Sturm-Liouville operator controlled by sectional curvature on Riemannian manifolds

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Abstract. The purpose of this paper is fourfold: (1) to introduce and study a second order PDE, determined accidentally by a Riemann wave, reflecting the connection between oriented parallelograms area and sectional curvature on Riemannian manifolds; (2) to introduce and study the asymptotic behavior of oriented parallelograms area controlled by the sectional curvature; (3) to study some partial differential inequalities describing the evolution of parallelogram area on pinched manifolds; (4) to find controlled minimum of total sectional curvature. This means to control some geometric quantities associated to a Riemannian metric as it evolves with respect to a parameter via a geometric PDE (partial differential equation) or PDI (partial differential inequality). This approach and our PDEs/PDIs on Riemannian manifolds inaugurate new understandings of certain interrelationships among fundamental geometrical concepts.

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1 Introduction

Let M be a smooth closed (compact and without boundary) manifold of dimension n. Recall that a Riemannian metric on M is a choice $g = (g_{ij})$ of inner product on each tangent space which varies smoothly from point to point. Any manifold admits an infinite dimensional family of Riemannian metrics, but the question of whether a manifold admits metrics with desired geometric properties is one of the basic questions of global Riemannian geometry.

The Riemannian metric $g = (g_{ij})$ defines the *bialternate product Riemannian metric*

$$G = g \odot g,\ G_{ijkl} = g_{ik}g_{jl} - g_{il}g_{jk},\ i,j,k,l = \overline{1,n}$$

and the curvature tensor $R = (R_{ijkl})$. Starting from the Riemann wave approach to area of parallelograms, here we study a dynamic behavior of oriented area controlled

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by the sectional curvature. This approach was introduced as an open problem in seminal work of the first author in the 2012s (see e.g. [6], [11], [15]). The first fundamental idea is to start with a given Riemannian manifold (M, g_0), and evolve the metric by the evolution equation

$$\frac{\partial^2 G}{\partial t^2}(x,t) = -2\,R(g(x,t)),\ g(x,0) = g_0(x),\ \frac{\partial g}{\partial t}(x,0) = h(x),$$

where R denotes the Riemann tensor of the time-dependent metric $g(x,t)$. The solution $g(x,t)$ of this PDE is called *Riemann wave*. The second step is to build a Sturm-Liouville operator controlled by sectional curvature and to analyze the Kerr of this operator (see also, [3]-[7], [21], [22]).

The objectives and targets of this work are: (1) to find dynamic properties of the PDE relating oriented parallelogram area and sectional curvature on Riemannian manifolds, (2) to introduce and study special PDIs on pinched Riemannian manifolds. They could be attained only melting PDEs Theory into Differential Geometry (see also, [8]-[20]).

Section 2 studies the Riemann waves on constant curvature manifolds. Section 3 finds fundamental properties of a dynamic PDE determined by a Riemann wave and sectional curvature. This PDE suggests a Sturm-Liouville operator controlled by sectional curvature, whence derives our theory. Section 4 analyzes the asymptotic behavior of oriented parallelogram area controlled by sectional curvature. Section 5 look for oscillatory behavior of oriented parallelogram area controlled by the sectional curvature. Section 6 shows that the minimum of total sectional curvature, constrained by a an appropriate evolution PDE, is obtained by a bang-bang procedure.

2 Riemann waves on constant curvature manifolds

We take a metric g_0 such that $\mathrm{Riem}(g_0) = \lambda G_0$ for some constant $\lambda \in R$ (these metrics are known as constant curvature metrics). Then a solution $g(t)$ of PDE (2) with $g(0) = g_0$ is of the form $g(t) = f(t)\, g_0,\ f(t) > 0, f(0) = 1, f'(0) = v$, and hence $G(t)$ is of the form $G(t) = f^2(t)G_0$, if and only if [15]

$$f'^2(t) + f(t)f''(t) + \lambda f(t) = 0,\ f(0) = 1, f'(0) = v.$$

If $\lambda < 0$, then the solution is the polynom $f(t) = 1 + vt - \frac{\lambda}{6}\,t^2$. In particular, let g_0 be a hyperbolic metric, that is, a metric of constant sectional curvature -1. In this case, $n \geq 2$, $\mathrm{Riem}(g_0) = -(n-1)G_0$, the evolution metric is $g(t) = (1 + vt - \frac{\lambda}{6}\,t^2)\,g_0$ and the manifold expands homothetically for all time.

If $\lambda > 0$, then there exists T depending on v such that the solution $f(t)$, $t \in [0,T)$ is a concave function with $\lim_{t\to T} f(t) = 0$. In particular, for the round unit sphere (S^n, g_0), $n \geq 2$, we have $\mathrm{Riem}(g_0) = (n-1)G_0$, so the evolution metric is $g(t) = f(t)\,g_0$ and the sphere collapses to a point when $t \to T$.

If the initial metric $g(x,0)$ is Riemann flat, i.e., $\mathrm{Riem}(g(x,0)) = 0$, then $g(x,t) = g(x,0)$ is obviously a solution of the evolution PDE (2). Consequently, each Riemann flat metric $g(x)$ is a steady solution of the wave ultrahyperbolic PDEs system. The most general solution of this type is $g(x,t) = \sqrt{1+2vt}\,g(x,0)$ since the function $f(t) = \sqrt{1+2vt}$ is solution for the foregoing Cauchy problem with $\lambda = 0$.

3 Dynamic PDE determined by a Riemann wave and sectional curvature

Now let us introduce a second order PDE, determined by a Riemann wave, reflecting the connection between area of parallelograms and sectional curvature on Riemannian manifolds. To analyze the dynamic behavior of parallelogram area controlled by sectional curvature, let M be a smooth closed (compact and without boundary) manifold.

Let $R(g(x,t))$ be the Riemann curvature tensor associated to the evolution metric $g(x,t)$. The Riemann wave $g(x,t)$ is a solution of the PDE

$$\frac{\partial^2 G}{\partial t^2}(x,t) = -2\, R(g(x,t)).$$

Given two local linearly independent vector fields X and Y, the *sectional curvature* is defined by

$$K(X,Y) = \frac{R(X,Y,X,Y)}{G(X,Y,X,Y)}\,.$$

The sectional curvature is a further, equivalent but more geometrical, description of the curvature of Riemannian manifolds. It is a continuous real-valued function on M and a smooth real-valued function on the 2-Grassmannian bundle over the manifold.

Supposing $g(x,t)$ is a Riemann wave and introducing the area

$$\sigma(x,t) = G(x,t)(X(x),Y(x),X(x),Y(x)) > 0,$$

of a parallelogram $X(x) \wedge Y(x)$, we obtain the linear second order partial differential equation

$$(PDE) \qquad \sigma_{t^2}(x,t) + 2K(x,t)\sigma(x,t) = 0.$$

This PDE is just the Euler-Lagrange equation of the *functional*

$$\int_a^b \left(\frac{1}{2}\sigma_t^2(x,t) - K(x,t)\sigma^2(x,t)\right) dt.$$

If the sectional curvature is positive, i.e., $K(x,t) \geq 0$, then the function $t \to \sigma(x,t)$ is concave. If the sectional curvature is negative, i.e., $K(x,t) \leq 0$, then the function $t \to \sigma(x,t)$ is convex.

Suppose $0 < a(x) \leq K(x,t) \leq b(x)$, for $t \geq 0$. Let $\sigma(x,t) > 0$ be a solution of the PDE, with $\sigma_t(x,t) > 0$, for $t \geq 0$. The double ineguality is changed into

$$-2b(x)\sigma(x,t)\sigma_t(x,t) \leq \sigma_t^2(x,t)\sigma_t(x,t) \leq -2a(x)\sigma(x,t)\sigma_t(x,t).$$

Integrating on the interval $[0,t]$, and denoting

$$\alpha^2 = \sigma^2(x,0) + \frac{\sigma_t^2(x,0)}{2b(x)},\ \beta^2 = 2b(x)\alpha^2$$

$$\gamma^2 = \sigma^2(x,0) + \frac{\sigma_t^2(x,0)}{2a(x)},\ \delta^2 = 2a(x)\alpha^2,$$

we obtain a domain in the phase space

$$\frac{\sigma^2(x,t)}{\alpha^2} + \frac{\sigma_t^2(x,t)}{\beta^2} \geq 1, \ \frac{\sigma^2(x,t)}{\gamma^2} + \frac{\sigma_t^2(x,t)}{\delta^2} \leq 1.$$

Since $\alpha^2 \leq \gamma^2$ and $\beta^2 \geq \delta^2$, with equlity only if $a(x) = b(x)$, our domain is the part from the first quadrant situated in the exterior of the first ellipse and in the interior of the second one. If $a(x) = 0$, then instead of the second ellipse we have an horizontal band $|\sigma(x,t)| \leq \sigma_t(x,0)$.

If the sectional curvature is constant, i.e., $K(x,t) = a(x) = b(x) = c$, then we have an *energy first integral*

$$\frac{1}{2}\,\sigma_t^2(x,t) + c\,\sigma^2(x,t) = k_1(x).$$

Geometrically, the orbits $\{(\sigma(t), \sigma_t(t)) \,:\, t \in \mathbb{R}_+\}$ are either semi-hyperbolas (for $c < 0$) or semi-ellipses for ($c > 0$). In this case, the positivennes of σ_t is not necessary.

To understand the geometric and dynamic role of (PDE) and to justify our next results, we recall three Theorems regarding pinched (positive or negative) curvature (see, [1], [2], [4])

Definition 3.1. A Riemannian manifold (M, g) is said to be *weakly δ-pinched* in the global sense if the sectional curvature K of (M, g) satisfies $0 < \delta \leq K \leq 1$. If the strict inequality holds, we say that (M, g) is strictly *δ-pinched* in the global sense.

Theorem 3.1. *(The Sphere Theorem, Rauch-Berger-Klingenberg, 1952-1961) If the sectional curvature K of a simply connected, complete, Riemannian manifold (M, g) satisfies $1/4 < K \leq 1$, then the manifold is homeomorphic to a sphere.*

Theorem 3.2. *Let M be a smooth manifold with virtually abelian fundamental group. The following statements are equivalent: (i) M admits a complete metric g of $K \equiv -1$; (ii) M admits a complete metric g of pinched negative curvature, i.e., $-1 \leq K \leq 0$.*

The following Theorems describe the dynamic evolution of area when the curvature is bounded.

Theorem 3.3. *Suppose $0 < a(x) < K(x,t) \leq b(x)$, for $t > 0$, and $\lim_{t\to 0} K(x,t)$ does not exist. If $\lim_{t\to 0} \sigma(x,t) = 0$ and $\lim_{t\to 0} \sigma_t(x,t) = 0$, then the derivative function $\sigma_t(x,t)$ is oscillatory and consequently, the solution $\sigma(x,t)$ is not monotonic in any interval $[a, \infty)$.*

Proof. From the first hypothesis it follows

$$\sigma_{t^2}(x,t) + a(x)\,\sigma(x,t) < 0, \ \sigma_{t^2}(x,t) + b(x)\,\sigma(x,t) \geq 0.$$

Suppose that $\sigma_t(x,t)$ is positive throughout. Multiplying the first inequality with $\sigma_t(x,t)$ and integrating on the interval $[\epsilon, t]$, we find

$$\sigma_t^2(x,t) + a(x)\,\sigma^2(x,t) < \sigma_t^2(x,\epsilon) + a(x)\,\sigma(x,\epsilon),$$

Taking the limit when $\epsilon \to 0$, we obtain a contradiction.

Supposing that $\sigma_t(x,t)$ is negative throughout and using the second inequality, we attend again a contradiction. □

4 Asymptotic behavior of oriented parallelogram area controlled by sectional curvature

Suppose we equippe the manifold M with a family of smooth Riemannian metrics $g(t,x)$ satisfying the evolution PDE

$$(1) \qquad u_{t^2}(x,t) + 2K(x,t)u(x,t) = 0,$$

where $t \to K(x,t)$ as function in $L_{loc}([a,b],\mathbb{R})$ represents the sectional curvature with respect to a metric $g(x,t)$. For example, M is a smooth closed (compact and without boundary) manifold.

To adapt the Sturm-Liouville theory (see, [3]-[7], [21], [22]) to this geometric PDE, we denote $p(t) = -2K(x,t)$ and we accept that the unknown function $u(x,t)$ is a prolongation of $\sigma(x,t)$ (the negative values correspond to oriented area and the value zero is accepted for prolongation by continuity).

To the PDE (1), we can attach either the initial conditions $u(x,0) = c(x)$, $u_t(x,0) = v(x)$ or the boundary conditions $u(x,0) = c_0(x)$, $u(x,T) = c_T(x)$. If we look for periodic solutions we must impose $u(x,0) = u(x,T)$, $u_t(x,0) = u_t(x,T)$.

To analyze the asymptotic behavior of the solutions of ODE

$$(2) \qquad u''(t) = p(t)u(t),$$

we introduce the following spaces of functions: (1) $V(p)$ *the set of solutions satisfying the conditions*

$$\int^{+\infty} \left(\frac{u(t)}{t}\right)^2 dt < +\infty, \quad \int^{+\infty} u_t(t)^2\, dt < +\infty;$$

(2) $W(p)$ *the set of solutions satisfying the conditions*

$$\int^{+\infty} u(t)^2\, dt < +\infty, \quad \int^{+\infty} t^2 u_t(t)^2\, dt < +\infty;$$

(3) $Z(p)$ *the set of solutions for which* $\lim_{t\to\infty} u(t) = 0$.

Theorem 4.1. *(i) If* $p(t) \geq 0$ *for large* t*, then* $dim\, V(p) = 1$*. (ii) Moreover, if* $\lim_{t\to\infty} t^2 p(t) = +\infty$*, then* $W(p) = V(p)$*.*

Proof. (i) Let S be the set of solutions of the ODE (2). The set S is a bidimensional vector space. Then $\dim V = 1$ shows that $\{0\} \neq V \neq S$.

Assume there exists $a \in \mathbb{R}_+$ such that $p(t) \geq 0$ for $t \in [a,\infty)$. If $p(t) = 0$, for $t \geq a$, then the subspace V consists only in constant functions and hence $\dim V = 1$. It remains to consider the case $p(t) \neq 0$ on a set of positive measure, in any neighborhood of $+\infty$. Let $b \in (a,\infty)$ and u a solution of the ODE (2). Multiplying both members by $u(t)$ and integrating on the interval $[a,b]$, we obtain

$$\int_a^b p(t)(u(t))^2 dt + \int_a^b (u'(t))^2 dt = u'(b)u(b) - u'(a)u(a)$$

$$\leq |u'(b)||u(b)| + |u'(a)||u(a)| \leq (|u(b)| + |u'(b)|)\,|u(b)| + (|u(a)| + |u'(a)|)\,|u(a)|.$$

Since each integrand is positive, we find

$$\int_a^b p(t)(u(t))^2 dt \leq (|u(b)| + |u'(b)|)\,|u(b)| + (|u(a)| + |u'(a)|)\,|u(a)|$$

$$\int_a^b (u'(t))^2 dt \leq (|u(b)| + |u'(b)|)\,|u(b)| + (|u(a)| + |u'(a)|)\,|u(a)|.$$

Let $c \in \mathbb{R}$. We denote by u_b the solution which satisfies the boundary conditions $u_b(a) = c$, $u_b(b) = 0$. Then

$$\int_a^b p(t)(u_b(t))^2 dt \leq (|u_b(a)| + |u_b'(a)|)\,|c| = \rho(b)|c|,$$

$$\int_a^b (u'(t))^2 dt \leq (|u_b(a)| + |u_b'(a)|)\,|c| = \rho(b)|c|.$$

Let us show that the subset

$$\{\rho(b) = |u_b(a)| + |u_b'(a)| \mid b \geq a\}$$

is bounded in $\mathbb{R}$. Assume the opposite: there exist a sequence (b_n) such that $\rho(b_n) = \rho_n > n$. Denote $v_n(t) = \frac{1}{\rho_n} u_{b_n}(t)$. Obviously $|v_n(a)| + |v_n'(a)| = 1$ and

$$\int_a^{b_0} p(t)(v_n(t))^2 dt \leq \int_a^{b_n} p(t)(v_n(t))^2 dt = \frac{1}{\rho_n^2}\int_a^{b_n} p(t)(u_{b_n}(t))^2 dt \leq \frac{|c|}{\rho_n} < \frac{|c|}{n}. \tag{3}$$

Without loss of generality, we can assume that the sequences $(v_n(a))$ and $(v_n'(a))$ are convergent. Then there exists $\lim_{n\to\infty} v_n(t) = v(t)$, uniformly on $[a, b_0]$, the function $v(t)$ being a solution of the ODE (2) and satisfying $|v(a)| + |v'(a)| = 1$. On the other hand, passing to limit in (3), we find $v(t) = 0$. This contradiction proves that the set $\{\rho(b) \mid b \geq a\}$ is bounded.

Now we consider a sequence (b_n) with $\lim_{n\to\infty} b_n = +\infty$. The bounded sequence $(u_{b_n}'(a))$ contains a subsequence convergent to c'. We can assume just like that $\lim_{n\to\infty} u_{b_n}'(a) = c'$. Then there exists $\lim_{n\to\infty} u_{b_n}(t) = u(t)$, uniformly on any compact interval, where u is a solution of the ODE (2) satisfying

$$u(a) = c, \quad \int_a^{+\infty} (u'(t))^2 dt < +\infty.$$

By construction, the above solution is bounded on $[a, +\infty)$. Consequently

$$\int_a^{+\infty} \left(\frac{u(t)}{t}\right)^2 dt < +\infty$$

and we have $u \in V(p)$.

Let us show that the obtained solution is unique. This revert to prove that the ODE (2), together the conditions $u(a) = 0$, $\int_a^{+\infty} (u'(t))^2 dt < +\infty$ has only the trivial solution $u(t) = 0$. Assume the opposite: there exists a nontrivial solution $u(t)$. First we remark that

$$\lim_{b\to\infty} \frac{1}{(b-a)^2} \int_a^b (u(t))^2 dt = 0. \tag{4}$$

Second, let $w(t) = u'(t)u(t)$. Then $w(a) = 0$, $w'(t) = (u'(t))^2 + p(t)(u(t))^2 \geq 0$ and the derivative w' is nonzero on a set of positive measure. Consequently, there exists $t_0 > a$ and $\alpha > 0$ such that $w(t) > \alpha$ for $t \geq t_0$. We find

$$\int_a^b (b-t)w(t)dt \geq \alpha \int_a^b (b-t)dt = \frac{\alpha}{2}(b-a)^2 > 0.$$

On the other hand,

$$\int_a^b (b-t)w(t)dt = \frac{1}{2}(b-t)(u(t))^2|_a^b + \frac{1}{2}\int_a^b (u(t))^2 dt = \frac{1}{2}\int_a^b (u(t))^2 dt.$$

It follows

$$\frac{1}{(b-a)^2}\int_a^b (u(t))^2 dt \geq \alpha > 0,$$

for any $b > a$, which contradicts the relation (4).

We have proved the existence of a unique function $u \in V(p)$ with $u(a) = c$; so $\dim V(p) = 1$.

(ii) Finally, suppose that $\lim_{t\to\infty} t^2 p(t) = +\infty$. Clearly $W(p) \subseteq V(p)$. Then, it is sufficient to prove that there exists a nontrivial solution of the equation (1) which belongs to the set $W(p)$. To do that, we assume

$$t^2 p(t) > \alpha + \frac{3+\beta}{4} > 0, \text{ for } t \geq a, \text{ with suitable constants } \alpha > 0,\ \beta \in (0,1)$$

and consider, for a fixed $b > a$, a solution of the equation (1), which satisfies the boundary conditions $u(a) = c$, $u(b) = 0$. The identity

$$\int_a^b t^2 p(t)(u(t))^2\,dt + \int_a^b t^2 (u'(t))^2\,dt - \int_a^b (u(t))^2\,dt = a(u(a))^2 - a^2 u'(a)u(a)$$

leads us to

$$\int_a^b (t^2(u'(t))^2 + \alpha(u(t))^2)\,dt - \frac{1-\beta}{4}\int_a^b (u(t))^2\,dt \leq a(u(a))^2 - a^2 u'(a)u(a)\,.$$

Since we can write

$$\frac{1-\beta}{4}\int_a^b (u(t))^2\,dt = -\frac{1-\beta}{2}a(u(a))^2 + (1-\beta)\int_a^b t^2(u'(t))^2\,dt$$

$$-\frac{1-\beta}{4}\int_a^b (2tu'(t) + u(t))^2\,dt \leq -\frac{1-\beta}{2}a(u(a))^2 + (1-\beta)\int_a^b t^2(u'(t))^2\,dt,$$

we obtain

$$\beta \int_a^b t^2(u'(t))^2\,dt + \alpha\int_a^b (u(t))^2\,dt \leq \frac{3-\beta}{2}\,a(u(a))^2 - a^2 u'(a)u(a)$$

$$\leq a^2|u(a)|\,(|u(a)| + |u'(a)|)\,.$$

Then we proceed as in the first part of the proof and we obtain a nontrivial solution $u \in W(p)$. $\square$

Theorem 4.2. *Let $p(t) \geq 0$ for large t. The equality $Z(p) = V(p)$ holds if and only if*

$$\int_0^{+\infty} tp(t)\,dt = +\infty. \tag{5}$$

Furthermore, in the same assumption, if $\lim_{t\to\infty} inf\, t^2 p(t) > \frac{3}{4}$, then we have the equality $Z(p) = W(p)$.

Proof. Let $u \in V(p)$ be a nonzero solution of the equation (2). Since $p(t) \geq 0$, for t sufficiently large, we can find $a \in (0, \infty)$ such that $u(t) \neq 0$ and $u(t)u'(t) \leq 0$, $t \in [a, \infty)$. Assume $p(t) \geq 0$, $t \in [a, \infty)$ and that $u(t) > 0$, $u'(t) \leq 0$. Then, from the equality

$$\int_a^t sp(s)u(s)ds = tu'(t) - u(t) - au'(a) + u(a),$$

we find

$$c_0 = u(a) - au'(a) \geq u(t) \int_a^t sp(s)ds,$$

since $tu'(t) - u(t) \leq 0$, and $u(t)$ is decreasing. If the condition (5) is satisfied, then $\lim t \to \infty u(t) = 0$, i.e., $u \in Z(p)$. Hence $V(p) \subset Z(p)$. On the other hand, $\dim Z(p) \leq 1$, since any solution satisfying $u(a) > 0$ and $u'(a) > 0$ is unbounded. According the foregoing Theorem $\dim V(p) = 1$. Consequently, the previous inclusion gives the equality $Z(p) = V(p)$.

Conversely, let us show that the condition (5) is also necessary for the equality $Z(p) = V(p)$. Assume the opposite, i.e., $\int_0^\infty tp(t)dt < \infty$. Then for a function $u \in V(p) = Z(p)$, with $u(t) > 0$ and $u'(t) \leq 0$, we find

$$u(t) = \int_t^\infty (s-t)p(s)u(s)ds \leq u(t) \int_t^\infty sp(s)ds, \ t \geq a.$$

Hence

$$\int_t^\infty sp(s)ds \geq 1, \forall t \geq a,$$

which contradicts the convergence of the improper integral.

Finally, suppose that $\lim_{t\to\infty} \inf t^2 p(t) > \frac{3}{4}$. Then it suffices to prove that any solution of the ODE (2) under conditions $u(t) > 0, u'(t) \leq 0$, belongs to the set $W(p)$. For this aim, assume that

$$t^2 p(t) > \frac{3+5\varepsilon}{4} \quad \text{for } t \geq a$$

with a suitable constant $\varepsilon \in (0, 1)$. If we denote $c_1 = a(u(a))^2 - a^2 u'(a)u(a)$ and take into account the properties of the function u, the identity

$$\int_a^t s^2 p(s)(u(s))^2\,ds + \int_a^t s^2(u'(s))^2\,ds - \int_a^t (u(s))^2\,ds = t^2 u'(t)u(t) - t(u(t))^2 + c_1$$

leads us to

$$t(u(t))^2 + \int_a^t (s^2(u'(s))^2 + \varepsilon(u(s))^2)\,ds - \frac{1-\varepsilon}{4}\int_a^t (u(s))^2\,ds \leq c_1 \quad \text{for } t \geq a.$$

But we can write

$$\frac{1-\varepsilon}{4}\int_a^t (u(s))^2\,ds = \frac{1-\varepsilon}{2}(t(u(t))^2 - a(u(a))^2) + (1-\varepsilon)\int_a^t s^2(u'(s))^2\,ds$$

$$-\frac{1-\varepsilon}{4}\int_a^t (2su'(s)+u(s))^2\,ds \le \frac{1-\varepsilon}{2}\,t(u(t))^2 + (1-\varepsilon)\int_a^t s^2(u'(s))^2\,ds \quad \text{for } t \ge a,$$

and we obtain

$$\int_a^{+\infty} (s^2(u'(s))^2 + (u(s))^2)\,ds \le \frac{c_1}{\varepsilon}.$$

Therefore, $u \in W(p)$. □

5 Oscillatory behavior of oriented parallelogram area controlled by the sectional curvature

We borrow a condition from [3] guaranteeing non-oscillatory solutions of ODE (2).

Theorem 5.1. *If* $\int_a^\infty |p(t)|dt < \infty$, *then every solution of ODE (2) is non-oscillatory.*

Proof. We use the Prufer Transformation

$$u(t) = \rho(t)\sin\theta(t),\ u'(t) = -\rho(t)\cos\theta(t)$$

to change the equation $-u''(t) + p(t)u(t) = 0$ into polar coordinates. It follows

$$\theta'(t) = \cos^2\theta(t) - p(t)\sin^2\theta(t),\ \rho'(t) = (1+p(t))\rho(t)\sin\theta(t)\cos\theta(t).$$

A zero t_0 of $u(t)$ is a zero of $\sin\theta(t)$. It follows $\theta'(t) > 0$. Thus, consecutive zeros of the function $u(t)$ correspond to consecutive multiples of π as values of θ. Consequently the solution $u(t)$ is oscillatory if and only if $\lim_{t\to\infty}\theta(t) = \infty$.

The Cauchy problem

$$\phi'(t) = 1 + |p(t)|,\ \phi(a) = \theta_0$$

shows

$$\phi(t) - \theta_0 = \int_a^t (1+|p(s)|)ds < \infty,$$

by hypothesis. Thus $\phi(t)$ is bounded as $t \to \infty$. On the other hand,

$$|\cos^2\theta(t) - p(t)\sin^2\theta(t)| < 1 + |p(t)|$$

shows that $\theta(t)$ is less than some $\phi(t)$ and is therefore bounded. Consequently, the solution $u(t)$ is non-oscillatory. □

Examples (1) Let us consider the ODE $u''(t) + e^t u(t) = 0$. The general solution

$$u(t) = c_1\text{Bessel}\,J(0, 2e^{\frac{1}{2}t}) + c_2\text{Bessel}\,Y(0, 2e^{\frac{1}{2}t})$$

has an oscillatory behavior.

(2) Now we consider the ODE $u''(t) - e^t u(t) = 0$. The general solution

$$u(t) = c_1 \text{Bessel}\, I(0, 2e^{\frac{1}{2}t}) + c_2 \text{Bessel}\, K(0, 2e^{\frac{1}{2}t})$$

has an non-oscillatory behavior.

(3) Another interesting example is the ODE $u''(t) - (1+t^2)u(t) = 0$. The general solution

$$u(t) = \left(c_1 + c_2 \int e^{-t^2} dt\right) e^{\frac{t^2}{2}}$$

has an non-oscillatory behavior.

6 Minimum of total sectional curvature

Using optimal control theory, we present some global optimality results connected with the unique solvability for the Sturm-Liouville problem. Since the PDE (1) is linear, it coincides with its *infinitesimal deformation*, around a solution $u(x,t)$. This PDE is also *auto-adjoint* since $vp_{t^2} - pv_{t^2} = 0$, for any two solutions $v(x,t)$ and $p(x,t)$. If it is used as adjoint equation, then a solution $p(x,t)$ is called the *costate function.*

Define an admissible control set Ω of functions $K : M \times [0,1] \to R$ by two conditions: (1) $K(x,\cdot) \in L[0,1]$, for $x \in M$; (2) there exist two functions $A, B : M \to R$ such that $A(x) \leq K(x,t) \leq B(x)$. Our goal is to seek a solution $K^*(x,t) \in \Omega$ of the following optimal control problem (for similar problems see [8] - [10] , [12] - [14], [16]-[20]):

$$\text{Find } \min J(K(x,\cdot)) = \int_0^1 K(x,t)dt \text{ (total sectional curvature)}$$

$$\text{subject to} \quad u_{t^2}(x,t) + 2K(x,t)u(x,t) = 0.$$

Theorem 6.1. *The previous control problem has an optimal control $K^*(x,t) \in \Omega$. This $K^*(x,t)$ is a bang-bang control.*

Proof. To prove the existence of a bang-bang control, we use the single-time Pontryaguin minimum principle. The Hamiltonian $H(u,p,K) := (1+2p(x,t)u(x,t))K(x,t)$ gives the initial PDE $u_{t^2} = -H_p$ and the adjoint PDE $p_{t^2} = -H_u$. The extremum of the linear function $K \to H$ exists since the control belongs to the interval $[A(x), B(x)]$; for optimum, the control must be at $A(x)$ or $B(x)$ (see, linear optimization, simplex method).

If $Q(x,t) = 1 + 2p(x,t)u(x,t)$, then the optimal control K^* must be the function $t \to K^*(x,t)$, where

$$K^*(x,t) = \begin{cases} B(x) & \text{for } Q(x,t) < 0 : \text{ bang-bang control} \\ \text{undetermined} & \text{for } Q(x,t) = 0 : \text{ singular control} \\ A(x) & \text{for } Q(x,t) > 0 : \text{ bang-bang control.} \end{cases}$$

Suppose that the Lebesgue measure of the set $\{t \in [0,1] : Q(x,t) = 0\}$ vanishes. Then, the singular control is ruled out and the remaining possibilities are bang-bang controls. This optimal control $K^*(x,t)$ is discontinuous with respect to the variable t since the control jumps from a minimum to a maximum and vice-versa, in response to each change in the sign of $Q(x,t)$. The function $t \to Q(x,t)$ is called *switching function*.

Without loss of generality, we accept $A(x) < 0 < B(x)$. The *optimal evolution* and the *optimal costate* are either

$$u(x,t) = c_1(x) \cos \sqrt{2B(x)}\ t + c_2(x) \sin \sqrt{2B(x)}\ t,$$

$$p(x,t) = k_1(x) \cos \sqrt{2B(x)}\ t + k_2(x) \sin \sqrt{2B(x)}\ t,\ \ t \in \mathbb{R},$$

for $K^*(x,t) = B(x)$, or

$$u(x,t) = c_1(x) e^{\sqrt{-2A(x)}\ t} + c_2(x) e^{-\sqrt{-2A(x)}\ t},$$

$$p(x,t) = k_1(x) e^{\sqrt{-2A(x)}\ t} + k_2(x) e^{-\sqrt{-2A(x)}\ t},\ \ t \in \mathbb{R},$$

for $K^*(x,t) = A(x)$. The constants (with respect to t) $c_1(x)$, $c_2(x)$, $k_1(x)$, $k_2(x)$ are determined by Cauchy data.

The switching function $t \to Q(x,t) = 1 + 2p(x,t)u(x,t)$ cannot vanish identically. Consequently the singular control is ruled out. In the generic case, the bang-bang control is the only possibility, i.e., the optimal $K^*(x,t)$ must fall into one of the following four cases: (i) $B(x)$ for $t \in [0,1]$; (ii) $A(x)$ for $t \in [0,1]$; (iii) $B(x)$ for $t \in [0, t_s(x)]$ and $A(x)$ for $t \in [t_s(x), 1]$; (iv) $A(x)$ for $t \in [0, t_s(x)]$ and $B(x)$ for $t \in [t_s(x), 1]$, where $t_s(x)$ is the switching time. □

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