On the products of certain toric folded symplectic manifolds

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Abstract. An origami manifold is a smooth, compact, and connected manifold equipped with a closed 2-form which is symplectic except on a hypersurface such that the restriction of the 2-form to the hypersurface has the maximal rank and such that the kernel fibrates with oriented circle fibers over a compact symplectic base. On the other hand, a toric origami manifold is an origami manifold with an effective Hamiltonian torus $\mathbb{T}^n$-action, where $n = \frac{1}{2}\dim M$. The product of two toric origami manifolds may not be a toric origami manifold with the product action and product form. The aim of this paper is to show that, whenever the product of two orientable toric origami manifolds with the product action admits an orientable toric origami form, either one of two toric origami manifolds actually should be a toric symplectic manifold. In addition, we also discuss some interesting consequences of our main results.

M.S.C. 2010: 7S15, 53D20.
Key words: toric symplectic manifold, toric origami manifold, Delzant polytope, toric origami template, product of two toric origami manifolds.

1 Introduction

An *origami manifold* M of dimension $2n$ is a smooth, compact, and connected manifold equipped with a closed 2-form ω which is symplectic except on a hypersurface Z, where the restriction $\omega|_Z$ of ω to Z satisfies $(\omega|_Z)^{n-1} \neq 0$ and, in addition, Z is a principal S^1-bundle over a compact symplectic base with oriented circles fibers generated by the kernel of $\omega|_Z$. These manifolds form a special class of *folded symplectic manifolds*, where we simply require the condition that the closed 2-form ω be symplectic on $M\backslash Z$ and that $(\omega|_Z)^{n-1} \neq 0$. In this case, the codimension one embedded submanifold Z which is closed in M is called a *folding hypersurface* or simply *fold*, and admits a *null foliation* which consists of the line fields generated by the one-dimensional kernel of $\omega|_Z$ (see [2], [3], [8], and [9] for more details).

For an oriented origami manifold M with fold Z, there is an operation, called the *unfolding*, which can convert M into a (disconnected) compact symplectic manifold

Balkan Journal of Geometry and Its Applications, Vol.20, No.2, 2015, pp. 41-48.

M_0 by taking the closures of the connected components of $M \backslash Z$ and identifying boundary points on the same leaf of the null foliation. In fact, M_0 has been obtained by the so-called symplectic cutting techniques. The resulting manifold M_0 is called the *symplectic cut space*, while each of its connected components is called a *symplectic cut piece*.

Let B_0 be the codimension two submanifold of M_0 which has been obtained by identifying the boundary points in the same circle leaf of the foliation. Then it is easy to see that B_0 is a symplectic submanifold of M_0 and that, in fact, we can recover the original origami manifold M by taking the fiber connected sum (or called a *radial blow-up* in [3], Section 2.3) of the connected components of M_0 along the symplectic submanifold B_0. However, this fiber connected sum operation does not preserve the symplectic condition of M_0, since the normal Euler classes of the normal bundles of B_0 are not opposite (refer to [6], Section 1 for more details).

A *toric symplectic manifold* of dimension $2n$ is a compact connected symplectic manifold with an effective Hamiltonian torus $\mathbb{T}^n$-action. It is well known by a result of Delzant in [5] that there is a one-to-one correspondence between compact toric symplectic manifolds up to equivariant symplectomorphisms and Delzant polytopes up to affine equivalence. As in toric symplectic manifolds, a *toric origami manifold* of dimension $2n$ is an origami manifold with an effective Hamiltonian torus $\mathbb{T}^n$-action. In this case, the circle fibers of the fold are orbits of a circle subgroup of the torus $\mathbb{T}^n$, and the moment map μ induces moment maps μ_i on each connected component of a toric symplectic manifold M_0. The image of each μ_i is a Delzant polytope (or moment polytope).

An *origami template* is a collection of Delzant polytopes with certain folding data consisting of the template graph G and a pair (Ψ_V, Ψ_E) of maps $\Psi_V : V \to \mathcal{P}$ and $\Psi_E : E \to \mathcal{E}$ satisfying certain compatibility conditions, where V (resp. E) denotes the vertex (resp. edge) set and $\mathcal{P}$ (resp. $\mathcal{E}$) denotes the set of all Delzant polytopes (resp. the set of the facets of elements of $\mathcal{P}$) (see [8], Definition 1.6 for more details). The polytopes in the image of Ψ_V are the Delzant polytopes of the symplectic cut pieces, and for each edge $e \in E$ the set $\Psi_E(e)$ is a facet, called the *fold facet*, of the polytopes corresponding to the end vertices of e.

Analogously to toric symplectic manifolds, it has been shown in [3] that toric origami manifolds correspond bijectively to origami templates. An toric origami template is called *acyclic* if the graph G is acyclic, i.e., G is a tree. Note that for an acyclic toric origami template its corresponding toric origami manifold always has an isolated fixed point. It will be also important to recall that every compact toric symplectic manifold always has an isolated fixed point.

In the paper [8], Holm and Pires studied the topology of acyclic toric origami manifolds. As consequences, among other things, they proved that the cohomology of such a toric origami manifold is concentrated in even degrees and that the equivariant cohomology satisfies the GKM-condition. Quite recently, in the paper [9] they also determined the fundamental groups and some Betti numbers of general toric origami manifolds including the non-simply connected case. In the paper [11], Masuda and Park studied the toric origami manifolds earlier than [9], and associated toric origami manifolds to multi-fans introduced by Masuda and Hattori in the papers [7] and [10]. By using the notion of a multi-fan, they also studied the fundamental group of toric origami manifolds in the paper [11] (see also [1] for more recent results).

With these understood, the aim of this paper is to study some general properties of the products of two toric origami manifolds. To be more precise, let (M_1, ω_1) be a toric origami manifold with non-empty fold Z_1, and let (M_2, ω_2) be a toric symplectic manifold. Then the product manifold

$$(M, \omega_1 + \omega_2) := (M_1, \omega_1) \times (M_2, \omega_2)$$

is again a toric origami manifold with the product torus action whose fold is $Z_1 \times M_2$. However, it is easy to see that the product of two toric origami manifolds with non-empty folds Z_1 and Z_2 is never a toric origami manifold with the product action and product form $\omega_1 + \omega_2$, since $\omega_1 + \omega_2$ does not have maximal rank at a point in $Z_1 \times Z_2$. In view of this observation, Masuda and Park asked if either one of two toric origami manifolds is actually a toric symplectic manifold, whenever the product of two orientable toric origami manifolds with the product action admits an orientable toric origami form. It is not difficult to expect that the resolution of this question will significantly clarify the structure of the products of two toric origami manifolds, as we show in Section 2.

Our main result is to affirmatively answer their question ([11], Section 5, Problem), as follows.

Theorem 1.1. *Let M_1 (resp. M_2) be an orientable toric origami manifold of dimension $2n_1$ (resp. $2n_2$) with respect to the torus action of $\mathbb{T}_1$ (resp. $\mathbb{T}_2$). Assume that $M_1 \times M_2$ admits an orientable toric origami form with respect to the product action of $\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2$. Then either one of M_1 and M_2 always admits a toric symplectic form.*

This paper is organized as follows. In Section 2, we give a proof of Theorem 1.1. In Section 3, we also provide some interesting consequences of Theorem 1.1. Especially, we prove that several interesting manifolds with the half-dimensional torus action including some known examples do not admit any toric origami form. It seems to us that, compared to the known proofs, our proofs are not only much simpler but also more transparent, though.

Throughout this paper, for simplicity all toric origami manifolds are orientable, closed (compact without boundary), connected, and smooth, unless stated otherwise.

2 Proof of Theorem 1.1

The aim of this section is to give a proof of Theorem 1.1, and we will provide its interesting consequences in Section 3.

First, we begin with stating the following theorem which is the contrapositive of Theorem 1.1, as follows.

Theorem 2.1. *Assume that M_1 (resp. M_2) of dimension $2n_1$ (resp. $2n_2$) does not admit any toric symplectic form with respect to the torus action of $\mathbb{T}_1$ (resp. $\mathbb{T}_2$). Then $M_1 \times M_2$ of dimension $2n = 2(n_1 + n_2)$ cannot admit any orientable toric origami form with respect to the product action of $\mathbb{T} := \mathbb{T}_1 \times \mathbb{T}_2$, either.*

Proof of Theorem 1.1 and Theorem 2.1. To prove it, let Z be a non-empty fold of the toric origami manifold $M_1 \times M_2$ of dimension $2n = 2(n_1 + n_2)$ with a toric origami form

Ω. For simplicity, let us assume without loss of generality that Z is connected. By the definition of the fold Z, there is a principal S^1-bundle $Z \to B$ with the orientable circle fibers over a closed symplectic manifold B of dimension $2n-2$. Here the circle fibers are orbits for a circle subgroup S^1 of the torus $\mathbb{T}$, since the action of $\mathbb{T}$ is toric (see [3], Theorem 3.2 or Corollary 3.7). More precisely, we have the following lemma.

Lemma 2.2. *The circle subgroup S^1 of the torus $\mathbb{T}$ inducing the circle orbits of Z is entirely contained in either $\mathbb{T}_1$ or $\mathbb{T}_2$.*

Proof. To prove it, as in Section 1, we first convert $M_1 \times M_2$ into a (disconnected) compact symplectic manifold $(M_1 \times M_2)_0$ by taking the closures of the connected components of $M_1 \times M_2 \backslash Z$ and identifying boundary points on the same leaf of the null foliation. As before, let B_0 be the codimension two submanifold of $(M_1 \times M_2)_0$ which has been obtained by identifying the boundary points in the same circle leaf of the foliation. Then it is important to observe that $B_0 \subset (M_1 \times M_2)_0$ is fixed pointwise under the circle subgroup S^1 of the torus $\mathbb{T}$. But, since the action of the torus $\mathbb{T}$ on $M_1 \times M_2$ is a product one and the $\mathbb{T}_i$-action on each M_i is locally standard ([8], Lemma 5.1), this implies that the circle subgroup S^1 of the torus $\mathbb{T}$ inducing the circle orbits should be contained in either one of $\mathbb{T}_1$ and $\mathbb{T}_2$. Indeed, in other case it is easy to see that we cannot have such a real codimension two submanifold B_0 in $(M_1 \times M_2)_0$ fixed pointwise under a circle subgroup of $\mathbb{T}$. This completes the proof of Lemma 2.2. $\square$

So, from now on we assume without loss of generality that the circle subgroup S^1 is contained in $\mathbb{T}_1$.

For $i = 1, 2$, let $\pi_i : M_1 \times M_2 \to M_i$ denote the natural projection on the i-th factor. Then take any p_0 in M_1 such that (p_0, q_0) lies in Z for some $q_0 \in M_2$. Let $\mathfrak{N}$ be the set of all points q in M_2 containing q_0 for which $\Omega_{(p_0,q)}$ has maximal rank $2n-2$. By the choice of p_0, $\mathfrak{N}$ is a *non-empty* subset of $\pi_2(Z) \subset M_2$. Then, we have the following lemma.

Lemma 2.3. *The set $\mathfrak{N}$ actually coincides with M_2. In particular, $\mathfrak{N}$ also coincides with $\pi_2(Z)$.*

Proof. To prove it, we first claim that $\mathfrak{N}$ is actually open and closed in M_2. To see it, let q be an element of $\mathfrak{N}$. Then it follows from the definition of $\mathfrak{N}$ that $\Omega_{(p_0,q)}$ has maximal rank $2n-2$. By Lemma 2.2, note also that the circle subgroup S^1 of the torus $\mathbb{T}$ inducing the circle orbits of Z is entirely contained in either $\mathbb{T}_1$ or $\mathbb{T}_2$.

Since the action of the torus $\mathbb{T}$ on $M_1 \times M_2$ is a product one and the $\mathbb{T}_i$-action on M_i is locally standard, it follows from an analogue of Darboux's theorem for folded symplectic forms ([4]) and Lemma 2.2 that there is an invariant open subset V_1 of a coordinate chart

$$(U_1, (x_1, y_1, \ldots, x_{n_1}, y_{n_1}))$$

centered at the origin in $\mathbb{R}^{2n_1}$, equipped with the obvious standard torus action (resp. an invariant open subset V_2 of a coordinate chart

$$(U_2, (x_{n_1+1}, y_{n_1+1}, \ldots, x_{n_1+n_2}, y_{n_1+n_2}))$$

centered at the origin in $\mathbb{R}^{2n_2}$) which is equivariantly isomorphic to an invariant open neighborhood of p_0 (resp. q) with respect to the standard torus action of $\mathbb{T}_1$ (resp. $\mathbb{T}_2$) such that over $V_1 \times V_2$ we have

$$\begin{aligned}\Omega &= x_1 dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \ldots + dx_{n_1} \wedge dy_{n_1} \\ &\quad + dx_{n_1+1} \wedge dy_{n_1+1} + \ldots + dx_{n_1+n_2} \wedge dy_{n_1+n_2}.\end{aligned} \tag{2.1}$$

Here we need to remark that the circle S^1 acts on $(\mathbb{R}^2, (x_1, y_1))$ by the rule

$$(\theta, (x_1, y_1)) \mapsto (x_1, y_1 + \theta), \quad y_1, \theta \in \mathbb{R}/\mathbb{Z}.$$

Since the fold Z is locally given by $x_1 = 0$ by (2.1), this implies that there should be an invariant open neighborhood W of q in M_2 such that Ω over $\{p_0\} \times W$ has maximal rank $2n - 2$. So $\mathfrak{N}$ should be open in M_2.

On the other hand, since Z is a closed subset of a compact product manifold $M_1 \times M_2$, Z is compact and π_2 is continuous, $\pi_2(Z)$ is also compact. In particular, this implies that $\pi_2(Z)$ is a closed subset of M_2. Since $\mathfrak{N} \subset \pi_2(Z)$, the closure $\bar{\mathfrak{N}}$ of $\mathfrak{N}$ is a subset of the closure $\overline{\pi_2(Z)}$ of $\pi_2(Z)$ that is equal to $\pi_2(Z)$. So, if we take an element q of $\bar{\mathfrak{N}}$, then there is a sequence $\{q_n\}_{n=1}^{\infty}$ in $\mathfrak{N}$ such that q_n converges to q, as n goes to infinity. That is, we can obtain a sequence $\{(p_0, q_n)\}_{n=1}^{\infty}$ in Z such that (p_0, q_n) converges to (p_0, q) in $M_1 \times M_2$, as n goes to infinity. Since Z is compact and so complete, actually (p_0, q) should be an element of Z, so that $\Omega_{(p_0,q)}$ has maximal rank $2n - 2$. This means that q is an element of $\mathfrak{N}$. That is, $\mathfrak{N}$ is always closed in M_2.

Since M_2 is connected and $\mathfrak{N}$ is non-empty, this implies that $\mathfrak{N}$ should be equal to M_2, as desired. □

Now, we are ready to finish the proof of Theorem 1.1. That is, let $k : M_2 \hookrightarrow M_1 \times M_2$ be the inclusion given by $q \mapsto (p_0, q)$, where, as in Lemma 2.3, p_0 is any element of M_1 such that (p_0, q_0) lies in Z for some $q_0 \in M_2$. Then it follows from Lemma 2.3 that $\Omega_{(p_0,q)}$ has maximal rank $2n - 2$ for all $q \in M_2$. This implies that $\{p_0\} \times M_2$ is contained in the fold Z. Moreover, we have the following lemma.

Lemma 2.4. *The fold Z is of the form $Z_1 \times M_2$, where Z_1 is a closed codimension one submanifold of M_1, and so Z is a principal S^1-bundle over $B_1 \times M_2 = B$, where B_1 is a closed symplectic submanifold of dimension $2n_1 - 2$ of the symplectic cut space of M_1.*

Proof. To prove it, note that the fold Z for an orientable toric origami product manifold $M_1 \times M_2$ with respect to the product action of the torus $\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2$ is of the form $Z_1 \times M_2$, where Z_1 is a codimension one submanifold of M_1. Since the closure $\overline{Z_1 \times M_2}$ of $Z_1 \times M_2$ is equal to $\overline{Z_1} \times M_2$ and Z is closed in $M_1 \times M_2$, $\overline{Z_1}$ should be same as Z_1. That is, Z_1 is, in fact, closed in M_1.

Recall now that the circle subgroup S^1 of the torus $\mathbb{T}$ inducing the circle orbits in Z is entirely contained in $\mathbb{T}_1$. So Z_1 is actually a principal S^1-bundle over a closed symplectic codimension two submanifold B_1 of the symplectic cut space of M_1. So we can obtain a principal S^1-bundle $Z \to B_1 \times M_2 = B$. This completes the proof of Lemma 2.4. □

Finally, since the fold Z is of the form $Z_1 \times M_2$ by Lemma 2.4, it follows again from an analogue of Darboux's theorem for folded symplectic forms that the pullback closed 2-form $k^*\Omega$ is non-degenerate over all of M_2. Moreover, by the definition of a toric origami form Ω on $M_1 \times M_2$, $k^*\Omega$ is preserved by the torus action of $\mathbb{T}_2$. Thus, the pullback closed 2-form $k^*\Omega$ defines a toric symplectic form on M_2. This completes the proof of Theorem 1.1. □

Remark 2.1. The proof of Theorem 1.1 shows that the theorem also holds under the assumption that both M_1 (resp. M_2) are orientable manifolds equipped with a locally standard torus action of $\mathbb{T}_1$ (resp. $\mathbb{T}_2$), where $\dim \mathbb{T}_i = \frac{1}{2}\dim M_i$ $(i = 1, 2)$.

3 Some Applications

The aim of this section is to collect some interesting consequences of Theorem 1.1 (or Theorem 2.1) and its proof.

To do so, we first show that we can determine whether or not many product manifolds admits a toric origami form. For example, let S^4 be the standard 4-sphere with the standard $\mathbb{T}^2$-action. Then the product $S^4 \times S^4$ does not admit a toric origami form, since S^4 does not admit an orientable toric symplectic form.

Since S^{2n} $(n \geq 2)$ does not admit any symplectic form, it is easy to see that $S^{2n} \times S^2$ with the standard product $\mathbb{T}^{n+1}$-action cannot admit any toric symplectic form, either (e.g., apply the Künneth formula for de Rham cohomology to the product manifold $S^{2n} \times S^2$, and then notice that any possible symplectic form on $S^{2n} \times S^2$ should be represented as a sum of an exact form on S^{2n_i} and a symplectic form on S^2. This would yield a contradiction to the Stokes' theorem). So, if we apply Theorem 2.1 to the product manifold $\prod_{i=1}^k S^{2n_i}$ with the standard product action of $\prod_{i=1}^k \mathbb{T}^{n_i}$, it is easy to reprove the following result in [11], Theorem 5.2 in a different way, as follows.

Proposition 3.1. *Let S^{2n_i} $(n_i \geq 1)$ be the standard unit sphere in $\mathbb{R}^{2n_i+1}$ with the standard action of $\mathbb{T}^{n_i}$. Then the product manifold $\prod_{i=1}^k S^{2n_i}$ with the standard product action of $\prod_{i=1}^k \mathbb{T}^{n_i}$ admits a toric origami form if and only if all of the n_i except for one index i are equal to one.*

Proof. It suffices to prove "only if" part, since "if" part is clearly true. To do so, assume that $\prod_{i=1}^k S^{2n_i}$ with the standard product action of $\prod_{i=1}^k \mathbb{T}^{n_i}$ admits a toric origami form, and suppose without loss of generality that both n_1 and n_2 are not equal to one. Then, we apply Theorem 2.1 to $S^{2n_1} \times \prod_{i=2}^k S^{2n_i}$ with the standard product torus action. As already observed above, it is easy to show that $\prod_{i=2}^k S^{2n_i}$ with the standard product torus action cannot admit a toric symplectic form. Since S^{2n_1} $(n_1 \geq 2)$ does not admit any toric symplectic form with respect to the standard $\mathbb{T}^{n_1}$-action, Theorem 2.1 implies that $\prod_{i=1}^k S^{2n_i}$ with the standard product torus action cannot admit a toric symplectic form, either. This is a contradiction, which completes the proof of Proposition 3.1. □

From now on, we shall denote by T^m the compact torus $(S^1)^m$ in $\mathbb{C}^m$ with the standard torus action of $\mathbb{T}^m$. As briefly mentioned in Section 1, recall that the orbit space of a toric origami manifold is realized as the topological space obtained by gluing the Delzant polytopes along the fold facets as specified by a certain rule. In fact, a toric origami manifold can be obtained as the equivariant fiber connected sum of toric symplectic manifolds along the symplectic codimension two submanifolds associated to the fold facets. If we combine this fact with Lemma 2.4, it is easy to show that $S^{2n} \times T^2$ with respect to the standard product torus action does not admit any toric origami form (see [11], Example 2.1 and Proposition 5.3 for a different proof). In fact, we have the following more general result.

Theorem 3.2. *The product manifold* $S^{2n} \times T^{2m}$ ($n \geq 2, m \geq 1$ *or* $n = 1, m \geq 2$) *with respect to the standard product torus action does not admit any toric origami form.*

Proof. We prove the theorem by contradiction. So, suppose that $S^{2n} \times T^{2m}$ with respect to the standard product torus action admits a toric origami form. If $n \geq 2$ and $m = 1$, then it follows from Lemma 2.4 that the fold Z would be of the form $S^{2n-1} \times T^2$, where S^{2n-1} is regarded as the equator of S^{2n}. Thus $S^{2n} \times T^2$ should be obtained by taking the equivariant fiber connected sum of two copies of a toric symplectic product manifold $\mathbb{CP}^n \times T^2$ along the symplectic submanifold $\mathbb{CP}^{n-1} \times T^2$. However, it is easy to see that $\mathbb{CP}^n \times T^2$ with respect to the standard product torus action cannot admit any toric symplectic form, since the $\mathbb{T}^1$-action on T^2 is free and so does not have any fixed point (toric symplectic manifolds of dimension $2n$ always have at least $n+1$ fixed points. This can be seen for example by noting that fixed points correspond to vertcies of the moment polytope, and an n-dimensional polytope has at least $n+1$ vertices). In fact, this case ($n \geq 2$ and $m = 1$) can be proved by the same technique as in the second case ($n \geq 1$ and $m \geq 2$) below.

On the other hand, if $n \geq 1$ and $m \geq 2$, then we want to make use of Theorem 1.1 and its Remark 2.1. Indeed, for this particular case we apply Theorem 1.1 and its Remark 2.1 to the product manifold $(S^{2n} \times T^{2(m-1)}) \times T^2$. Then, since our torus actions are all locally standard, either $S^{2n} \times T^{2(m-1)}$ or T^2 should admit a toric symplectic form by Remark 2.1. However, this is not the case, since the standard torus actions on $S^{2n} \times T^{2(m-1)}$ and T^2 do not have any fixed point. This completes the proof of Theorem 3.2. $\square$

Recall that $S^2 \times T^2$ with the standard product torus action does admit a toric origami form whose N/N_Δ is isomorphic to $\mathbb{Z}$, where $N = H_2(B\mathbb{T}^2; \mathbb{Z})$ is a lattice and N_Δ is the sublattice of N generated by primitive vectors in the one-dimensional cones in the multi-fan (see [11], Section 3 and [9], Section 2, Table 2.17).

Finally, we close this section with one more example. By applying Theorem 1.1 and Remark 2.1 as in the proof of Theorem 3.2, we can show that T^{2m} ($m \geq 2$) does not admit any orientable toric origami form, while T^2 does ([8], Figure 2.4 or this can be seen by taking equivariant fiber connected sum of two copies of a toric symplectic manifold S^2 along the north and south poles). This can be also seen by looking at the orbit space of a toric origami manifold. Recall that the orbit space of a toric origami manifold is homotopy equivalent to a bouquet of S^1's. In case of T^{2m} with the standard torus action, the orbit space is just the $(2m-1)$-dimensional torus which is not homotopy equivalent to a bouquet of circles (see [11], Section 4).

Acknowledgements. This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2014R1A1A2054683).

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