary foliations L, K whose leaves intersect perpendicularly. If the leaves of L are totally geodesics and the leaves of K totally umbilic, then (M, g) is locally isometric to a twisted product $M' \times_\lambda M''$ such that M' and M'' are leaves of L and K, respectively.

Balkan Journal of Geometry and Its Applications, Vol.17, No.1, 2012, pp. 17-29.

Moreover, if the leaves of K are extrinsic spheres, then $M' \times M''$ is a warped product. This last statement corresponds to the decomposition theorem of Hiepko ([13]).

In this paper, involving a.H. and a.c.m. manifolds, we provide a new link between the Gray-Hervella work on a.H. manifolds and the Chinea-Gonzales classification of a.c.m. manifolds ([12, 5]).

More precisely, let $(F, \widehat{J}, \widehat{g})$ be an a.H. manifold and $\lambda : I \times F \to \mathbb{R}$ a positive smooth function, $I \subset \mathbb{R}$ being an open interval. On $I \times F$ one considers the twisted product metric g_λ of the Euclidean metric on I and $\widehat{g}$ by λ and the a.c.m. structure $(\varphi, \xi, \eta, g_\lambda)$ naturally induced by $(\widehat{J}, \widehat{g})$ as in (2.1). The a.c.m. manifold $I \times_\lambda F = (I \times F, \varphi, \xi, \eta, g_\lambda)$ is called the twisted product of I and F by λ. Firstly, we prove that $I \times_\lambda F$ belongs to the Chinea-Gonzales class $\underset{1\leq i\leq 5}{\oplus} C_i$, briefly denoted by C_{1-5}. An algebraic characterization of a.c.m. manifolds which fall in the class $\mathcal{C}_{1-5}$ is obtained, also. Combining this result with the Ponge and Reckziegel theorem, one proves that any $\mathcal{C}_{1-5}$-manifold is locally realized as a twisted product $]-\varepsilon.\varepsilon[\times_\lambda F$, $\varepsilon > 0$, F being an a.H. manifold and $\lambda :]-\varepsilon, \varepsilon[\times F \to \mathbb{R}$ a smooth positive function. A differential equation involving $\omega(\xi)$, where ω is the Lee form, specifies the $\mathcal{C}_{1-5}$-manifolds that are, locally, warped products.
Then, we point our attention to the classes $\mathcal{C}_h \oplus \mathcal{C}_5$, $h \in \{1, 2, 3, 4\}$. We prove that $\mathcal{C}_h \oplus \mathcal{C}_5$ consists of the $\mathcal{C}_{1-5}$-manifolds that are, locally, a twisted product $]-\varepsilon, \varepsilon[\times_\lambda F$, where F belongs to the Gray-Hervella class $\mathcal{W}_h$. Moreover, any $\mathcal{C}_h \oplus \mathcal{C}_5$-manifold such that $\omega(\xi) = -1$ is locally a warped product $]-\varepsilon, \varepsilon[\times_\lambda F$, F being a $\mathcal{W}_h$-manifold and $\lambda :]-\varepsilon, \varepsilon[\to \mathbb{R}$ acting as $\lambda(t) = Ce^t$, $C > 0$.
The last section deals with g.S. space-forms $M(f_1, f_2, f_3)$ that fall in the class $\mathcal{C}_{1-5}$. By repeated applications of the second Bianchi identity, we prove that M is, locally, a warped product manifold. Moreover, if $\dim M \geq 7$ and f_2 never vanishes, then M falls in the class $\mathcal{C}_5$ and is, locally, a warped product $]-\varepsilon, \varepsilon[\times_\lambda F$, F being a complex space-form. Finally, we establish a local classification in the case $f_2 = 0$.

In this article all manifolds are assumed to be connected.

2 Twisted product manifolds

Given an a.H. manifold $(F, \widehat{J}, \widehat{g})$, an open interval $I \subset \mathbb{R}$ and a smooth function $\lambda : I \times F \to \mathbb{R}$, $\lambda > 0$, on $I \times F$ we consider the a.c.m. structure $(\varphi, \xi, \eta, g_\lambda)$ such that

$$(2.1) \qquad \begin{array}{l} \varphi(a\frac{\partial}{\partial t}, U) = (0, \widehat{J}U), \quad \eta(a\frac{\partial}{\partial t}, U) = a, \ a \in \mathcal{F}(I \times F), U \in \mathcal{X}(F), \\ \xi = (\frac{\partial}{\partial t}, 0), \quad g_\lambda = \pi^*(dt \otimes dt) + \lambda^2 \sigma^*(\widehat{g}), \end{array}$$

$\pi : I \times F \to I$, $\sigma : I \times F \to F$ denoting the canonical projections.
Note that g_λ is the twisted product metric of the Euclidean metric g_0 and $\widehat{g}$. If λ only depends on the coordinate t, then g_λ is the warped product metric of g_0 and $\widehat{g}$. Then the a.c.m. manifold $I \times_\lambda F = (I \times F, \varphi, \xi, \eta, g_\lambda)$ is called, respectively, the twisted product manifold and the warped product manifold of (I, g_0) and $(F, \widehat{J}, \widehat{g})$ by λ.
Through the paper, we'll identify any vector field U on F with $(0, U) \in \mathcal{X}(I \times F)$. The Levi-Civita connections ∇ of $I \times_\lambda F$ and $\widehat{\nabla}$ of F are related by:

$$(2.2) \quad \nabla_U V = \widehat{\nabla}_U V - g_\lambda(U, V) grad \log \lambda + g_\lambda(U, grad \log \lambda) V + g_\lambda(V, grad \log \lambda) U,$$

for any vector fields U, V on F, where $grad$ stands for $grad_{g_\lambda}$ ([21]).
The following relations are well-known, also

$$\nabla_\xi \xi = 0, \qquad \nabla_\xi U = \nabla_U \xi = \xi(\log\lambda)U, \quad U \in \mathcal{X}(F). \tag{2.3}$$

Now, we recall some basic data involving a.c.m. and a.H. manifolds.
Given an a.c.m. manifold $(M, \varphi, \xi, \eta, g)$ with fundamental form Φ, $\Phi(X,Y) = g(X, \varphi Y)$, and Levi-Civita connection ∇, for any $h \in \{1, ..., 12\}$ one considers the projection τ_h of $\nabla\Phi$ on the vector bundle $\mathcal{C}_h(M)$ whose fibre at any $x \in M$ is the linear space $\mathcal{C}_h(T_xM)$ considered in [5]. Putting $\mathcal{C}(M) = \underset{1\le h\le 12}{\oplus} \mathcal{C}_h(M)$, to any section α of $\mathcal{C}(M)$ are associated the 1-forms $c(\alpha)$, $\bar{c}(\alpha)$ given, in a local orthonormal frame on M, by $c(\alpha)(X) = \sum_{i=1}^{2n+1} \alpha(e_i, e_i, X)$ and $\bar{c}(a)(X) = \sum_{i=1}^{2n+1} \alpha(e_i, \varphi e_i, X)$.
The Lee form ω of M, defined by $\omega = -\frac{1}{2(n-1)}(\delta\Phi \circ \varphi + \nabla_\xi \eta) + \frac{\delta\eta}{2n}\eta$, if $n \ge 2$, and $\omega = \nabla_\xi \eta + \frac{\delta\eta}{2}\eta$, if $n = 1$, depends on the projections τ_4,τ_5 and τ_{12} according to the formulas:

$$\omega(X) = \frac{1}{2(n-1)} c(\tau_4)(\varphi X) + \frac{1}{2n}\bar{c}(\tau_5)(\xi)\eta(X), \text{ if } n \ge 2,$$

$$\omega(X) = \tau_{12}(\xi, \xi, \varphi X) + \frac{1}{2}\bar{c}(\tau_5)(\xi)\eta(X), \text{ if } n = 1.$$

Let (N, J', g') be an a.H. manifold with Levi-Civita connection ∇' and fundamental form Ω', $\Omega'(X,Y) = g'(X, J'Y)$. For any $h \in \{1, ..., 4\}$, one considers the component τ'_h of $\nabla'\Omega'$ on the vector bundle $\mathcal{W}_h(N)$ over N whose fibre at each point $p \in N$ is the linear space $\mathcal{W}_h(T_pN)$ introduced in [12].

If $\dim N = 2m \ge 4$, the 1-form $\omega' = -\frac{1}{2(m-1)}\delta'\Omega' \circ J'$ is called the Lee form and depends on the projection τ'_4. In fact, with respect to a local orthonormal frame $\{E_i\}_{1\le i\le 2m}$, one has $\omega'(X) = \frac{1}{2(m-1)} \sum_{i=1}^{2m} \tau'_4(E_i, E_i, J'X)$.

The next result is useful in determining the Chinea-Gonzales class of a twisted product manifold $I \times_\lambda F$ and in relating the covariant derivatives $\widehat{\nabla}\widehat{\Omega}, \nabla\Phi_\lambda$, where $\widehat{\Omega}, \Phi_\lambda$ denote the fundamental forms of F, $I \times_\lambda F$, respectively. The Lee forms of $F, I \times_\lambda F$ are denoted by $\widehat{\omega}, \omega_\lambda$.

Proposition 2.1. *Let $(F, \widehat{J}, \widehat{g})$ be a $2n$-dimensional a.H. manifold, $I \subset \mathbb{R}$ an open interval and $\lambda : I \times F \to \mathbb{R}$ a smooth positive function. Then, for the twisted product manifold $I \times_\lambda F$ the following relations hold*
i) $\nabla_\xi \varphi = 0$,
ii) $\nabla_X \xi = -\xi(\log\lambda)\varphi^2 X, \ X \in \mathcal{X}(I \times F)$,
iii) $\delta\eta = -2n\xi(\log\lambda)$ *and* $\delta\Phi_\lambda(\xi) = 0$,
iv) $\omega_\lambda = \sigma^*(\widehat{\omega}) - d(\log\lambda)$, *if* $n \ge 2$, *and* $\omega_\lambda = -\xi(\log\lambda)\eta$, *if* $n = 1$.

Proof. Formula (2.3) implies i), ii). Let $\{U_i\}_{1\le i\le 2n}$ be a local $\widehat{g}$-orthonormal frame on F. For any $i \in \{1, ..., 2n\}$ one puts $e_i = \frac{1}{\lambda}U_i$, so that $\{\xi, e_1, ..., e_{2n}\}$ is an adapted local orthonormal frame on $I \times_\lambda F$. Applying ii), one easily obtains $\delta\eta = -2n\xi(\log\lambda)$. Furthermore, considering $U, V \in \mathcal{X}(F)$, by (2.2) we have

$$\begin{aligned}(\nabla_U \varphi)V &= (\widehat{\nabla}_U \widehat{J})V + \varphi V(\log\lambda)U - V(\log\lambda)\varphi U \\ &+ g_\lambda(U,V)\varphi(grad \log\lambda) - g_\lambda(U, \varphi V) grad \log\lambda.\end{aligned} \tag{2.4}$$

So, considering an adapted frame as above, by (2.4) and i) we obtain $\delta\Phi_\lambda(\xi) = 0$, and $\delta\Phi_\lambda(U) = \frac{1}{\lambda^2}\sum_{i=!}^{2n} g_\lambda((\nabla_{U_i}\varphi)U_i, U) = \widehat{\delta}\widehat{\Omega}(U) - 2(n-1)\varphi U(\log\lambda)$, $U \in \mathcal{X}(F)$. Hence, if $n \geq 2$, one gets $\omega_\lambda(U) = \widehat{\omega}(U) - U(\log\lambda)$, $\omega_\lambda(\xi) = \frac{\delta\eta}{2n} = -\xi(\log\lambda)$. Finally, if $n = 1$, ii) and iii) give $\omega_\lambda = -\xi(\log\lambda)\eta$ and iv) follows. □

Remark 2.1. By Proposition 2.1 it follows that, if $\dim F \geq 4$, the Lee form of $I \times_\lambda F$ vanishes if and only if there exists a smooth positive function μ on F such that $\mu \circ \sigma = \lambda$ and $\widehat{\omega} = d(\log\mu)$. Furthermore, one easily obtains that the $\mathcal{C}_4$-component of the covariant derivative $\nabla\Phi_\lambda$ vanishes if and only if $\sigma^*(\widehat{\omega}) = d(\log\lambda) - \xi(\log\lambda)\eta$.

Proposition 2.2. *In the same hypothesis of Proposition 2.1, for any $i \in \{1, 2, 3\}$, the $\mathcal{C}_i$-component of $\nabla\Phi_\lambda$ vanishes if and only if the $\mathcal{W}_i$-component of $\widehat{\nabla}\widehat{\Omega}$ vanishes.*

Proof. Firstly, we point out that the statement holds if $\dim F = 2$. In fact, in this case, for any $i \in \{1, 2, 3\}$, the $\mathcal{W}_i$-component of $\widehat{\nabla}\widehat{\Omega}$ as well as the $\mathcal{C}_i$-component of $\nabla\Phi_\lambda$ vanish. Now, we assume that $\dim F = 2n \geq 4$ and we consider the $\mathcal{W}_i$-projection τ_i of $\widehat{\nabla}\widehat{\Omega}$ and the $\mathcal{C}_i$-projection $\widehat{\tau}_i$ of $\nabla\Phi_\lambda$. Let U, V, W be vector fields on F. Applying the theory developed in [5, 12] and Proposition 2.1 it is easy to obtain

$$\begin{aligned} \tau_4(U,V,W) &= -\omega_\lambda(\varphi W)g_\lambda(U,V) + \omega_\lambda(\varphi V)g_\lambda(U,W) \\ &\quad -\omega_\lambda(W)g_\lambda(U,\varphi V) + \omega_\lambda(V)g_\lambda(U,\varphi W) \\ &= \lambda^2\widehat{\tau}_4(U,V,W) + \varphi W(\log\lambda)g_\lambda(U,V) - \varphi V(\log\lambda)g_\lambda(U,W) \\ &\quad +W(\log\lambda)g_\lambda(U,\varphi V) - V(\log\lambda)g_\lambda(U,\varphi W) \\ \tau_i(U,V,W) &= 0, \quad i \in \{5, ..., 12\}. \end{aligned}$$

Furthermore by (2.4) we get

$$\begin{aligned} (\nabla_U\Phi_\lambda)(V,W) &= \lambda^2(\widehat{\nabla}_U\widehat{\Omega})(V,W) - \varphi V(\log\lambda)g_\lambda(U,W) - V(\log\lambda)g_\lambda(U,\varphi W) \\ &\quad +\varphi W(\log\lambda)g_\lambda(U,V) + W(\log\lambda)g_\lambda(U,\varphi V). \end{aligned}$$

This implies $\sum_{i=1}^3 \tau_i(U,V,W) = \lambda^2\sum_{i=1}^3 \widehat{\tau}_i(U,V,W)$, and $\tau_i(U,V,W) = \lambda^2\widehat{\tau}_i(U,V,W)$, $i \in \{1, 2, 3\}$. Then, the statement follows since for any $i \in \{1, 2, 3\}$ and X, Y tangent to $I \times_\lambda F$, one has $\tau_i(\xi, X, Y) = \tau_i(X, Y, \xi) = 0$. □

Proposition 2.3. *Given an a.c.m. manifold $(M, \varphi, \xi, \eta, g)$ with $\dim M = 2n+1$ the following conditions are equivalent*
i) *M is a $\mathcal{C}_{1-5}$-manifold,*
ii) $\nabla\eta = -\frac{1}{2n}\delta\eta(g - \eta\otimes\eta)$, $\quad \nabla_\xi\varphi = 0$,
iii) $\nabla\eta = -\frac{1}{2n}\delta\eta(g - \eta\otimes\eta)$, $\quad L_\xi\varphi = 0$,
L_ξ denoting the Lie derivative with respect to ξ.

Proof. In the hypothesis i) one puts $\nabla\Phi = \sum_{i=1}^5 \tau_i$ and applies the theory developed in [5] to evaluate the contribution of each projection τ_i in the calculus of $\nabla\eta$, $\nabla_\xi\varphi$. Since, for any $i \in \{1, ..., 5\}$ and X, Y tangent to M one has $\tau_i(\xi, X, Y) = 0$, we get $\nabla_\xi\varphi = 0$. Moreover, from the relations $\tau_i(X, \xi, Y) = 0$, $\quad c(\tau_i)(\xi) = 0$, $\quad i \in \{1, 2, 3, 4\}$ and $\tau_5(X, \xi, Y) = \frac{1}{2n}\overline{c}(\tau_5)(\xi)g(X, \varphi Y) = \frac{1}{2n}\delta\eta g(X, \varphi Y)$, $\quad c(\tau_5)(\xi) = 0$ one obtains $(\nabla_X\eta)Y = (\nabla_X\Phi)(\xi, \varphi Y) = -\frac{1}{2n}\delta\eta(g(X, Y) - \eta(X)\eta(Y))$ and ii) follows.
The equivalence ii) ⇔ iii) is an easy consequence of the relation
$(L_\xi\varphi)X = (\nabla_\xi\varphi)X - \nabla_{\varphi X}\xi + \varphi(\nabla_X\xi)$, $\quad X \in \mathcal{X}(M)$.

Finally, we assume ii) and write $\nabla\Phi = \sum_{i=1}^{12} \tau_i$. Considering X, Y tangent to M, by direct calculus we have $0 = (\nabla_\xi \Phi)(\varphi X, \varphi Y) = -\tau_{11}(\xi, X, Y)$. This implies $\tau_{11} = 0$. Since $\nabla_\xi \eta = 0$, we also have $\tau_{12} = 0$ and $(\nabla_X \Phi)(\xi, \varphi Y) = (\nabla_X \eta)Y = \tau_5(X, \xi, \varphi Y)$ entails $\sum_{i=6}^{10} \tau_i(X, \xi, \varphi Y) = 0$. In particular, this implies $c(\tau_6)(\xi) = 0$, so $\tau_6 = 0$. Hence, we get

$$(\tau_7 + \tau_8 + \tau_9 + \tau_{10})(X, \xi, \varphi Y) = 0, \quad X, Y \in \mathcal{X}(M).$$

Finally, the properties
$(\tau_7+\tau_8)(\varphi X, \xi, Y) + (\tau_7+\tau_8)(X, \xi, \varphi Y) = 0$, $(\tau_9+\tau_{10})(\varphi X, \xi, Y) = (\tau_9+\tau_{10})(X, \xi, \varphi Y)$,
$\tau_i(X, \xi, \varphi Y) = \tau_i(Y, \xi, \varphi X)$, $i \in \{8, 9\}$, $\tau_i(X, \xi, \varphi Y) = -\tau_i(Y, \xi, \varphi X)$, $i \in \{7, 10\}$,
imply the vanishing of τ_7, τ_8, τ_9, τ_{10}. □

We recall that, if M is a 5-dimensional a.c.m. manifold, the vector bundles $\mathcal{C}_1(M)$ and $\mathcal{C}_3(M)$ are trivial. Hence, in dimensions five, Proposition 2.3 gives a characterization of the class $\mathcal{C}_2 \oplus \mathcal{C}_4 \oplus \mathcal{C}_5$. In dimensions three the total class is $\mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{12}$, therefore the class $\mathcal{C}_{1-5}$ reduces to $\mathcal{C}_5$. More generally, in any dimensions, $2n + 1$, $\mathcal{C}_5$-manifolds are characterized by $(\nabla_X \varphi)Y = \frac{1}{2n}\delta\eta(\eta(Y)\varphi X + g(X, \varphi Y)\xi)$ and are called f-Kenmotsu manifolds ($f = -\frac{1}{2n}\delta\eta$). If $f = 1$, one obtains Kenmotsu manifolds ([15]). Moreover, in dimensions three, the relation $\nabla\eta = -\frac{1}{2}\delta\eta(g - \eta\otimes\eta)$ implies $\nabla_\xi \varphi = 0$ and by Proposition 2.3, we get the next result.

Corollary 2.4. *Let* $(M, \varphi, \xi, \eta, g)$ *be an a.c.m. manifold such that* $\dim M = 3$. *Then* M *is a* $\mathcal{C}_5$*-manifold if and only if* $\nabla\eta = -\frac{1}{2}(g - \eta \otimes \eta)$.

Now, we are able in specifying the class of twisted product manifolds.

Let $(F, \widehat{J}, \widehat{g})$ be a $2n$-dimensional manifold and $\lambda : I \times F \to \mathbb{R}$ a smooth positive function, $I \subset \mathbb{R}$ being an open interval. By Propositions 2.1, 2.3 and Corollary 2.4 it follows that $I \times_\lambda F$ is a $\mathcal{C}_5$-manifold if $n = 1$, a $\mathcal{C}_2 \oplus \mathcal{C}_4 \oplus \mathcal{C}_5$-manifold if $n = 2$, as well as $I \times_\lambda F$ belongs to the class $\mathcal{C}_{1-5}$ for any $n \geq 3$. Via Remark 2.1 and Proposition 2.2, under suitable restrictions on the class of $(F, \widehat{J}, \widehat{g})$, one can state that $I \times_\lambda F$ belongs to a particular subclass of $\mathcal{C}_{1-5}$. For instance, if $n \geq 2$ and $(\widehat{J}, \widehat{g})$ is a Kähler structure, then $I \times_\lambda F$ is a $\mathcal{C}_4 \oplus \mathcal{C}_5$-manifold. For any $i \in \{1, 2, 3\}$, $I \times_\lambda F$ belongs to the class $\mathcal{C}_i \oplus \mathcal{C}_4 \oplus \mathcal{C}_5$, provided that $(F, \widehat{J}, \widehat{g})$ is a $\mathcal{W}_i$-manifold.
Finally, we consider a warped product manifold $I \times_\lambda F$ and assume that the Lee form of F vanishes. Then, since $d\lambda = \xi(\lambda)\eta$, by Proposition 2.1 one has $\omega_\lambda = -\xi(\log\lambda)\eta$ and the $\mathcal{C}_4$-component of $\nabla\Phi_\lambda$ vanishes. It follows that, for any $i \in \{1, 2, 3\}$, $I \times_\lambda F$ is a $\mathcal{C}_i \oplus \mathcal{C}_5$-manifold, provided that $(F, \widehat{J}, \widehat{g})$ is a $\mathcal{W}_i$-manifold.

3 Local description of $\mathcal{C}_{1-5}$-manifolds

In this section we give a local description of $\mathcal{C}_{1-5}$-manifolds and a characterization of those manifolds which belong to the classes $\mathcal{C}_5$, $\mathcal{C}_h \oplus \mathcal{C}_5$, for any $h \in \{1, 2, 3, 4\}$.
Following ([6]), an isometry $f(M, \varphi, \xi, \eta, g) \to (M', \varphi', \xi', \eta', g')$ between a.c.m. manifolds is said to be an almost contact (a.c.) isometry if $f_* \circ \varphi = \varphi' \circ f_*$, $f_*\xi = \xi'$.

Theorem 3.1. *Let* $(M, \varphi, \xi, \eta, g)$ *be an a.c.m. manifold in the class* $\mathcal{C}_{1-5}$. *Then the distribution* D *associated with the subbundle* $\ker\eta$ *of* TM *is integrable and totally* *umbilic and the orthogonal distribution* $D^{\perp}$ *is totally geodesic. The manifold* M *is, locally, a.c. isometric to a twisted product manifold* $]-\varepsilon,\varepsilon[\times_\lambda F$, $\varepsilon>0$, F *being an a.H. manifold and* $\lambda:]-\varepsilon,\varepsilon[\times F\to\mathbb{R}$ *a smooth function,* $\lambda>0$. *Furthermore,* M *is, locally, a warped product if and only if* $d\omega(\xi)=\xi(\omega(\xi))\eta$, ω *denoting the Lee form.*

Proof. By Proposition 2.3 one has $\nabla\eta=-\omega(\xi)(g-\eta\otimes\eta)$, hence η is closed and $\nabla_\xi\xi=0$. It follows that D is integrable and $D^{\perp}$ is totally geodesic. Let N be a leaf of D, denote by g' the metric induced by g and put $J'=\varphi_{|TN}$. Then (N,J',g') is an a.H. manifold. Since for any $X\in\mathcal{X}(N)$ one has $\nabla_X\xi=-\omega(\xi)X$, (N,g') is an umbilic submanifold with mean curvature vector field $H=\omega(\xi)\xi_{|N}$. It follows that D is a totally umbilic foliation. Moreover, D is a spheric foliation, i.e. each leaf of D is an extrinsic sphere, if and only if $0=\nabla^{\perp}_X(\omega(\xi)\xi)=X(\omega(\xi))\xi$, for any section X of D. It follows that D is spheric if and only if $d\omega(\xi)=\xi(\omega(\xi))\eta$.
By Theorem 1 and Proposition 3 in [21], (M,g) is locally isometric to a twisted product. Hence, considering $p\in M$, there exist a (connected) open neighborhood U of p, $\varepsilon>0$, a Riemannian manifold $(F,\widehat{g})$, a smooth function $\lambda:]-\varepsilon,\varepsilon[\times F\to\mathbf{R}$, $\lambda>0$, and an isometry $f:]-\varepsilon,\varepsilon[\times_\lambda F\to U$ such that the canonical foliations of the product manifold $]-\varepsilon,\varepsilon[\times F$ correspond, via f, to the foliations D, $D^{\perp}$. Hence, we have $f^*(g_{|U})=dt\otimes dt+\lambda^2\widehat{g}$, $f_*(\frac{\partial}{\partial t})=\xi_{|U}$ and, for any $t\in]-\varepsilon,\varepsilon[$, $f_t(F)$ is an integral manifold of D, where $f_t=f(t,\cdot)$. So, one defines an almost complex structure $\widehat{J}$ on F which makes $(F,\widehat{J},\widehat{g})$ an a.H. manifold and proves that f realizes an a.c. isometry between the twisted product manifold $]-\varepsilon,\varepsilon[\times_\lambda F$ and $(U,\varphi_{|U},\xi_{|U},\eta_{|U},g_{|U})$. □

As remarked in Section 2, in dimensions three the class $\mathcal{C}_{1-5}$ reduces to $\mathcal{C}_5$. So, Theorem 3.1 entails that any $\mathcal{C}_5$-manifold (M,φ,ξ,η,g) is, locally, a.c. isometric to a twisted product $]-\varepsilon,\varepsilon[\times_\lambda F$, F being an a.H. manifold. Since $\dim F=2$, F is a Kähler manifold, as well as any leaf of D inherits from M a Kähler structure.
Considering $i\in\{1,2,3,4\}$, a $\mathcal{C}_{1-5}$-manifold M is said to be foliated by $\mathcal{W}_i$-leaves if each leaf $(N,g'=g_{|TN\times TN},J'=\varphi_{|TN})$ of D is in the Gray-Hervella class $\mathcal{W}_i$.
In order to characterize, in dimension $2n+1$, the $\mathcal{C}_{1-5}$-manifolds that are foliated by $\mathcal{W}_i$-leaves, we put our attention to the classes $\mathcal{C}_i\oplus\mathcal{C}_5$, for any $i\in\{1,2,3,4\}$, and list the defining conditions, that are easily obtained applying the theory developed in [5] and related results ([8, 9]).
$\mathcal{C}_1\oplus\mathcal{C}_5:\quad (\nabla_X\varphi)X=\frac{\delta\eta}{2n}\eta(X)\varphi X,\quad (\nabla_X\eta)Y=-\frac{\delta\eta}{2n}g(\varphi X,\varphi Y)$
$\mathcal{C}_2\oplus\mathcal{C}_5:\quad d\Phi=-\frac{\delta\eta}{n}\eta\wedge\Phi,\quad d\eta=0,\quad L_\xi\varphi=0$
$\mathcal{C}_3\oplus\mathcal{C}_5:\quad (\nabla_X\varphi)Y=(\nabla_{\varphi X}\varphi)\varphi Y+\frac{\delta\eta}{2n}\eta(Y)\varphi X,\quad \delta\Phi=0$
$\mathcal{C}_4\oplus\mathcal{C}_5:\quad (\nabla_X\varphi)Y=\omega(Y)\varphi X+\omega(\varphi Y)\varphi^2X+g(X,\varphi Y)B-g(\varphi X,\varphi Y)\varphi B,\quad B=\omega^\sharp.$
The class $\mathcal{C}_1\oplus\mathcal{C}_5$ contains nearly Kenmotsu manifolds, which are realized putting $\delta\eta=-2n$ in the defining condition. Putting $\delta\eta=-2n$ in the defining condition of $\mathcal{C}_2\oplus\mathcal{C}_5$ one obtains the almost Kenmotsu manifolds such that $L_\xi\varphi=0$. These manifolds are locally described in [7] and recently studied in different settings ([20]).

Proposition 3.2. *Let* (M,φ,ξ,η,g) *be a* $\mathcal{C}_{1-5}$*-manifold with* $\dim M=2n+1\geq 5$. *For any* $i\in\{1,2,3,4\}$ *the following conditions are equivalent*
i) M *is foliated by* $\mathcal{W}_i$*-leaves;*
ii) M *is a* $\mathcal{C}_i\oplus\mathcal{C}_5$*-manifold.*

Proof. Let (N,J',g') be a leaf of D and denote by ∇' its Levi-Civita connection. Since N is a totally umbilical submanifold of M with mean curvature vector field

$H = \frac{\delta\eta}{2n}\xi_{|N}$, for any X', $Y' \in \mathcal{X}(N)$ one has

$$(\nabla_{X'}\varphi)Y' = (\nabla'_{X'}J')Y' + g'(X', J'Y')H. \tag{3.1}$$

Hence, considering two vector fields X, Y such that $\varphi^2 X$,$\varphi^2 Y$ are tangent to N and writing $X = -\varphi^2 X + \eta(X)\xi$, $Y = -\varphi^2 Y + \eta(Y)\xi$, by polarization, (3.1) and Proposition 2.3 one obtains

$$(\nabla_X\varphi)Y = (\nabla'_{\varphi^2 X}J')\varphi^2 Y + \frac{\delta\eta}{2n}(\eta(Y)\varphi X + g(X, \varphi Y)\xi). \tag{3.2}$$

So, in each case, the equivalence i)$\Longleftrightarrow$ ii) is obtained by a routine calculus using Proposition 2.3, (3.1), (3.2) and the defining condition of $\mathcal{W}_i$-manifold ([12]). □

Corollary 3.3. *Let $(M, \varphi, \xi, \eta, g)$ be a $\mathcal{C}_{1-5}$-manifold. Then M is foliated by Kähler leaves if and only if M is a $\mathcal{C}_5$-manifold.*

Finally, we consider a $\mathcal{C}_{1-5}$-manifold $(M, \varphi, \xi, \eta, g)$ such that $\dim M = 2n+1 \geq 5$ and $\delta\eta = -2n$. Since $\omega(\xi) = -1$ is constant, M is, locally, a warped product manifold. More precisely, given $p \in M$, there exist an open neighborhood U of p, an a.H. manifold $(F, \widehat{J}, \widehat{g})$, a smooth positive function $\lambda :]-\varepsilon, \varepsilon[\to \mathbb{R}$ and an a.c. isometry $f :]-\varepsilon, \varepsilon[\times_\lambda F \to U$ such that $f^*(g_{|U}) = dt \otimes dt + \lambda^2 \widehat{g}$, $f_*(\frac{\partial}{\partial t}) = \xi_{|U}$. Then one has $f^*(\eta) = dt$ and, by Proposition 2.1, we obtain $-2n = \delta\eta \circ f = -2n\frac{d\log\lambda}{dt}$. It follows that λ acts as $\lambda(t) = Ce^t$, for some constant $C > 0$.
Clearly, given $i \in \{1, 2, 3\}$ and M in the class $\mathcal{C}_i \oplus \mathcal{C}_5$, then M is, locally, a warped product manifold $]-\varepsilon, \varepsilon[\times_\lambda F$ where F is a $\mathcal{W}_i$-manifold and $\lambda(t) = Ce^t$, $C > 0$.
Note that, in the case $i = 2$, we reobtain the local classification of almost Kenmotsu manifolds such that $L_\xi\varphi = 0$ ([7]).

4 Local description of generalized Sasakian-space-forms

In [1] the authors call generalized Sasakian-space-form (g.S. space-form), denoted $M(f_1, f_2, f_3)$, an a.c.m. manifold $(M, \varphi, \xi, \eta, g)$ which admits three smooth functions f_1, f_2, f_3 such that the curvature tensor R satisfies

$$R = f_1\pi_1 + f_2 S + f_3 T \tag{4.1}$$

π_1, S, T being the algebraic curvature tensor fields defined by
$\pi_1(X, Y, Z) = g(Y, Z)X - g(X, Z)Y$,
$S(X, Y, Z) = 2g(X, \varphi Y)\varphi Z + g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X$,
$T(X, Y, Z) = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi$.
In [11] we proved that g.S. space-forms are characterized as the $N(k)$-manifolds with pointwise constant (p.c.) φ-sectional curvature c admitting a smooth function l such that $R(X, Y, X, Y) - R(X, Y, \varphi X, \varphi Y) = l(\| X \|^2 \| Y \|^2 - g(X, Y)^2 - g(X, \varphi Y)^2)$, for any vector fields X,Y orthogonal to ξ. Moreover, the functions f_1, f_2, f_3, c, k, l are related by $f_1 = \frac{c+3l}{4}$, $f_2 = \frac{c-l}{4}$, $f_3 = \frac{c+3l}{4} - k$.

Now, we describe g.S. space-forms which fall in the class $\mathcal{C}_{1-5}$, stating two theorems in dimension $2n+1 \geq 7$. Firstly, we prove some preliminary results.

Proposition 4.1. *Let (M,φ,ξ,η,g) be a $\mathcal{C}_{1-5}$-manifold with Lee form ω and assume that $M(f_1,f_2,f_3)$ is a g.S. space-form. Then, the functions $k=f_1-f_3$ and $\omega(\xi)$ are constant on each leaf of D and are related by $k+\omega(\xi)^2=\xi(\omega(\xi))$.*

Proof. By direct calculus, applying Proposition 2.3, one has

$$R(X,Y,\xi)=Y(\omega(\xi))(X-\eta(X)\xi)-X(\omega(\xi))(Y-\eta(Y)\xi)-\omega(\xi)^2(\eta(Y)X-\eta(X)Y),$$

and comparing with the $N(k)$-condition, $R(X,Y,\xi)=k(\eta(Y)X-\eta(X)Y)$, one gets

$$(k+\omega(\xi)^2)(\eta(Y)X-\eta(X)Y)=Y(\omega(\xi))(X-\eta(X)\xi)-X(\omega(\xi))(Y-\eta(Y)\xi). \tag{4.2}$$

Hence, for two orthogonal sections X, Y of D, one has $Y(\omega(\xi))X-X(\omega(\xi))Y=0$ and this implies the constancy of the function $\omega(\xi)$ on each leaf of D. Putting $X=\xi$ in (4.2), for any section Y of D we have $(k+\omega(\xi)^2)Y=\xi(\omega(\xi))Y$. Hence, we get $d\omega(\xi)=\xi(\omega(\xi))\eta=(k+\omega(\xi)^2)\eta$. Differentiating, since $d\eta=0$, one obtains $0=dk\wedge\eta+2\omega(\xi)d\omega(\xi)\wedge\eta=dk\wedge\eta$ and the constancy of k on the leaves of D follows. □

Let $M(f_1,f_2,f_3)$ be a manifold as in Proposition 4.1. By Theorem 3.1, M is, locally, a warped product manifold $]-\varepsilon,\varepsilon[\times_\lambda F$, $(F,\widehat{J},\widehat{g})$ being an a.H. manifold and $\lambda:]-\varepsilon,\varepsilon[\to\mathbb{R}$ a positive smooth function. Let $f:]-\varepsilon,\varepsilon[\times_\lambda F\to U$ be an a.c. isometry and evaluate the curvature $\widehat{R}$ of F. So, considering $t\in]-\varepsilon,\varepsilon[$, for any $x\in F$, $X,Y,Z\in T_xF$, we have

$$\begin{aligned}\widehat{R}_x(X,Y,Z) &= (\lambda(t)^2(f_1\circ f)(t,x)-\lambda'(t)^2)(\widehat{g}_x(Y,Z)X-\widehat{g}_x(X,Z)Y)\\ &+\lambda(t)^2(f_2\circ f)(t,x)(2\widehat{g}_x(X,\widehat{J}Y)\widehat{J}Z+\widehat{g}_x(X,\widehat{J}Z)\widehat{J}Y-\widehat{g}_x(Y,\widehat{J}Z)\widehat{J}X).\end{aligned}$$

It follows that $(F,\widehat{J},\widehat{g})$ is a generalized complex space-form ([22]). Therefore, applying the results stated in [22, 18], under suitable restrictions on the dimension, one classifies the a.H. structure on F. Anyway, to get all the possible information on the a.c.m. structure on M, we apply the second Bianchi identity, starting by (4.1).

Considering vector fields U,X,Y,Z on M, by Proposition 2.3, one has

$$\begin{aligned}(\nabla_U S)(X,Y,Z) &= 2g(X,(\nabla_U\varphi)Y)\varphi Z+2g(X,\varphi Y)(\nabla_U\varphi)Z\\ &+g(X,(\nabla_U\varphi)Z)\varphi Y+g(X,\varphi Z)(\nabla_U\varphi)Y\\ &-g(Y,(\nabla_U\varphi)Z)\varphi X-g(Y,\varphi Z)(\nabla_U\varphi)X.\end{aligned} \tag{4.3}$$

$$\begin{aligned}(\nabla_U T)(X,Y,Z) &= -\omega(\xi)\eta(Z)(g(\varphi U,\varphi X)Y-g(\varphi U,\varphi Y)X)\\ &-\omega(\xi)g(\varphi U,\varphi Z)(\eta(X)Y-\eta(Y)X)+\omega(\xi)(g(X,Z)\eta(Y)\\ &-g(Y,Z)\eta(X))\varphi^2U-\omega(\xi)(g(X,Z)g(\varphi U,\varphi Y)\\ &-g(Y,Z)g(\varphi U,\varphi X))\xi.\end{aligned} \tag{4.4}$$

Lemma 4.2. *Let $M(f_1,f_2,f_3)$ be a g.S. space-form, with $\dim M=2n+1\geq 5$ and Lee form ω. Assume that M is a $\mathcal{C}_{1-5}$-manifold. Then, for any unit section X of D, one has*
i) $X(f_1)=-X(f_2)=-3f_2\omega(X)$,
ii) $f_2(\omega(X)+g((\nabla_Y\varphi)Y,\varphi X))=0$, *$Y$ unit section of D orthogonal to X, φX.*

Proof. Let U, X, Y, Z be sections of D. Applying the second Bianchi identity, (4.1), (4.3) and (4.4), one has

$$
\begin{aligned}
0 = \;& U(f_1)\pi_1(X,Y,Z) + U(f_2)S(X,Y,Z) + X(f_1)\pi_1(Y,U,Z) \\
&+X(f_2)S(Y,U,Z) + Y(f_1)\pi_1(U,X,Z) + Y(f_2)S(U,X,Z) \\
&+f_2\{2(g(X,(\nabla_U\varphi)Y) + g(Y,(\nabla_X\varphi)U) \\
&+g(U,(\nabla_Y\varphi)X))\varphi Z + 2(g(X,\varphi Y)(\nabla_U\varphi)Z \\
&+g(Y,\varphi U)(\nabla_X\varphi)Z + g(U,\varphi X)(\nabla_Y\varphi)Z) \\
&+(g(X,(\nabla_U\varphi)Z) - g(U,(\nabla_X\varphi)Z))\varphi Y \\
&+(g(Y,(\nabla_X\varphi)Z) - g(X,(\nabla_Y\varphi)Z))\varphi U + (g(U,(\nabla_Y\varphi)Z) \\
&-g(Y,(\nabla_U\varphi)Z))\varphi X + g(X,\varphi Z)((\nabla_U\varphi)Y - (\nabla_Y\varphi)U) \\
&+g(Y,\varphi Z)((\nabla_X\varphi)U - (\nabla_U\varphi)X) \\
&+g(U,\varphi Z)((\nabla_Y\varphi)X - (\nabla_X\varphi)Y)\}.
\end{aligned} \tag{4.5}
$$

We choose unit vector fields X and Y orthogonal to X,φX. Putting $Z = X$, $U = \varphi Y$ in (4.5) one obtains

$$
\begin{aligned}
&\varphi Y(f_1)Y + 2X(f_2)\varphi X - Y(f_1)\varphi Y - f_2(3g(X,(\nabla_{\varphi Y}\varphi)Y - (\nabla_Y\varphi)\varphi Y)\varphi X \\
&-2(\nabla_X\varphi)X - g(\varphi Y,(\nabla_X\varphi)X)\varphi Y - g(Y,(\nabla_X\varphi)X)Y) = 0.
\end{aligned}
$$

Taking the scalar product by φY and φX we have

$$Y(f_1) - 3f_2 g(\varphi Y,(\nabla_X\varphi)X) = 0 \tag{4.6}$$

$$2X(f_2) - 3f_2 g(X,(\nabla_{\varphi Y}\varphi)Y - (\nabla_Y\varphi)\varphi Y) = 0. \tag{4.7}$$

These relations imply $X(f_1+f_2) = 0$, for any unit section X of D. Let Y be a unit section of D and $\{e_1,...,e_n,\varphi e_1,...,\varphi e_n,\xi\}$ a local orthonormal frame with $e_1 = Y$. By (4.6) one has

$$
\begin{aligned}
2(n-1)Y(f_1) - 3f_2\delta\Phi(\varphi Y) &= 2(n-1)Y(f_1) - 3f_2\textstyle\sum_{i=2}^n(g((\nabla_{e_i}\varphi)e_i,\varphi Y) \\
&\quad+g((\nabla_{\varphi e_i}\varphi)\varphi e_i,\varphi Y)) = 0,
\end{aligned}
$$

so $3f_2\omega(Y) = -\frac{3}{2(n-1)}f_2\delta\Phi(\varphi Y) = -Y(f_1)$, hence i) and ii) follow. □

Proposition 4.3. *Let $M(f_1,f_2,f_3)$ be a g.S. space-form as in Lemma 4.2. If $n \geq 3$, the following properties hold*
i) *the functions f_1, f_2 are constant on each leaf of D,*
ii) $f_2(\omega - \omega(\xi)\eta) = 0$,
iii) *For any vector fields X,Y one has* $f_2((\nabla_X\varphi)Y - \omega(\xi)(\eta(Y)\varphi X + g(X,\varphi Y)\xi)) = 0$.

Proof. Let U,Y be sections of D and $\{e_1,...,e_{2n},\xi\}$ a local orthonormal frame. We put $Z = X = e_i$ in (4.5) and sum over $i \in \{1,...,2n\}$. Applying Lemma 4.2 and Proposition 2.3, one has

$$
\begin{aligned}
0 \;&= (2n-5)(Y(f_1)U - U(f_1)Y) + \varphi Y(f_1)\varphi U - \varphi U(f_1)\varphi Y \\
&-2g(Y,\varphi U)\textstyle\sum_{i=1}^{2n} e_i(f_1)\varphi e_i + f_2\{2\sum_{i=1}^{2n} g(Y,(\nabla_{e_i}\varphi)U)\varphi e_i \\
&+2g(Y,\varphi U)\textstyle\sum_{i=1}^{2n}(\nabla_{e_i}\varphi)e_i + (\nabla_{\varphi U}\varphi)Y - (\nabla_{\varphi Y}\varphi)U \\
&-\delta\Phi(U)\varphi Y + \delta\Phi(Y)\varphi U\}.
\end{aligned} \tag{4.8}
$$

We assume that $\| Y \|= 1$, $g(Y,U) = g(Y,\varphi U) = 0$, take in (4.8) the scalar product by φY and obtain

$$\varphi U(f_1) + f_2(2g((\nabla_Y\varphi)Y,U) - g((\nabla_{\varphi Y}\varphi)\varphi Y,U) + \delta\Phi(U)) = 0.$$

Applying Lemma 4.2, for any section U of D we have $(n-2)f_2\omega(U) = 0$ and ii) follows. So, also applying Lemma 4.2, we obtain i). Considering three sections U, Y, Z of D, by (4.8), i) and ii) we get

$$f_2(-2g(Y,(\nabla_{\varphi Z}\varphi)U) + g((\nabla_{\varphi U}\varphi)Y,Z) - g((\nabla_{\varphi Y}\varphi)U,Z)) = 0.$$

This also implies

$$\begin{aligned}0 = f_2(-2g(Y,(\nabla_{\varphi Z}\varphi)U) + 2g(U,(\nabla_{\varphi Y}\varphi)Z) + g((\nabla_{\varphi U}\varphi)Y,Z) - g((\nabla_{\varphi Z}\varphi)U,Y)\\ -g((\nabla_{\varphi Y}\varphi)U,Z) + g((\nabla_{\varphi U}\varphi)Z,Y)) = -3f_2g((\nabla_{\varphi Z}\varphi)\varphi Y + (\nabla_{\varphi Y}\varphi)\varphi Z,\varphi U).\end{aligned}$$

Hence, for any sections X,Y,Z of D we have $f_2g((\nabla_X\varphi)Y + (\nabla_Y\varphi)X,Z) = 0$.
Let $\{e_1,...,e_{2n},\xi\}$ be a local orthonormal frame. For any $i \in \{1,...,2n\}$ we put $Y = e_i$ in (4.5), take the scalar product with φe_i and sum the obtained expressions. Since f_1 and f_2 are constant on the leaves of D, using the last formula, for any sections X, U, Z of D, we have $f_2g((\nabla_X\varphi)U,Z) = 0$. Hence, also applying Proposition 2.3, for any sections X, U of D, one obtains

$$f_2(\nabla_X\varphi)U = -f_2(\nabla_X\eta)\varphi U\xi = f_2\omega(\xi)g(X,\varphi U)\xi.$$

Finally, considering X, $Y \in \mathcal{X}(M)$, one writes $X = -\varphi^2X + \eta(X)\xi$, $Y = -\varphi^2Y + \eta(Y)\xi$, applies polarization, Proposition 2.3 and the above formula and gets iii). □

Lemma 4.4. *Let $M(f_1,f_2,f_3)$ be a g.S. space-form as in Lemma 4.2. If $\dim M \geq 7$, one has $df_1 = 2f_3\omega(\xi)\eta$, $df_2 = 2f_2\omega(\xi)\eta$, $df_3 = \xi(f_3)\eta$.*

Proof. Let Z be a vector field on M and X,Y sections of D. One applies

$$(\nabla_\xi R)(X,Y,Z) + (\nabla_X R)(Y,\xi,Z) + (\nabla_Y R)(\xi,X,Z) = 0,$$

(4.1), (4.3), (4.4), Proposition 4.3 and

$$(\nabla_X S)(Y,\xi,Z) - (\nabla_Y S)(X,\xi,Z) = -2\omega(\xi)S(X,Y,Z),$$

$$(\nabla_X T)(Y,\xi,Z) - (\nabla_Y T)(X,\xi,Z) = -2\omega(\xi)\pi_1(X,Y,Z).$$

Then, we obtain

$$\begin{aligned}(\xi(f_1) - 2f_3\omega(\xi))\pi_1(X,Y,Z) + (\xi(f_2) - 2f_2\omega(\xi))S(X,Y,Z)\\ +X(f_3)T(Y,\xi,Z) - Y(f_3)T(X,\xi,Z) = 0.\end{aligned} \tag{4.9}$$

Putting $Z = \xi$ in (4.9) we have $X(f_3)Y - Y(f_3)X = 0$. It follows that f_3 is constant on any leaf of D and $df_3 = \xi(f_3)\eta$. Furthermore, (4.9) reduces to

$$(\xi(f_1) - 2f_3\omega(\xi))\pi_1(X,Y,Z) + (\xi(f_2) - 2f_2\omega(\xi))S(X,Y,Z) = 0.$$

This implies $\xi(f_1) = 2f_3\omega(\xi)$, $\xi(f_2) = 2f_2\omega(\xi)$ and by Proposition 4.3 the proof is completed. □

Theorem 4.5. *Let $(M, \varphi, \xi, \eta, g)$ be a $\mathcal{C}_{1-5}$-manifold such that $\dim M \geq 7$. Assume that $M(f_1, f_2, f_3)$ is a g.S. space-form. If f_2 never vanishes, then*
i) M is a $\mathcal{C}_5$-manifold and admits a cosymplectic structure with constant φ-sectional curvature $sign(f_2)$,
ii) $(M, \varphi, \xi, \eta, g)$ is, locally, a.c. isometric to a warped product $]-\varepsilon, \varepsilon[\times_\lambda F$, where $\varepsilon > 0$, $\lambda > 0$ is a smooth function and F is a Kähler manifold with non-zero constant holomorphic sectional curvature.

Proof. By Proposition 4.3 and Lemma 4.4 we have

$$\omega = \omega(\xi)\eta,\ df_2 = 2f_2\omega,\ (\nabla_X\varphi)Y = \omega(\xi)(\eta(Y)\varphi X + g(X, \varphi Y)\xi),\ X, Y \in \mathcal{X}(M).$$

Hence M is a $\mathcal{C}_5$-manifold with exact Lee form $\omega = d\log|f_2|^{\frac{1}{2}}$. It follows that the a.c.m. structure $(\varphi, |f_2|^{-\frac{1}{2}}\xi, |f_2|^{\frac{1}{2}}\eta, |f_2|g)$ on M is cosymplectic and has constant φ-sectional curvature $\frac{f_2}{|f_2|} = sign f_2$ ([10]). Moreover, M is foliated by Kähler leaves and one easily proves that each leaf (N, J', g') of D has constant holomorphic sectional curvature $c' = 4f_{2|N}$. By Theorem 3.1, M is, locally, a warped product manifold $]-\varepsilon, \varepsilon[\times_\lambda F$, where F is biholomorphic to a leaf of D. Hence F is a Kähler manifold with non-zero constant holomorphic sectional curvature. □

Finally, we describe the conformally flat g.S. space-forms in $\mathcal{C}_{1-5}$.
As stated by Kim, in dimensions $2n+1 \geq 5$, the conformal flatness of a g.S. space-form $M(f_1, f_2, f_3)$ is equivalent to $f_2 = 0$. These spaces are described in [16], under the hypothesis that the Reeb vector field is Killing. Note that, if M is a $\mathcal{C}_{1-5}$-manifold, we have $(L_\xi g)(X, Y) = -\frac{1}{n}\delta\eta g(\varphi X, \varphi Y)$. Hence ξ is Killing if and only if $\delta\eta = 0$. It follows that the result in [16] cannot be directly applied. Examples of g.S. space-forms in the class $\mathcal{C}_{1-5}$ can be constructed. For instance, as in [16], given $\widehat{c} > 0$, one considers the nearly Kähler manifold $(S^6, \widehat{J}, \widehat{g})$, $\widehat{g}$ denoting the metric of constant curvature $\widehat{c}$. Given a smooth, non constant, positive function $\lambda : \mathbb{R} \to \mathbb{R}$, the warped product manifold $\mathbb{R} \times_\lambda S^6$ belongs to $\mathcal{C}_1 \oplus \mathcal{C}_5$ and is a g.S. space-form with functions $f_1 = \frac{\widehat{c} - \lambda'^2}{\lambda^2}$, $f_2 = 0$, $f_3 = \frac{\widehat{c} - \lambda'^2}{\lambda^2} + \frac{\lambda''}{\lambda}$.

Theorem 4.6. *Let $(M, \varphi, \xi, \eta, g)$ be a $\mathcal{C}_{1-5}$-manifold with $\dim M \geq 7$ and Lee form ω. Assume that M is a conformally flat g.S. space-form with p.c. φ-sectional curvature c. Then, one of the cases occurs*
i) $c = -\omega(\xi)^2$ and M is, locally, a warped product $]-\varepsilon, \varepsilon[\times_\lambda F$, where $\varepsilon > 0$, $\lambda > 0$ is a smooth function and F is a flat a.H. manifold,
ii) $c + \omega(\xi)^2$ is a non-zero constant. Then, $\omega(\xi) = 0$ and M is, locally, a Riemannian product $]-\varepsilon, \varepsilon[\times F$, where $\varepsilon > 0$ and F is an a.H. manifold with non-zero constant sectional curvature,
iii) $c + \omega(\xi)^2$ is non-constant and never vanishes. Then M is, locally, a warped product $]-\varepsilon, \varepsilon[\times_\lambda F$, $\lambda > 0$ being a smooth function and F an a.H. manifold with non-zero constant sectional curvature.

Proof. Since M is conformally flat, we have $f_2 = 0$, $c = f_1$, $dc = 2f_3\omega(\xi)\eta$ and M is an $N(k)$-manifold such that $c - f_3 = k = \xi(\omega(\xi)) - \omega(\xi)^2$. These relations imply $d(c + \omega(\xi)^2) = 2\omega(\xi)(f_3 + \xi(\omega(\xi)))\eta$. Hence, we have

$$d(c + \omega(\xi)^2) = 2(c + \omega(\xi)^2)\omega(\xi)\eta. \tag{4.10}$$

Note that $\omega(\xi)\eta$ is closed, $\omega(\xi)$ being constant on the leaves of D and η closed. Therefore, locally, $\omega(\xi)\eta$ can be expressed as $-\frac{1}{2}d(\log\tau)$, for some positive function τ. Then, (4.10) implies the existence of a real number a such that $\frac{a}{\tau} = c + \omega(\xi)^2$. Together with the connectedness of M this means that either $c + \omega(\xi)^2 = 0$ or $c + \omega(\xi)^2 \neq 0$. Furthermore, any leaf (N, J', g') of D has constant sectional curvature $c' = (c + \omega(\xi)^2)_{|N}$.

Now, we discuss the cases a) $c + \omega(\xi)^2 = 0$, b) $c + \omega(\xi)^2 \neq 0$.
In a) M is, locally, a.c. isometric to a warped product manifold $]-\varepsilon, \varepsilon[\, \times_\lambda F$, where F is a flat a.H. manifold. In fact, F is biholomorphic to a leaf of D.
In b), if $c+\omega(\xi)^2$ is constant, by (4.10) we have $\omega(\xi) = 0$. It follows that any leaf of D is a totally geodesic submanifold of M and has constant sectional curvature $c \neq 0$. So, both the distributions D and $D^\perp$ are totally geodesic and ii) is realized. If $c + \omega(\xi)^2$ is non-constant, we obtain iii), applying Theorem 3.1, also. □

Acknowledgments. The author thanks Professor Anna Maria Pastore for the valuable remarks and comments on the subject.

References

[1] P. Alegre, D.E. Blair and A. Carriazo, *Generalized Sasakian-space-forms*, Israel J. Math. 141 (2004), 157-183.

[2] P. Alegre and A. Carriazo, *Structures on generalized Sasakian-space-forms*, Differential Geom. Appl. 26 (2008), 656-666.

[3] A. Besse, *Einstein Manifolds*, Ergeb. Math. Grenzgeb. (3) 10, Springer-Verlag, Berlin, 1987.

[4] R. L. Bishop, *Clairaut submersions*, in *Geometry in Honor of K. Yano*, *Kinokuniya*, Tokio (1972), 21-31.

[5] D. Chinea and C. Gonzales, *A classification of almost contact metric manifolds*, Ann. Mat. Pura Appl. (4) 156 (1990), 15-36.

[6] D. Chinea, J.C. Marrero and J. Rocha, *Almost contact submersions with total space a locally conformal cosymplectic manifold*, Ann. Fac. Sc. Toulouse, (6), Math. 4 (1995), 473-517.

[7] G. Dileo and A.M. Pastore, *Almost Kenmotsu manifolds and local symmetry*, Bull. Belg. Math. Soc. Simon Stevin 14 (2007), 343-354.

[8] M. Falcitelli, *Some classes of almost contact metric manifolds and contact Riemannian submersions*, Acta Math. Hungar. 105 (2004), 291-312.

[9] M. Falcitelli, *A wide class of almost contact metric manifolds and contact Riemannian submersions*, Rev. Roumaine Math. Pures Appl. 51 (2006), 181-203.

[10] M. Falcitelli, *A class of almost contact metric manifolds with pointwise constant φ-sectional curvature*, Math. Balkanica 22 (2008), 133-154.

[11] M. Falcitelli, *Locally conformal C_6-manifolds and Generalized Sasakian Space-Forms*, Mediterr. J. Math. 7 (2010), 19-36.

[12] A. Gray and L.M. Hervella, *The Sixteen Classes of Almost Hermitian Manifolds and their linear invariants*, Ann. Mat. Pura Appl. (4) 123 (1980), 35-58.

[13] Hiepko, *Eine innere Kennzeichnung der verzerrten Produkte*, Math. Ann. 241 (1979), 209-215.

[14] D. Janssens and L. Vanhecke, *Almost contact structures and curvature tensors*, Kodai Math. J. 4 (1981), 1-27.

[15] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tôhoku Math. J. 24 (1972), 93-103.

[16] U.K. Kim, *Conformally flat generalized Sasakian-space-forms and locally symmetric generalized Sasakian-space-forms*, Note Mat. 26 (2006), 55-67.

[17] T.W.Kim and H.K. Pak, *Canonical foliations of certain classes of almost contact metric structures*, Acta Math. Sin. (Engl. Ser.) 21 (2005), 841-846.

[18] Z. Olszak, *On the existence of generalized complex space forms*, Israel J. Math. 65 (1989), 214-218.

[19] Z. Olszak and R. Rosca, *Normal locally conformal almost cosymplectic manifolds*, Publ. Math. Debrecen 39 (1991), 315-323.

[20] A.M. Pastore and V. Saltarelli, *Almost Kenmotsu manifolds with conformal Reeb foliation*, Bull. Belg. Math. Soc. Simon Stevin 18 (2011), 655-666.

[21] R. Ponge and H. Reckziegel, *Twisted products in pseudo-Riemannian Geometry*, Geometriae Dedicata 48 (1993), 15-25.

[22] F. Tricerri and L. Vanhecke, *Curvature tensors on almost Hermitian manifolds*, Trans. Am. Math. Soc. 267 (1981), 365-398.

Author's address:

Maria Falcitelli
Università degli Studi di Bari,
Dipartimento di Matematica,
Via E. Orabona 4, 70125 Bari, Italy.
E-mail: falci@dm.uniba.it