Computing the Rodrigues coefficients of the exponential map of the Lie groups of matrices

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Abstract. In Theorem 2.1 we present, in the case when the eigenvalues of the matrix are pairwise distinct, a direct way to determine the Rodrigues coefficients of the exponential map for the linear general group $GL(n, \mathbb{R})$ by reducing the Rodrigues problem to the system (2.3). The method is illustrated for the special orthogonal group $\mathbf{SO(n)}$, when $n = 2, 3, 4$.

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1 Introduction

The exponential map $\exp : gl(n, \mathbb{R}) = M_n(\mathbb{R}) \to \mathbf{GL}(n, \mathbb{R})$, where $\mathbf{GL}(n, \mathbb{R})$ denotes the Lie group of real invertible $m \times n$ matrices, is defined by (see for instance C. Chevalley [4], J.E. Marsden and T.S. Ratiu [11], or F. Warner [16])

$$\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k. \tag{1.1}$$

According to the well-known Hamilton-Cayley theorem, it follows that every power X^k, $k \geq n$, is a linear combination of X^0, X^1, ..., X^{n-1}, hence we can write

$$\exp(X) = \sum_{k=0}^{n-1} a_k(X) X^k, \tag{1.2}$$

where the real coefficients $a_0(X), \ldots, a_{n-1}(X)$ are uniquely defined and depend on the matrix X. From this formula, it follows that $\exp(X)$ is a polynomial of X. The problem to find a reasonable formula for $\exp(X)$ is reduced to the problem to determine the coefficients $a_0(X), \ldots, a_{n-1}(X)$. We will call this general question, the *Rodrigues problem*, and the numbers $a_0(X), \ldots, a_{n-1}(X)$ the *Rodrigues coefficients* of the exponential map with respect to the matrix $X \in M_n(\mathbb{R})$.

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The origin of this problem is the classical Rodriques formula (1840) for the special orthogonal group $\mathbf{SO(3)}$:

$$\exp(X) = I_3 + \frac{\sin\theta}{\theta}X + \frac{1-\cos\theta}{\theta^2}X^2,$$

where $\sqrt{2}\theta = ||X||$ is the Frobenius norm of the matrix X (for details see Subsection 3.1). There are at least two arguments pointing out the importance of this formula: the study of the rigid body rotation in $\mathbb{R}^3$, and the parametrization of the rotations in $\mathbb{R}^3$.

An important property of the Rodrigues coefficients is the invariance under the matrix similarity, that is for every invertible matrix U the following relations hold

$$a_k(UXU^{-1}) = a_k(X), k = 0, \ldots, n-1. \tag{1.3}$$

Indeed, if we assume that

$$\exp(UXU^{-1}) = \sum_{k=0}^{n-1} a'_k(UXU^{-1})^k,$$

where $a'_k = a'_k(UXU^{-1}), k = 0, \ldots, n-1$, then using the well-known property of the exponential map $\exp(UXU^{-1}) = U\exp(X)U^{-1}$ (see for instance J. Gallier [6]), we can write

$$\exp(UXU^{-1}) = U\exp(X)U^{-1} = U\left(\sum_{k=0}^{n-1} a_k(X)X^k\right)U^{-1} = \sum_{k=0}^{n-1} a_k(X)(UXU^{-1})^k,$$

and the property immediately follows from the uniqueness of the Rodrigues coefficients.

The invariance under matrix similarity points out the importance of the spectrum of the matrix X in relation (1.2). An important method to obtain the Rodrigues coefficients following this idea, is so-called *Putzer method* (the original reference is E. J. Putzer [15]). This method consists in the following steps. Firstly, consider the characteristic polynomial of matrix X,

$$f(t) = \det(tI_n - X) = t^n + c_{n-1}t^{n-1} + \ldots + c_1 t + c_0,$$

and define the Putzer matrix

$$C = \begin{pmatrix} c_1 & c_2 & \ldots & c_{n-1} & 1 \\ c_2 & c_3 & \ldots & 1 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{n-1} & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

In the second step, construct the scalar function z which is the solution of the linear homogeneous differential equation with constant coefficients

$$z^{(n)} + c_{n-1}z^{(n-1)} + \ldots c_1 z' + c_0 z = 0,$$

satisfying the initial conditions

$$z(0) = z'(0) = \ldots = z^{(n-2)}(0) = 0,\ z^{(n-1)}(0) = 1.$$

The following relation holds

$$A = C \cdot Z, \tag{1.4}$$

where A is the $n\times 1$ matrix with entries the Rodrigues coefficients $a_0(X), \ldots, a_{n-1}(X)$, and Z is the $n \times 1$ matrix with the entries $z(1), z'(1), \ldots, z^{(n-1)}(1)$.

2 The Rodrigues formula for $\exp : gl(n, \mathbb{R}) \to GL(n, \mathbb{R})$

In this section we will indicate a way to determine the Rodrigues coefficients $a_0(X), \ldots, a_{n-1}(X)$ in (1.2). The main idea consists in the reduction of (1.2) to a linear system with the unknowns $a_0(X), \ldots, a_{n-1}(X)$. In this respect we multiply both sides of (1.2) by the power X^j, $j = 0, \ldots, n-1$, and we obtain the matrix relations

$$X^j \exp(X) = \sum_{k=0}^{n-1} a_k X^{k+j},\ j = 0, \ldots, n-1, \tag{2.1}$$

where $a_k = a_k(X)$, $k = 0, \ldots, n-1$. Now, considering the matrix trace in the both sides of (2.1), we obtain the linear system

$$\sum_{k=0}^{n-1} tr(X^{k+j}) a_k = tr(X^j \exp(X)),\ j = 0, \ldots, n-1, \tag{2.2}$$

with the coefficients functions of the matrix X. Now, assume that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of matrix X. Then, it is well-known that the matrix X^{k+j} has the eigenvalues $\lambda_1^{k+j}, \ldots, \lambda_n^{k+j}$, and the matrix $X^j \exp(X)$ has the eigenvalues $\lambda_1^j e^{\lambda_1}, \ldots, \lambda_n^j e^{\lambda_n}$. Indeed, the function $f_j : \mathbb{C} \to \mathbb{C}$, $f(z) = z^j e^z$, is analytic, hence the eigenvalues of the matrix $f_j(X)$ are $f_j(\lambda_1), \ldots, f_j(\lambda_n)$. But, clearly we have $f_j(\lambda_s) = \lambda_s^j e^{\lambda_s}, s = 1, \ldots, n$, and the property is proved.

According to the considerations above, the system (2.2) is equivalent to

$$\sum_{k=0}^{n-1} \left(\sum_{s=1}^{n} \lambda_s^{k+j} \right) a_k = \sum_{s=1}^{n} \lambda_s^j e^{\lambda_s}, j = 0, \ldots, n-1. \tag{2.3}$$

From the system (2.3) we obtain the following result concerning the solution to the Rodrigues problem for the group $\mathbf{GL}(n, \mathbb{R})$:

Theorem 2.1. *1) The Rodrigues coefficients in formula* (1.2) are solutions to the system (2.3).

2) If the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix X are pairwise distinct, then the Rodrigues coefficients $a_0, \ldots, a_{n-1}$ are perfectly determined by the system (2.3) *and they are linear combinations of $e^{\lambda_1}, \ldots, e^{\lambda_n}$ having the coefficients rational functions of $\lambda_1, \ldots, \lambda_n$, i.e. we have*

$$a_k = b_k^{(1)} e^{\lambda_1} + \ldots + b_k^{(n)} e^{\lambda_n}, k = 0, \ldots, n-1.$$

Proof. The first statement was already proved.

For the second statement, observe that the determinant of the system (2.3) can be written as

$$D_n = \det \begin{pmatrix} S_0 & S_1 & \dots & S_{n-1} \\ S_1 & S_2 & \dots & S_n \\ \dots & \dots & \dots & \dots \\ S_{n-1} & S_n & \dots & S_{2n-1}, \end{pmatrix}$$

where $S_l = S_l(\lambda_1, \dots, \lambda_n) = \lambda_1^l + \dots + \lambda_n^l, l = 0, \dots, 2n-1$.

It is clear that

$$D_n = \det \begin{pmatrix} 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_n \\ \dots & \dots & \dots \\ \lambda_1^{n-1} & \dots & \lambda_n^{n-1} \end{pmatrix} \cdot \det \begin{pmatrix} 1 & \lambda_1 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \dots & \lambda_2^{n-1} \\ \dots & \dots & \dots & \dots \\ 1 & \lambda_n & \dots & \lambda_n^{n-1} \end{pmatrix}$$

$$= V_n^2 = \prod_{1 \le i < j \le n} (\lambda_i - \lambda_j)^2,$$

where $V_n = V_n(\lambda_1, \dots, \lambda_n)$ is the Vandermonde determinant of order n. According to the well-known formulas giving the solution $a_0, \dots, a_{n-1}$ to the system (2.3), the conclusion follows. □

The following consequence shows how to determine directly the matrix Z in the Putzer method, in the case when the eigenvalues of X are pairwise distinct, only in terms of eigenvalues of X.

Corollary 2.2. *If the eigenvalues $\lambda_1, \dots, \lambda_n$ of the matrix X are pairwise distinct, then the $n \times 1$ matrix Z in the Putzer method is given by*

$$Z = (SC)^{-1} B,$$

where the matrix S is defined by

$$\begin{pmatrix} S_0 & S_1 & \dots & S_{n-1} \\ S_1 & S_2 & \dots & S_n \\ \vdots & \vdots & \ddots & \vdots \\ S_{n-1} & S_n & \dots & S_{2n-1} \end{pmatrix},$$

C is the Putzer matrix, and B is the $n \times 1$ matrix having the entries

$$b_j = \sum_{s=1}^{n} \lambda_s^j e^{\lambda_s}, \ j = 0, \dots, n-1.$$

Proof. According to the Putzer method we have $A = C \cdot Z$, and from the system (2.3) we have $S \cdot A = B$. Because the eigenvalues of the matrix X are distinct, it follows that the matrix S is invertible, hence we obtain $C \cdot Z = S^{-1} \cdot B$. Therefore,

$$Z = C^{-1} \cdot S^{-1} = (SC)^{-1} B,$$

and we are done. □

Remark 2.1. Comparing with the Putzer method, our result contained in Theorem 2.1 is simpler in the case when the eigenvalues $\lambda_1, \ldots, \lambda_n$ of matrix X are pairwise distinct, because in this case we have just to solve the linear system (2.3). The Putzer method is better in the situations when we have multiplicities of the eigenvalues of matrix X. In concrete situations, when the multiplicities of the eigenvalues are also involved, we need to combine both methods (see the subsection 3.2).

3 The Rodrigues coefficients of the special orthogonal group $\mathbf{SO}(n)$

It is easy to check that the set of real $n \times n$ orthogonal matrices forms a Lie group under multiplication, denoted by $\mathbf{O}(n)$. The subset of $\mathbf{O}(n)$ consisting of those matrices having the determinant equal to $+1$ is a subgroup, denoted by $\mathbf{SO}(n)$ and called the *Special Orthogonal Group* of the Euclidean space $\mathbb{R}^n$. Due to geometric reasons, the matrices in $\mathbf{SO}(n)$ are also called *rotation matrices*.

It is well-known that the Lie algebra $\mathfrak{so}(n)$ of $SO(n)$ consists in all skew-symmetric matrices in $M_n(\mathbb{R})$ and the Lie bracket is the standard matrices commutator $[A, B] = AB - BA$. The exponential map $\exp : \mathfrak{so}(n) \to \mathbf{SO}(n)$ is defined by the same formula (1.1) because it is given by the restriction $\exp|_{\mathfrak{so}(n)}$ of the exponential map $\exp : \mathfrak{gl}(n, \mathbb{R}) \to \mathbf{GL}(n, \mathbb{R})$. It is known that for every compact connected Lie group the exponential map is surjective (see T. Bröcker, T. tom Dieck [3], D. Andrica, I.N. Casu [1] for the standard proof, or R.-A. Rohan [16] for a new idea of proof given by T. Tao), that is every compact connected Lie group is exponential (see the monograph of M. Wüstner [18] for details about the exponential groups). Because the group $\mathbf{SO(n)}$ is compact it follows that the exponential map $\exp : \mathfrak{so}(n) \to \mathbf{SO}(n)$ is surjective. The surjectivity of exp for the group $\mathbf{SO(n)}$ is an important property. Indeed, it implies the existence of a locally inverse function $\log : \mathbf{SO}(n) \to \mathfrak{so}(n)$, and this has interesting applications. In the paper of J.Gallier, D.Xu [5] is mentioned that the functions exp and log for the group $\mathbf{SO(n)}$ can be used for motion interpolation (see M.-J. Kim, M.-S. Shin [9], [10] and F.C. Park, B. Ravani [12], [13]). Motion interpolation and rational motions have also been investigated by B. Jüttler [7], [8]. Also, the surjectivity of the exponential map for the group $\mathbf{SO(n)}$ gives the possibility to describe the rotations of the Euclidean space $\mathbb{R}^n$ (see R.-A. Rohan [16]). The connection with the noncommutative differential geometry is given the paper of L.I. Piscoran [14].

The matrices in $\mathfrak{so}(n)$ have two essential properties which simplify the computation of the Rodrigues coefficients:

- If n is odd, then they are singular, i.e. they have one eigenvalue equal to 0 (possible with a multiplicity);
- The non-zero eigenvalues are purely imaginary and, of course, conjugated.

3.1 Illustrating the classical cases $n = 2$ and $n = 3$

Clearly, when $X = \mathbf{O}_n$, we have $\exp(X) = I_n$ hence, in this situation we have $a_0 = 1$, $a_1 = \ldots = a_{n-1} = 0$.

When $n = 2$, a skew-symmetric matrix $X \neq O_2$ can be written as

$$X = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, a \in \mathbb{R}^*,$$

having the eigenvalues $\lambda_1 = ai$, $\lambda_2 = -ai$.

The system (2.3) is in this case

$$\begin{cases} 2a_0 + (\lambda_1 + \lambda_2)a_1 = e^{\lambda_1} + e^{\lambda_2} \\ (\lambda_1 + \lambda_2)a_0 + (\lambda_1^2 + \lambda_2^2)a_1 = \lambda_1 e^{\lambda_1} + \lambda_2 e^{\lambda_2}, \end{cases}$$

hence immediately we obtain

$$a_0 = \frac{1}{2}\left(e^{ai} + e^{-ai}\right) = \cos a,$$

$$a_1 = \frac{\lambda_1 e^{\lambda_1} + \lambda_2 e^{\lambda_2}}{\lambda_1^2 + \lambda_2^2} = \frac{e^{ai} - e^{-ai}}{2a} = \frac{\sin a}{a},$$

and then

$$\exp(X) = (\cos a)I_2 + \frac{\sin a}{a}X.$$

It follows that

$$a_0(X) = \cos a,\ a_1(X) = \frac{\sin a}{a}.$$

When $n = 3$, a real skew-symmetric matrix X is of the form

$$X = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

having the characteristic polynomial

$$p_X(t) = t^3 + (a^2 + b^2 + c^2)t = t^3 + \theta^2 t,$$

where $\theta = \sqrt{a^2 + b^2 + c^2}$. The eigenvalues of X are $\lambda_1 = \theta i$, $\lambda_2 = -\theta i$, $\lambda_3 = 0$. It is clear that $X = O_3$ if and only if $\theta = 0$, hence it suffices to consider only the situation $\theta \neq 0$. The system (2.3) is equivalent to

$$\begin{cases} 3a_0 - 2\theta^2 a_2 = 1 + e^{\theta i} + e^{-\theta i} \\ -2\theta^2 a_1 = \theta i(e^{\theta i} - e^{-\theta i}) \\ -2\theta^2 a_0 + 2\theta^4 a_2 = -\theta^2(e^{\theta i} + e^{-\theta i}) \end{cases}$$

Because $\theta \neq 0$, it follows that

$$a_0 = 1,\ a_1 = \frac{\sin\theta}{\theta},\ a_2 = \frac{1 - \cos\theta}{\theta^2},$$

giving the well-known classical formula due to Rodrigues

$$\exp(X) = I_3 + \frac{\sin\theta}{\theta}X + \frac{1 - \cos\theta}{\theta^2}X^2.$$

3.2 The case $n = 4$

The general skew-symmetric matrix $X \in so(4)$ is

$$X = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix},$$

and the corresponding characteristic polynomial is given by

$$p_X(t) = t^4 + (a^2 + b^2 + c^2 + d^2 + e^2 + f^2)t^2 + (af - be + cd)^2.$$

Let $\lambda_{1,2} = \pm\alpha i$, $\lambda_{3,4} = \pm\beta i$ be the eigenvalues of the matrix X, where α, $\beta \in \mathbb{R}$. After simple algebraic manipulations, the system (2.3) becomes

$$\begin{cases} 2a_0 - (\alpha^2 + \beta^2)a_2 = \cos\alpha + \cos\beta \\ -(\alpha^2 + \beta^2)a_1 + (\alpha^4 + \beta^4)a_3 = -\alpha\sin\alpha - \beta\sin\beta \\ -(\alpha^2 + \beta^2)a_0 + (\alpha^4 + \beta^4)a_2 = -\alpha^2\sin\alpha - \beta^2\sin\beta \\ (\alpha^4 + \beta^4)a_1 - (\alpha^6 + \beta^6)a_3 = \alpha^3\sin\alpha + \beta^3\sin\beta \end{cases} \tag{3.1}$$

We consider the following three cases:

Case 1. If $\alpha \neq \beta$, α, $\beta \in \mathbb{R}^*$, then by grouping the first equation with the third one, and the second equation with the last one, we obtain the Rodrigues coefficients

$$a_0 = \frac{\beta^2\cos\alpha - \alpha^2\cos\beta}{\beta^2 - \alpha^2},$$

$$a_1 = \frac{\beta^3\sin\alpha - \alpha^3\sin\beta}{\alpha\beta(\beta^2 - \alpha^2)},$$

$$a_2 = \frac{\cos\alpha - \cos\beta}{\beta^2 - \alpha^2},$$

$$a_3 = \frac{\beta\sin\alpha - \alpha\sin\beta}{\alpha\beta(\beta^2 - \alpha^2)}.$$

In this case it follows the corresponding Rodrigues formula in the form:

$$\begin{aligned} \exp(X) = {} & \frac{\beta^2\cos\alpha - \alpha^2\cos\beta}{\beta^2 - \alpha^2} I_4 + \frac{\beta^3\sin\alpha - \alpha^3\sin\beta}{\alpha\beta(\beta^2 - \alpha^2)} X \\ & + \frac{\cos\alpha - \cos\beta}{\beta^2 - \alpha^2} X^2 + \frac{\beta\sin\alpha - \alpha\sin\beta}{\alpha\beta(\beta^2 - \alpha^2)} X^3. \end{aligned} \tag{3.2}$$

Case 2. If $\alpha \neq 0$ and $\beta = 0$, then we will use the Putzer method described in the first section. In this situation the characteristic polynomial simplifies to $p_X(t) = t^4 + \alpha^2 t^2$ and the Putzer matrix is given by

$$C = \begin{pmatrix} 0 & \alpha^2 & 0 & 1 \\ \alpha^2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The scalar function z, solution to the differential equation $z^{(4)} + \alpha^2 z^{(2)} = 0$ with the initial conditions $z(0) = z'(0) = z''(0) = 0, z^{(3)} = 1$, is $z(u) = -\frac{\sin \alpha u}{\alpha^3} + \frac{u}{\alpha^2}$. The 4×1 matrix Z is

$$Z = \begin{pmatrix} \frac{\alpha - \sin \alpha}{\alpha^3} \\ \frac{1 - \cos \alpha}{\alpha^2} \\ \frac{\sin \alpha}{\alpha} \\ \cos \alpha \end{pmatrix}.$$

Using the formula (1.4) we obtain

$$A = C \cdot Z = \begin{pmatrix} 1 \\ 1 \\ \frac{1 - \cos \alpha}{\alpha^2} \\ \frac{\alpha - \sin \alpha}{\alpha^3} \end{pmatrix},$$

therefore, the corresponding Rodrigues formula to this case is

$$\exp(X) = I_4 + X + \frac{1 - \cos \alpha}{\alpha^2} X^2 + \frac{\alpha - \sin \alpha}{\alpha^3} X^3. \tag{3.3}$$

Case 3. If $\alpha = \beta \neq 0$, then we will use again the Putzer method. The characteristic polynomial of matrix X is $p_X(t) = t^4 + 2\alpha^2 t^2 + \alpha^4$, and the Putzer matrix is defined by

$$C = \begin{pmatrix} 0 & 2\alpha^2 & 0 & 1 \\ 2\alpha^2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

According to the general theory of the linear homogeneous differential equations with constant coefficients, the scalar function z satisfying $z^{(4)} + 2\alpha^2 z^{(2)} + \alpha^4 = 0$ is of the form $z(u) = (C_1 + C_2 u) \cos \alpha u + (C_3 + C_4 u) \sin \alpha u$. From the initial conditions $z(0) = z'(0) = z''(0) = 0, z^{(3)} = 1$, after simple computations, we obtain the function

$$z(u) = -\frac{u}{2\alpha^2} \cos \alpha u + \frac{1}{2\alpha^3} \sin \alpha u.$$

The 4×1 matrix Z is in this case

$$Z = \begin{pmatrix} \frac{\sin \alpha - \alpha \cos \alpha}{2\alpha^3} \\ \frac{\sin \alpha}{2\alpha} \\ \frac{\sin \alpha + \alpha \cos \alpha}{2\alpha} \\ \frac{2 \cos \alpha - \alpha \sin \alpha}{2} \end{pmatrix}.$$

Using the formula (1.4) we obtain in this case

$$A = C \cdot Z = \begin{pmatrix} \frac{\alpha \sin \alpha + 2 \cos \alpha}{2} \\ \frac{3 \sin \alpha - \alpha \cos \alpha}{2\alpha} \\ \frac{\sin \alpha}{2\alpha} \\ \frac{\sin \alpha - \alpha \cos \alpha}{2\alpha^3} \end{pmatrix},$$

and the Rodrigues formula is

$$\exp(X) = \frac{\alpha\sin\alpha + 2\cos\alpha}{2}I_4 + \frac{3\sin\alpha - \alpha\cos\alpha}{2\alpha}X + \frac{\sin\alpha}{2\alpha}X^2 + \frac{\sin\alpha - \alpha\cos\alpha}{2\alpha^3}X^3. \tag{3.4}$$

Remark 3.1. J. Gallier and D. Xu [5, Theorem 2.2], has proved the following Rodrigues type formula for the group $\mathbf{SO}(n)$: Given any non-null skew-symmetric $n \times n$ matrix B, where $n \geq 3$, if $\{i\theta_1, -i\theta_1, \ldots, i\theta_p, -i\theta_p\}$ is the set of distinct eigenvalues of B, where $\theta_j > 0$ and each $i\theta_j$ (and $-i\theta_j$) has multiplicity $k_j \geq 1$, there are p unique skew-symmetric matrices $B_1, \ldots, B_p$ such that:

$$B = \theta_1 B_1 + \ldots + \theta_p B_p, B_i B_j = B_j B_i = O_n, i \neq j, B_i^3 = -B_i$$

for all i, j with $1 \leq i, j \leq p$, and $2p \leq n$. Furthermore:

$$\exp(B) = \exp(\theta_1 B_1 + \ldots + \theta_p B_p) = I_n + \sum_{i=1}^{p} [(\sin\theta_i)B_i + (1 - \cos\theta_i))B_i^2],$$

and $\{\theta_1, \ldots, \theta_p\}$ is the set of distinct eigenvalues of the symmetric matrix

$$-\frac{1}{4}(B - B^T)^2,$$

where $m = k_1 + \ldots + k_p$.

Because the difficulty to determine the matrices $B_1, \ldots, B_p$, this result is implicit. It is clear that these matrices depend on the eigenvalues of the matrix B, but it is not easy to write down the dependence.

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References

[1] D. Andrica, I. N. Casu, *Lie Groups, Exponential Map, and Geometric Mechanics* (Romanian), Cluj University Press, 2008.

[2] D. Andrica, R.-A. Rohan, *The image of the exponential map and some applications*, Proc. 8th Joint Conference on Mathematics and Computer Science MaCS, Komarno, Slovakia, July 14-17, 2010, 3-14.

[3] T. Bröcker, T. Dieck, *Representations of compact Lie groups*, Springer-Verlag, GTM, vol. 98, New York, 1985.

[4] C. Chevalley, *Theory of Lie groups I*, Princeton Mathematical Series, No.8, Princeton University Press, 1946.

[5] J. Gallier, D. Xu, *Computing exponentials of skew-symmetric matrices and logarithms of orthogonal matrices*, International Journal of Robotics and Automation, 17 (2002), No. 4, 2-11.

[6] J. Gallier, *Notes on Differential Geometry and Lie Groups,* University of Pennsylvannia, 2012.

[7] B. Jütler, *Visualization of moving objects using dual quaternion curves*, Computers and Graphics, 18 (3) (1994), 315-326.

[8] B. Jütler, *An osculating motion with second order contact for spacial Euclidean motions*, Mechanics and Machine Theory, 32 (7) (1997), 843-853.

[9] M.-J. Kim, M.-S. Kim, A. Shin, *A general construction scheme for unit quaternion curves with simple high order derivatives*, Computer Graphics Proceedings, Annual Conference Series, ACM, 1995, 369-376.

[10] M.-J. Kim, M.-S. Kim, A. Shin, *A compact differential formulafor the first derivative of a unit quaternion curve*, Journal of Visualization and Computer Animation, 7 (1996), 43-57.

[11] J. E. Marsden, T. S. Ratiu, *Introduction to Mechanics and Symmetry*, TAM, vol. 17, Springer-Verlag, 1994.

[12] F. C. Park, B. Ravani, *Bézier curves on Riemmanian manifolds and Lie groups with kinematic applications*, Mechanism, Synthesis and Analysis, 70 (1994), 15-21.

[13] F. C. Park, B. Ravani, *Smooth invariant interpolation of rotations*, ACM Transactions of Graphics, 16 (1997), 277-195.

[14] L. I. Piscoran, *Noncommutative differential direct Lie derivative and the algebra of special Euclidean group* $SE(2)$, Creative Math.and Inf., 17 (2008),No.3, 493-498.

[15] E. J. Putzer, *Avoiding the Jordan canonical form in the discussion of linear systems with constant coefficients*, Amer.Math.Monthly, 73 (1966), 2-7.

[16] R.-A. Rohan, *The exponential map and the Euclidean isometries*, Acta Universitatis Apulensis, 31 (2012), 199-204.

[17] F. Warner, *Foundations of Differential Manifolds and Lie Groups*, GTM 94, Springer-Verlag, 1983.

[18] M. Wüstner, *Lie groups with surjective exponential function*, Shaker Verlag, Berichte aus der Mathematik, Aachen 2001.

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