Slopes of Lefschetz fibrations and separating vanishing cycles

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Abstract. In this article we find upper and lower bounds for the slope of genus g hyperelliptic Lefschetz fibrations. We demonstrate the connection between the slope of genus g hyperelliptic Lefschetz fibrations and the number of separating vanishing cycles: we show that $\lambda > 4 - 4/g$ if and only if the fibration contains separating vanishing cycles. We also improve the existing bound on s/n, the ratio of number of separating vanishing cycles to the number of non-separating vanishing cycles, for hyperelliptic Lefschetz fibrations of genus $g \geq 2$. In particular we show that $s < n$ when $g \geq 6$.

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1 Introduction

A Lefschetz fibration over S^2 is a smooth map $f : X \to S^2$ from a compact, connected, oriented, smooth 4- manifold X with the following properties :

1. f has finitely many critical values $q_1, q_2, \ldots, q_k$ in S^2,
2. each of the preimages $f^{-1}(q_1), f^{-1}(q_2), \ldots, f^{-1}(q_k)$ consists of exactly one critical point, say $p_1, p_2, \ldots, p_k$ in X,
3. around each of the points $p_1, p_2, \ldots, p_k$ and $q_1, q_2, \ldots, q_k$ there are local charts, agreeing with the orientations of X and S^2, on which f is locally given as $(z_1, z_2) \mapsto z_1^2 + z_2^2$ in complex coordinates.

The fibers over $B = \{q_1, q_2, \ldots, q_k\}$ are called singular fibers. The points in $S^2 \setminus B$ are called regular values and the fibers over them are called regular fibers. It's a consequence of this definition that the restriction of f to $f^{-1}\left(S^2 \setminus B\right)$ is a fiber bundle over $S^2 \setminus B$ with fibers diffeomorphic to Σ_g, a compact, connected, oriented surface

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of genus g. We also refer to g as the genus of the fibration $f : X \longmapsto S^2$, [8].

The monodromy around each of the singular fibers is given by a positive Dehn twist about a simple closed curve in Σ_g, which is called a *vanishing cycle.* A vanishing cycle is a simple closed curve on regular fibers that collapses to a point on a singular fiber as one gets near a critical point. Choosing a reference point $q_* \in S^2 \setminus B$ one can characterize the fibration f by its *monodromy homomorphism*

$$\psi : \pi_1 \left(S^2 \setminus B\right) \longmapsto \mathcal{M}_g, \tag{1.1}$$

where $\mathcal{M}_g = \pi_0 \left(\mathrm{Diff}^+ \left(\Sigma_g\right)\right)$ is the mapping class group of Σ_g, [1, 5] .

We can assume that each vanishing cycle is homotopically nontrivial because we can eliminate those that are trivial by blowing down the fibration to obtain another one that is *relatively minimal*,[4, 5, 8]. We call a vanishing cycle γ *nonseparating* if $\Sigma_g \setminus \gamma$ is connected. Otherwise we call it *separating*.

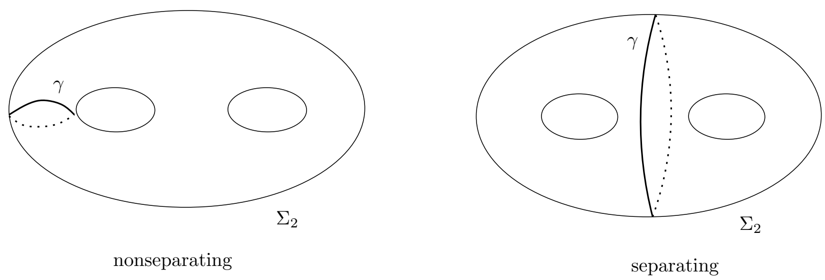

Figure 1: A separating and a nonseparating vanishing cycle on Σ_2

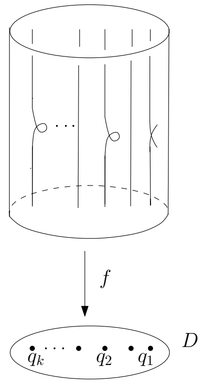

Figure 2: Fibration restricted to $D \subset S^2$

It's possible to arrange the critical points of $f : X \longmapsto S^2$ in such a way that they are distinct and each singular fiber contains only one of them as we assumed in the definition of Lefschetz fibration. Let's now take a disc D within S^2 including the set of critical values B and restrict the fibration to $f^{-1}(D)$, [4]. The crucial information about the fibration lies over this part of S^2, Figure (2).

Let's define now the monodromy homomorphism in (1.1) explicitly.

Let q_* be a regular value in D and choose $\alpha_1, \alpha_2, \ldots, \alpha_k$ in D as the generators of $\pi_1\left(S^2 \setminus B\right)$ as shown in Figure 3. As we go around q_1 along α_1 a smooth fiber bundle over α_1 with fiber Σ_g is formed and the way we identify the fibers over the inital and final points of α_1, both of which are q_*, with a model surface Σ_g gives us a diffeomorphism of Σ_g. The isotopy class of this map is an element of $\mathcal{M}_g$ by definition. It turns out that this mapping class is realized by a positive Dehn twist about a (homotopically) nontrivial simple closed curve in Σ_g, call it γ_1. Let's denote the Dehn twist about γ_1 by D_{γ_1}.

We do the same for α_2 and obtain another element of $\mathcal{M}_g$, call it D_{γ_2}. It's not difficult to see that the map ψ in (1.1) respects composition both in the domain and in the range and we have

$$\psi\left([\alpha_1 * \alpha_2]\right) = \psi([\alpha_1] * [\alpha_2]) = \psi([\alpha_1])\psi([\alpha_2]) = D_{\gamma_2} D_{\gamma_1}. \tag{1.2}$$

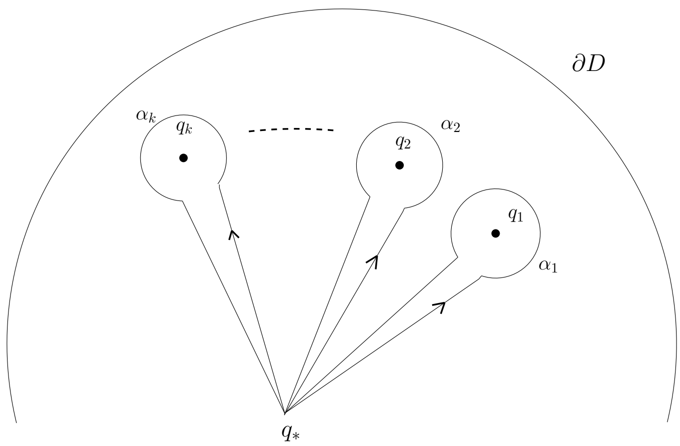

Figure 3: Generators of $\pi_1\left(S^2 \setminus B\right)$

Strictly speaking, the last equality in (1.2) must have $D_{\gamma_1} D_{\gamma_2}$ on the right hand side but it's customary to compose elements of the mapping class group from right to left. Therefore it shouldn't cause a problem as long as we keep that little detail in mind. Continuing the same way until we go around the last critical value q_k and composing along we obtain

$$\psi\left([\alpha_1 * \alpha_2 * \cdots \alpha_k]\right) = D_{\gamma_k} \cdots D_{\gamma_2} D_{\gamma_1}. \tag{1.3}$$

It's clear from Figure 3 that $[\alpha_1 * \alpha_2 * \cdots \alpha_k] = [\partial D]$ in $\pi_1\left(S^2 \setminus B\right)$ and $[\partial D] = \mathrm{Id}$ in $\pi_1\left(S^2 \setminus B\right)$; therefore we have $D_{\gamma_k} \cdots D_{\gamma_2} D_{\gamma_1} = \mathrm{Id}$ in $\mathcal{M}_g$. This shows that a genus g Lefschetz fibration over S^2 gives us a word that is equal to identity in the mapping class group of the fiber Σ_g. The converse is also true: Every such word that is equal to identity in the mapping class group of Σ_g defines a Σ_g fibration over S^2.

It's important to note two things here: First one is, this correspondence is not one-to-one. In order to have a one-to-one correspondence we have to consider equivalence classes of Lefschetz fibrations and words in the mapping class group. On one side we have isomorphism classes of genus g Lefschetz fibrations and on the other side we have equivalence classes of words with positive exponents that are equal to identity in $\mathcal{M}_g$. Two such words in $\mathcal{M}_g$ are equivalent if it's possible to obtain one from the other through cyclic permutation of the twists D_{γ_i}, conjugating the word by an element of $\mathcal{M}_g$, or through elementary transformations such as replacing $D_{\gamma_{i+1}} D_{\gamma_i}$ by $D_{\gamma_{i+1}} \left(D_{\gamma_{i+1}}^{-1} D_{\gamma_i} D_{\gamma_{i+1}} \right)$, [4, 5, 8]. The correspondence between isomorphism classes of Lefschetz fibrations and equivalence classes of words in the mapping class group as defined above would be one-to-one. Secondly the words in the mapping class group must carry only positive exponents. It's due to orientation preserving condition for the charts in the definition of Lefschetz fibrations that we allow only positive Dehn twists. If we relax the definition to allow orientation reversing ones then the 4-manifold X can no longer be shown to be symplectic, [4]. One of the goals that we seek in studying Lefschetz fibrations is to obtain information about symplectic 4-manifolds because Lefschetz fibrations are roughly topological descriptions of symplectic 4- manifolds due to the companion theorems of Donaldson and Gompf, [4, 5]. The reader is referred to [5] for a thorough review of Lefschetz fibrations and symplectic 4-manifolds.

All Lefschetz fibrations throughout this article are assumed to have genus $g \geq 2$ fibering over S^2. It's known that the 4- manifold X in the definition of Lefschetz fibration carries an almost complex structure, [5]; therefore it makes sense to define its holomorphic Euler characteristic χ_h and first Chern class c_1. Let

$$\chi_h := \frac{1}{4}(\sigma + e) \quad \text{and} \quad c_1^2 := 2e + 3\sigma, \tag{1.4}$$

where e is the Euler characteristic and σ is the signature of the 4- manifold X. The *slope* λ_f of a fibration $f : X \longmapsto S^2$ is defined as $\lambda_f := K_f^2 / \chi_f$, where

$$K_f^2 := c_1^2 + 8(g-1) \qquad \text{and} \qquad \chi_f := \chi_h + g - 1. \tag{1.5}$$

It is known that

$$\lambda_f \geq 4 - \frac{4}{g}$$

for a relatively minimal holomorphic genus g Lefschetz fibration, and this bound is sharp since all of the known hyperelliptic Lefschetz fibrations over S^2 with no separating vanishing cycle satisfy $\lambda_f = 4 - 4/g$ [10]. For example consider the words

$$\begin{aligned} \left(c_1 c_2 \cdots c_{2g+1}^2 \cdots c_2 c_1\right)^2 &= 1 \\ \left(c_{2g} \cdots c_2 c_1\right)^{4g+2} &= 1 \\ \left(c_{2g+1} \cdots c_2 c_1\right)^{2g+2} &= 1 \end{aligned} \tag{1.6}$$

in the hyperelliptic mapping class group $\mathcal{H}_g$.

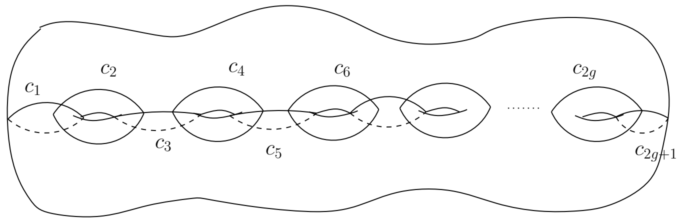

Figure 4: Generators of hyperelliptic mapping class group

Let X_1, X_2, X_3 be the Lefschetz fibrations defined by those words, respectively. The Euler characteristic of each of these 4-manifolds can be computed using the formula

$$e(X) = 2(2 - 2g) + \mu, \tag{1.7}$$

where μ is the number of singular fibers, which is equal to the number of twists, [4]. (Here we don't make a distinction between the simple closed curves c_i and the Dehn twists about them in order to keep the notation simple) Therefore

$$\begin{aligned} e(X_1) &= 2(2-2g) + 4(2g+1) = 4g + 8 \\ e(X_2) &= 2(2-2g) + 2g(4g+2) = 8g^2 + 4 \\ e(X_3) &= 2(2-2g) + (2g+1)(2g+2) = 4g^2 + 2g + 6. \end{aligned}$$

For hyperelliptic Lefschetz fibrations, "local signature" formulas have been computed by Endo, [3]. The "local contribution" of a nonseparating vanishing cycle to the signature is $-\dfrac{g+1}{2g+1}$. Therefore we can compute the signatures of X_1, X_2, X_3 as

$$\begin{aligned} \sigma(X_1) &= -\frac{g+1}{2g+1} \cdot 4(2g+1) = -4(g+1) \\ \sigma(X_2) &= -\frac{g+1}{2g+1} \cdot 2g(4g+2) = -4g(g+1) \\ \sigma(X_3) &= -\frac{g+1}{2g+1} \cdot (2g+1)(2g+2) = -2(g+1)^2. \end{aligned}$$

Using (1.4) we obtain

$$\begin{aligned} (\chi_h(X_1), c_1^2(X_1)) &= (1, 4-4g) \\ (\chi_h(X_2), c_1^2(X_2)) &= (g^2 - g + 1, 4g^2 - 12g + 8) \\ (\chi_h(X_2), c_1^2(X_2)) &= (\frac{1}{2}g^2 - \frac{1}{2}g + 1, 2g^2 - 8g + 6). \end{aligned}$$

When we compute the slopes of each of these three fibrations using (1.5) we see that they are all equal to $4 - 4/g$.

Monden gave examples of nonholomorphic Lefschetz fibrations violating the bound $\lambda_f \geq 4 - 4/g$ by using *inverse lantern substitution* to lower the slope, [7]. The reader is also referred to [7] for a short list of articles where more examples of Lefschetz fibrations proven to be nonholomorphic using various techniques can be found.

We will write λ instead of λ_f throughout this article for simplicity. The connection between λ and the number of separating vanishing cycles in an hyperelliptic Lefschetz fibration seems to be unaccounted for in the literature. In the next section we will prove Theorem 2.2 that reveals this connection.

Let s be the number of separating vanishing cycles and n be the number of those that are non-separating. Recall that a Lefschetz fibration over S^2 can not contain only separating vanishing cycles, (Corollary 8, [8]). Therefore Theorem 2.2 should be understood as a fibration containing a mixture of separating and non-separating vanishing cycles.

An interesting quantity that is worth calculating at this point is the proportion of the number of separating cycles within a fibration, in particular its ratio to the number of non-separating vanishing cycles, $\dfrac{s}{n}$. We do not find any estimates in the literature on this ratio except for

$$\frac{s}{n} \leq 5 \tag{1.8}$$

due to A.Stipsicz, [9], and

$$\frac{s}{n} \leq 5 - 6\frac{g}{n} \tag{1.9}$$

for Lefschetz pencils due to V. Braungardt and D. Kotschick, [2]. We'll assume that $n > 0$ throughout this article. Therefore s/n is always defined. Let $r_g := s/n$ for a Lefschetz fibration of genus g. There isn't enough evidence to justify that the bounds (1.8) and (1.9) could actually be sharp. On the contrary, all of the known examples suggest that r_g should not be too high. In this article we will improve the bound on r_g for hyperelliptic Lefschetz fibrations and prove

Theorem 1.1. *For an hyperelliptic Lefschetz fibration of genus $g \geq 2$ we have*

$$r_g \leq \frac{3g+2}{4\,(g-1)} - \frac{2g+1}{n(g-1)}.$$

We will also prove

Theorem 1.2. *For a relatively minimal holomorphic Lefschetz fibration of genus $g \geq 2$ we have*

$$r_g \leq 3 + \frac{2}{g} - \frac{4}{n} - \frac{2}{ng}.$$

2 Main Section

The signature of a genus g hyperelliptic Lefschetz fibration $X \to S^2$ is given by

$$\sigma = -\frac{g+1}{2g+1}n + \sum_{h=1}^{[g/2]} \frac{4h\,(g-h)\,s_h}{2g+1} - s,$$

where s_h is the number of separating vanishing cycles which separate the surface into two components one with genus $h \leq \left[\frac{g}{2}\right]$ and $s = \sum_{h=1}^{[g/2]} s_h$, [3]. Let

$$x = \sum_{h=1}^{[g/2]} h\,(g-h)\,s_h.$$

The other invariants of X that will be used throughout this article are:
Euler characteristic

$$e = n + s - 4\,(g-1)\,,$$

using (1.7), holomorphic Euler characteristic

$$\begin{aligned}\chi_h = \frac{1}{4}\,(e+\sigma) &= \frac{1}{4}\left(n + s - 4\,(g-1) - \frac{g+1}{2g+1}n + \frac{4x}{2g+1} - s\right)\\ &= \frac{ng+4x}{4\,(2g+1)} - (g-1),\end{aligned} \tag{2.1}$$

and square of the first Chern class c_1^2

$$\begin{aligned}c_1^2 = 2e + 3\sigma &= 2\,(n + s - 4\,(g-1)) + 3\left(-\frac{g+1}{2g+1}n + \frac{4x}{2g+1} - s\right)\\ &= 2n - s - 8\,(g-1) - 3\frac{g+1}{2g+1}n + \frac{12x}{2g+1},\end{aligned}$$

where s is the number of separating vanishing cycles and n is the number of non-separating vanishing cycles.

Lemma 2.1. *$sg \leq 2x$ for $g \geq 2$.*

Proof. It's not difficult to see that $s(g-1) \leq x$ by definition of x and s. Therefore

$$s \leq \frac{x}{g-1} \quad \text{and} \quad sg \leq \frac{gx}{g-1}.$$

The proof follows from the fact that $\frac{g}{g-1} \leq 2$ for $g \geq 2$. □

We will use this lemma to prove the theorem that shows the connection between λ and s :

Theorem 2.2. *A genus g hyperelliptic Lefschetz fibration $X \to S^2$ satisfies $\lambda > 4 - \frac{4}{g}$ if and only if $s \neq 0$, i.e., it contains separating vanishing cycles.*

Proof. The slope λ of the fibration is given as

$$\begin{aligned}\lambda &= \frac{c_1^2 + 8\,(g-1)}{\chi_h + (g-1)} = \frac{2n - s - 3\frac{g+1}{2g+1}n + \frac{12x}{2g+1}}{\frac{ng+4x}{4(2g+1)}}\\ &= 4\frac{n\,(g-1) - s\,(2g+1) + 12x}{ng+4x}\,.\end{aligned} \tag{2.2}$$

Assume $s \neq 0$. Then $x \neq 0$ and we have

$$\begin{aligned} \lambda - (4 - 4/g) &= 4\frac{n(g-1) - s(2g+1) + 12x}{ng + 4x} - 4 + 4/g \\ &= 4\frac{-2sg^2 - sg + 8gx + 4x}{(ng+4x)g} = 4\frac{(2g+1)(4x - sg)}{(ng+4x)g} > 0, \end{aligned}$$

because $4x > sg$ by Lemma 2.1. and all other factors are positive. Therefore $\lambda - (4 - 4/g) = 0$ if and only if $4x = sg$; i.e., if and only if $s = 0$. □

Note that if every fiber is smooth then $\lambda = 12$, otherwise $\lambda < 12$, [10].

Proposition 2.3. *For a genus g Lefschetz fibration the slope is given by*

$$\lambda = 12 - \frac{n+s}{\chi_h + g - 1}. \tag{2.3}$$

Proof. By definition

$$\begin{aligned} \lambda &= \frac{c_1^2 + 8(g-1)}{\chi_h + g - 1} = \frac{12\chi_h - e + 8(g-1)}{\chi_h + g - 1} \\ &= \frac{12\chi_h + 12g - 12 - e - 4(g-1)}{\chi_h + g - 1} \\ &= 12 + \frac{-(n + s - 4(g-1)) - 4(g-1)}{\chi_h + g - 1} \\ &= 12 - \frac{n+s}{\chi_h + g - 1} \end{aligned}$$

□

Remark 2.1. Using $\chi_h = \frac{1}{4}(e + \sigma) = \frac{1}{4}(n + s - 4(g-1) + \sigma)$, we can substitute

$$\sigma + n + s = 4(\chi_h + g - 1) \tag{2.4}$$

in (2.3), in order to obtain

$$\lambda = 12 - 4\frac{n+s}{\sigma + n + s} = 12 - \frac{4}{1 + \frac{\sigma}{n+s}}. \tag{2.5}$$

Solving the first equality for σ gives

$$\sigma = \frac{\lambda - 8}{12 - \lambda}(n+s), \text{ i.e., } \frac{\sigma}{n+s} = \frac{\lambda - 8}{12 - \lambda}, \tag{2.6}$$

which relates the signature of a Lefschetz fibration to the total number of vanishing cycles through scalar multiplication and the *average signature* $\frac{\sigma}{n+s}$ per vanishing cycle to the slope. In particular $\sigma > 0$ corresponds to $\lambda > 8$ and $\sigma < 0$ corresponds to $\lambda < 8$ just as $c_1^2 > 8\chi_h$ and $c_1^2 < 8\chi_h$ correspond to $\sigma > 0$ and $\sigma < 0$, respectively.

Remark 2.2. When $\lambda = 10$ the average signature $\dfrac{\sigma}{n+s}$ must be 1. This can never happen because the signature contribution of each vanishing cycle is either -1, or 0, or +1 and according to the handlebody decomposition of Lefschetz fibrations the first handle attached along the first vanishing cycle, which can be arranged to be a non-separating one by cyclically permuting, will always result in a 4- manifold with 0 signature, [8]. Therefore $\frac{\sigma}{n+s} < 1$. This proves

Proposition 2.4. *The slope of a Lefschetz fibration satisfies* $\lambda < 10$.

Corollary 2.5. *A genus g Lefschetz fibration satisfies the bound* $c_1^2 < 10\chi_h + 2g - 2$.

More is true if the Lefschetz fibration is hyperelliptic:

Proposition 2.6. *For a genus g hyperelliptic Lefschetz fibration we have*

$$\lambda \leq 10 - \frac{2+s}{\chi_h + g - 1}. \tag{2.7}$$

Proof. First we estimate χ_h as

$$\begin{aligned}
\chi_h &= \frac{1}{4}(\sigma + e) = \frac{1}{4}\left(-\frac{g+1}{2g+1}n + \sum_{h=1}^{[g/2]} \frac{4h(g-h)s_h}{2g+1} - s + n + s - 4(g-1)\right) \\
&\leq \frac{1}{4}\left(\frac{ng}{2g+1} + \frac{4\frac{g}{2}\left(g - \frac{g}{2}\right)s}{2g+1} - 4(g-1)\right) \\
&= \frac{1}{4}\frac{ng}{2g+1} + \frac{1}{4}\frac{sg^2}{2g+1} - (g-1) := M, \qquad (2.8)
\end{aligned}$$

using the fact that $h(g-h) \leq \frac{g}{2}(g - \frac{g}{2})$ and $\sum_{h=1}^{[g/2]} s_h = s$. Now, use this to write Euler characteristic as

$$\begin{aligned}
e &= n + s - 4(g-1) \\
&= \frac{4(2g+1)}{g}\left(\frac{1}{4}\frac{ng}{2g+1} + \frac{1}{4}\frac{sg^2}{2g+1} - (g-1)\right) + (1-g)s + 4g - \frac{4}{g} \\
&= \frac{4(2g+1)}{g}M + (1-g)s + 4g - \frac{4}{g}. \qquad (2.9)
\end{aligned}$$

The estimate

$$\sigma \leq n - s - 4 = n + s - 4(g-1) - 2s + 4(g-2) = e - 2s + 4(g-2),$$

(Corollary 9, [8]), can be used to write

$$\chi_h = \frac{1}{4}(\sigma + e) \leq \frac{1}{4}(e - 2s + 4(g-2) + e) = \frac{1}{2}e - \frac{1}{2}s + g - 2, \tag{2.10}$$

and using (2.9) we obtain

$$\begin{aligned}
\chi_h &\leq \frac{1}{2}\left(\frac{4(2g+1)}{g}M + (1-g)s + 4g - \frac{4}{g}\right) - \frac{1}{2}s + g - 2 \\
&= 2\frac{2g+1}{g}M - \frac{1}{2}sg + 3g - 2 - \frac{2}{g}.
\end{aligned}$$

We will solve this for sg

$$sg \leq 4\frac{2g+1}{g}M - 2\chi_h + 6g - 4 - \frac{4}{g}$$

and use it in estimating

$$\begin{aligned} c_1^2 &= 12\chi_h - e = 12\chi_h - \left(\frac{4\left(2g+1\right)}{g}M + \left(1-g\right)s + 4g - \frac{4}{g}\right) \\ &= 12\chi_h - 4\frac{2g+1}{g}M + \left(g-1\right)s - 4g + \frac{4}{g} \\ &\leq 12\chi_h - 4\frac{2g+1}{g}M + 4\frac{2g+1}{g}M - 2\chi_h + 6g - 4 - \frac{4}{g} - s - 4g + \frac{4}{g} \\ &= 10\chi_h + 2g - 4 - s. \end{aligned}$$

Now,

$$\lambda = \frac{c_1^2 + 8\left(g-1\right)}{\chi_h + g - 1} \leq \frac{10\chi_h + 2g - 4 - s + 8\left(g-1\right)}{\chi_h + g - 1} = \frac{10\chi_h + 10g - 10 - 2 - s}{\chi_h + g - 1},$$

and we have

$$\lambda \leq 10 - \frac{2+s}{\chi_h + g - 1}.$$

□

For hyperelliptic Lefschetz fibrations we can do even better:

Proposition 2.7. *The slope of an hyperelliptic genus g Lefschetz fibration satisfies*

$$4\frac{g-1}{g} + \frac{4s\left(2g+1\right)\left(3g-4\right)}{\left(ng + 4s\left(g-1\right)\right)g} \leq \lambda \leq 10 - 2\frac{2+s}{n-2}. \tag{2.11}$$

Proof. The signature satisfies the bound

$$\sigma = -\frac{g+1}{2g+1}n + \frac{4x}{2g+1} - s \geq -\frac{g+1}{2g+1}n + \frac{4s(g-1)}{2g+1} - s = -\frac{g+1}{2g+1}n + \frac{2g-5}{2g+1}s,$$

because $s(g-1) \leq x$ by definition of x and s. Now, using (2.6) we can write

$$-\frac{g+1}{2g+1}n + \frac{2g-5}{2g+1}s \leq -\frac{8-\lambda}{12-\lambda}\left(n+s\right),$$

and solving this for λ gives the first inequality. To prove the second inequality we begin with the fact that $\chi_h + g - 1 > 0$, as we can see it from (2.4) because $|\sigma| < n+s$ (see Remark 2.2 above or Corollary 9 in [8]). Also using (2.10) we can write

$$\chi_h \leq \frac{1}{2}e - \frac{1}{2}s + g - 2 = \frac{1}{2}\left(n + s - 4\left(g-1\right)\right) - \frac{1}{2}s + g - 2 = \frac{1}{2}n - g,$$

which can be rewritten as

$$-\frac{1}{\chi_h + g - 1} \leq \frac{-2}{n-2}.$$

Now, adding 10 to both sides after multiplying by $2+s$ proves the second inequality thanks to Proposition 2.6. □

Remark 2.3. We wrote (2.11) in that particular form instead of simplifying it in order to emphasize the fact that it is another proof for Theorem 2.2 and that $4-\frac{4}{g} \leq \lambda < 10$ for hyperelliptic Lefschetz fibrations.

Proof. (of Theorem 1.1) Using the bound $\sigma \leq n-s-4$ (Corollary 9, [8]) we get $\frac{1}{4}(n-s-\sigma) \geq 1$. Then

$$\frac{1}{4}(n-s-\sigma) = \frac{1}{4}\left(n-s-\left(-\frac{g+1}{2g+1}n+\frac{4x}{2g+1}-s\right)\right) = \frac{1}{4}\frac{(3g+2)\,n-4x}{2g+1}$$

gives

$$1 \leq \frac{1}{4}\frac{(3g+2)\,n-4x}{2g+1}, \quad \text{i.e.,} \quad x \leq \frac{1}{4}n\,(3g+2)-(2g+1)\,.$$

Using the estimate $(g-1)s \leq x$ one more time, we have

$$(g-1)\,s \leq \frac{1}{4}n\,(3g+2)-(2g+1)\,.$$

Dividing through by $n(g-1)$ gives

$$r = \frac{s}{n} \leq \frac{3g+2}{4\,(g-1)} - \frac{2g+1}{n\,(g-1)}.$$

□

Corollary 2.8. *For an hyperelliptic Lefschetz fibration of genus $g \geq 6$ we have $s < n$.*

Remark 2.4. One can prove Theorem 1.1 by solving

$$4\frac{g-1}{g} + \frac{4s\,(2g+1)\,(3g-4)}{(ng+4s\,(g-1))\,g} \leq 10-2\frac{2+s}{n-2}$$

for $\frac{s}{n}$ as well, (2.11). Also, solving

$$\lambda = 12-4\frac{n+s}{n+s+\sigma} \leq 10-2\frac{2+s}{n-2}$$

for σ results in $\sigma \leq n-s-4$, which is another proof for Proposition 2.6 thanks to (Corollary 9, [8]). Finally, solving

$$4\frac{g-1}{g} \leq \lambda = 12-\frac{4}{1+\frac{\sigma}{n+s}} \quad \text{for} \quad \frac{\sigma}{n+s} \quad \text{gives}$$

$$\frac{\sigma}{n+s} \geq -\frac{g+1}{2g+1}, \tag{2.12}$$

which shows that the average signature per vanishing cycle is at least $-\frac{g+1}{2g+1}$ for relatively minimal holomorphic Lefschetz fibrations. For hyperelliptic Lefschetz fibrations equality holds when $s=0$, [3], and it's strict inequality when $s>0$ by virtue of Theorem 2.2.

Note that violating the bound $\lambda \geq 4 - 4/g$ is equivalent to violating the average signature bound in (2.12). In other words the average signature bound (2.12) can be used as a simple tool to prove that a Lefschetz fibration is nonholomorphic. Xiao conjectured that $\lambda > 4 - 4/g$ for non-hyperelliptic holomorphic Lefschetz fibrations which was proved for some low genus by Konno, [6].

Now we will prove Theorem 1.2 using (2.12).

Proof. (of Theorem 1.2) By Corollary 7 in [8] we have $\sigma \leq n - s$. Since we assume $n > 0$ we can conclude that $\sigma \leq n - s - 2$, [9]. Combining this with (2.12) we get

$$-\frac{g+1}{2g+1} \leq \frac{\sigma}{n+s} \leq \frac{n-s-2}{n+s}.$$

Solving for s/n yields $r_g \leq 3 + \frac{2}{g} - \frac{4}{n} - \frac{2}{gn}$. □

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