Beil metrics in complex Finsler geometry

A. Szász

Abstract. In this paper we continue the study of the complex Beil metrics, in complex Finsler geometry, [18]. Primarily, we determine the main geometric objects corresponding to these metrics, e.g. the Chern-Finsler complex non-linear connection, the Chern-Finsler complex linear connection and the holomorphic curvature. We focus our study on the cases when a complex Finsler space, endowed with a complex Beil metric, becomes weakly Kähler and Kähler. Also, our study proves that a given complex Finsler metric is projectively related to its associated complex Beil metric. As an application of this theory, we set the variational problem of the complex Beil metric constructed with the weakly gravitational metric. In this case we find the Chern-Finsler complex non-linear connection by using another approach.

M.S.C. 2010: 53B40, 53C60.
Key words: complex Finsler space, Beil metric, projectively related complex Finsler metrics, weakly gravitational metric.

1 Introduction

The Beil metrics were introduced and studied by R. G. Beil in [9, 10] to develop a unified field theory. Beil's idea was, that if the connection contains the field, then the metric itself should contain the electromagnetic potential vectors. This is a natural extension of general relativity since the gravitational potentials are also part of the metric. The importance of this type of metric has been pointed out in many studies, [13, 19, 11, 14, 15], etc, but the configuration was given by M. Anastasiei and H. Shimada in [7].

The study of the complex version of the Beil metric is initiated by us in [18], considering a generalized complex Lagrange metric

$$^*g_{i\bar{j}} = g_{i\bar{j}} + \sigma B_i B_{\bar{j}}, \tag{1.1}$$

where $g_{i\bar{j}}$ is the fundamental metric tensor of a complex Finsler space, σ is a real valued function, and B_i is a covector. This metric is called the complex Beil metric, if $1 + \sigma B_i B^i \neq 0$. Here the evolution of this metric, i.e. the weakly regular and the

Balkan Journal of Geometry and Its Applications, Vol.20, No.2, 2015, pp. 72-83.

regular cases, is studied, and the situation when the metric is a complex Lagrange one is exemplified.

The aim of the present paper is to give an approach of the complex Beil metric in complex Finsler geometry. After a short introduction in complex Finsler geometry (Section 2), in Section 3 the necessary and sufficient conditions under which the tensor from (1.1) is the fundamental metric tensor of a complex Finsler space, (Theorem 3.1), are pointed out. As a result, we can construct the geometry related to this metric, more specifically, we express its main geometric objects: Chern-Finsler connection, holomorphic curvature, Käher and Berwald conditions and projectively related properties.

Moreover, our goal is to show that this type of treatment also attempts to emphasize physical interpretation. To serve this objective, we found an application of the metric given by (1.1). In this case, is constructed the Lagrangian of a complex Beil metric arising from the weakly gravitational metric perturbed by an electromagnetic potential. Solving the variational problem associated to this Lagrangian, we reobtain the Chern-Finsler complex non-linear connection (Theorem 4.2). The complex geodesics corresponding to the complex Beil metric are given in Theorem 4.3.

2 Preliminaries

Let M be an $n-$dimensional complex manifold. The complexified of the real tangent bundle $T_C M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$. The bundle $T'M$ is in its turn a complex manifold. The local coordinates in a chart will be denoted by $u = (z^k, \eta^k),\ k = 1, \dots, n$, which are changed by the following rules: $z'^k = z'^k(z),\ \eta'^k = \frac{\partial z'^k}{\partial z^j}\eta^j$. The complexified tangent bundle of $T'M$ is decomposed in the direct sum of $T'(T'M)$ and $T''(T'M)$, respectively. A natural local frame for $T'_u(T'M)$ is $\{\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \eta^k}\}$, and it changes according to the rules below:

$$\frac{\partial}{\partial z^k} = \frac{\partial z'k}{\partial z^h}\frac{\partial}{\partial z'^k} + \frac{\partial^2 z'k}{\partial z^j \partial z^h}\eta^j \frac{\partial}{\partial \eta'^k}; \quad \frac{\partial}{\partial \eta^k} = \frac{\partial z'k}{\partial z^h}\frac{\partial}{\partial \eta'^k}. \tag{2.1}$$

Let $V(T'M) = \text{Ker}(\pi^*) \subset T'(T'M)$ be the vertical bundle, spanned locally by $\frac{\partial}{\partial \eta^k}$. A complex non-linear connection, briefly (*c.n.c.*), determines a supplementary complex sub-bundle to $V(T'M)$, i.e. $T'(T'M) = V(T'M) \oplus H(T'M)$. It determines an adapted frame $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where $N_k^j(z, \eta)$ are the coefficients of the (*c.n.c.*). These functions have a special rule of change obtained by (2.1). Then $\{\delta_k := \frac{\delta}{\delta z^k}, \dot{\partial}_k := \frac{\partial}{\partial \eta^k}\}$ is an adapted basis of $H(T'M)$. For more details you can see [16, 1]. Moreover, the pair (M, F) is called a *complex Finsler space,* where $F : T'M \to \mathbb{R}^+$ is a continuous function which satisfies:

i) $L := F^2$ is smooth on $\widetilde{T'M} := T'M \setminus \{0\}$;

ii) $F(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;

iii) $F(z, \lambda\eta) = |\lambda| F(z, \eta)$ for $\lambda \in \mathbb{C}$;

iv) the following Hermitian matrix $(g_{i,\bar{j}}(z, \eta))$, with $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$, is positive definite on $\widetilde{T'M}$, and it is called *the fundamental metric tensor.*

If the iv)-th assumption is satisfied, then the Finsler metric F is strongly pseudo-convex, this means that the complex indicatrix $I_{F,z} = \{\eta \in T'_zM \mid F(z,\eta) < 1\}$ is strongly pseudo-convex.

A main problem in this geometry is to determine a (*c.n.c.*) related only to the fundamental metric tensor $g_{i\bar{j}}$ of the complex Finsler space (M,F), (for more details see [16]).

A Hermitian connection D on the sections of $T_C(T'M)$, of $(1,0)-$type, which satisfies in addition $D_{JX}Y = JD_XY$, for X horizontal vectors and J the natural complex structure of the manifold, is the *Chern-Finsler connection* (see [16]). This connection is locally given by the following coefficients:

$$N^i_j = g^{\bar{m}i}\frac{\partial g_{l\bar{m}}}{\partial z^j}\eta^l; \quad L^i_{jk} = g^{\bar{m}i}\delta_k g_{j\bar{m}} = \dot{\partial}_j N^i_k; \quad C^i_{jk} = g^{\bar{m}i}\dot{\partial}_k g_{j\bar{m}}, \tag{2.2}$$

and $L^i_{j\bar{k}} = C^i_{j\bar{k}} = 0$, where δ_k, here and subsequently, is the adapted frame of the Chern-Finsler (*c.n.c.*) and $D_{\delta_k}\delta_j = L^i_{jk}\delta_i$, $D_{\dot{\partial}_k}\dot{\partial}_j = C^i_{jk}\dot{\partial}_i$, etc. The $h-$, $v-$, $\bar{h}-$, $\bar{v}-$ covariant derivatives with respect to Chern-Finsler connection is noted by "$_|$","$|$","$_{\bar{|}}$" and "$\bar{|}$", respectively.

The *complex Cartan tensors* are the following $C_{i\bar{j}k} = \dot{\partial}_k g_{i\bar{j}}$ and $C_{i\bar{j}\bar{k}} = \dot{\partial}_{\bar{k}} g_{i\bar{j}}$.

In [1] a complex Finsler space (M,F) is *weakly Kähler* if $g_{i\bar{l}}T^i_{jk}\eta^j\bar{\eta}^l = 0$, *Kähler* if $T^i_{jk}\eta^j = 0$, and *strongly Kähler* if $T^i_{jk} = 0$, where $T^i_{jk} = L^i_{jk} - L^i_{kj}$. In [12] it is proved that the strongly Kähler and the Kähler notions coincide. In the particular case when the complex Finsler space is purely Hermitian, i.e. $g_{i\bar{l}} = g_{i\bar{l}}(z)$, all those nuances of Kähler are the same.

According to [1, 16, 2], the holomorphic curvature of the complex Finsler space (M,F) in direction η is $\mathcal{K}_F(z,\eta) = \frac{2}{L^2}\mathbf{G}(\mathbf{R}(\chi,\overline{\chi})\chi,\overline{\chi})$, where $\chi := \eta^k\delta_k$ is the horizontal lift. Locally it has the following expression (see [2])

$$\mathcal{K}_F(z,\eta) = \frac{2}{L^2}R_{j\bar{k}}\bar{\eta}^j\eta^k, \text{ where } R_{\bar{j}k} = -g_{m\bar{j}}(\delta_{\bar{h}}N^m_k)\bar{\eta}^h. \tag{2.3}$$

Generally the Chern-Finsler (*c.n.c.*), does not derive from a spray, but it always determines a complex spray, with local coefficients $G^i = \frac{1}{2}N^i_j\eta^j$.

In [5] it is proved, that the complex Finsler space (M,F) is *generalized Berwald* if and only if $\dot{\partial}_{\bar{h}}G^i = 0$, and (M,F) is a *complex Berwald* space if and only if it is Kähler and generalized Berwald.

In [1] a complex geodesic curve is given by $D_{T^h+\overline{T^h}}T^h = \theta^*(T^h,\overline{T^h})$, where $\theta^* = g^{\bar{m}k}g_{i\bar{p}}(L^{\bar{p}}_{\bar{j}\bar{m}} - L^{\bar{p}}_{\bar{m}\bar{j}})\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j$. Locally, the equations of a complex geodesic $z = z(t)$ of (M,L), with t as a real parameter, in [1]'s sense can be rewritten as

$$\frac{\mathrm{d}^2z^i}{\mathrm{d}t^2} + 2G^k(z(t),\frac{\mathrm{d}z}{\mathrm{d}t}) = \theta^{*i}(z(t),\frac{\mathrm{d}z}{\mathrm{d}t}); \; i = 1,\dots,n,$$

where by $z^i(t)$, $i = 1,\dots,n$, are denoted the coordinates along the curve $z = z(t)$.

Let $\tilde{L}$ be another complex Finsler metric on the underlying manifold M.

Definition 2.1. [4] The complex Finsler metrics L and $\tilde{L}$ on the manifold M, are called *projectively related* if they have the same complex geodesics as point sets.

In [4] several necessary and sufficient conditions are given for when two complex Finsler metrics are projectively related:

Theorem 2.1. *[4] Let L and $\tilde{L}$ be complex Finsler metrics on the manifold M. Then L and $\tilde{L}$ are projectively related if and only if there is a smooth function P in $T'M$ with complex values, such as $\tilde{G}^i = G^i + Q^i + P\eta^i,\ i = 1, \dots, n$.*

Theorem 2.2. *[4] Let L and $\tilde{L}$ be the complex Finsler metrics on the same manifold M. Then, L and $\tilde{L}$ are projectively related if and only if*

$$\dot{\partial}_{\bar{r}}(\delta_k \tilde{L})\eta^k + 2(\dot{\partial}_{\bar{r}} G^l)(\dot{\partial}_l \tilde{L}) = \frac{1}{\tilde{L}}(\delta_k \tilde{L})\eta^k(\dot{\partial}_{\bar{r}} \tilde{L}); \tag{2.4}$$

$$Q^r = -\frac{1}{2\tilde{L}}\theta^{*l}(\dot{\partial}_l \tilde{L})\eta^r;\ P = \frac{1}{2\tilde{L}}[(\delta_k \tilde{L})\eta^k + \theta^{*i}(\dot{\partial}_i \tilde{L})]. \tag{2.5}$$

$(r = 1, \dots, n)$ Moreover, the projective change is $\tilde{G}^i = G^i + \frac{1}{2\tilde{L}}(\delta_k \tilde{L})\eta^k\eta^i$.

3 The complex Beil metric on a complex Finsler space

Following the ideas from real cases, [7, 8, 11], we shall introduce a new class of complex metrics. Let (M, F) be an $n-$dimensional complex Finsler space, and $g_{j\bar{k}}$ its fundamental metric tensor. Assume that (M, F) is endowed with a complex Finsler vector field $B = B^k(z, \eta)\dot{\partial}_k$ and let $B_k(z, \eta)dz^k$ be a differential $(1, 0)-$form with $B_k = g_{k\bar{m}}B^{\bar{m}}$, where $B^{\bar{m}} := \overline{B^m}$. The lowering and rising of indices will be done with $(g_{i\bar{j}})$ and $(g^{\bar{j}k})$, where $g_{i\bar{j}}g^{\bar{j}k} = \delta^i_k$, respectively.

Also, we consider $\sigma : T'M \to \mathbb{R}$, a real valued function, on $T'M$. By these objects we set

$${}^*g_{i\bar{j}}(z, \eta) = g_{i\bar{j}}(z, \eta) + \sigma(z, \eta)B_i(z, \eta)B_{\bar{j}}(z, \eta). \tag{3.1}$$

We have proved in [18] that

Proposition 3.1. *For the $d-$tensor ${}^*g_{i\bar{j}}$ from (3.1) we have,*

i) $\det({}^*g_{i\bar{j}}) = (1 + \sigma\mathbb{B}^2)\det(g_{i\bar{j}})$;

ii) If $1 + \sigma\mathbb{B}^2 \neq 0$, the $d-$tensor $g_{i\bar{j}}$ is non-degenerate, and its inverse has the following expression ${}^*g^{\bar{j}i} = g^{\bar{j}i} - {}^*\sigma B^i B^{\bar{j}}$, with ${}^*\sigma = \frac{\sigma}{1+\sigma\mathbb{B}^2}$,

where $\mathbb{B}^2 = B_i B^i = g_{i\bar{j}}B^i B^{\bar{j}}$ (the length of $\mathbb{B}$ with respect to $g_{i\bar{j}}$).

Under the assumption $1 + \sigma\mathbb{B}^2 \neq 0$ the functions $({}^*g_{i\bar{j}})$ from (3.1) are called *the complex Beil metric.*

From [16] we know that ${}^*g_{i\bar{j}}$ is reducible to a complex Finsler metric, if and only if the complex Cartan tensor fields associated to this metric ${}^*C_{i\bar{j}k} = \dot{\partial}_k {}^*g_{i\bar{j}}$ and ${}^*C_{i\bar{j}\bar{k}} = \dot{\partial}_{\bar{k}} {}^*g_{i\bar{j}}$ satisfy the following conditions:

(i) ${}^*C_{i\bar{j}k} = {}^*C_{k\bar{j}i},\ {}^*C_{i\bar{j}\bar{k}} = {}^*C_{i\bar{k}\bar{j}}$;

(ii) ${}^*C_{i\bar{j}k} = \overline{{}^*C_{j\bar{i}\bar{k}}}$;

(iii) ${}^*C_{i\bar{j}k}\eta^k = {}^*C_{k\bar{j}i}\eta^i = {}^*C_{i\bar{j}\bar{k}}\bar{\eta}^j = {}^*C_{i\bar{k}\bar{j}}\bar{\eta}^k = 0.$

Using (3.1) we can prove

Theorem 3.2. *The complex Beil metric defined in (3.1) is the fundamental metric tensor of a complex Finsler space $(M,\ {}^*F)$ if and only if the following system of equations is satisfied*

$$(\dot{\partial}_k\sigma)B_iB_{\bar{j}} + \sigma(\dot{\partial}_kB_i\cdot B_{\bar{j}} + \dot{\partial}_kB_{\bar{j}}\cdot B_i) = (\dot{\partial}_i\sigma)B_kB_{\bar{j}} + \sigma(\dot{\partial}_iB_k\cdot B_{\bar{j}} + \dot{\partial}_iB_{\bar{j}}\cdot B_k);$$
$$(\dot{\partial}_k\sigma)B_iB_{\bar{j}}\eta^k + \sigma(\dot{\partial}_kB_i\cdot B_{\bar{j}} + \dot{\partial}_kB_{\bar{j}}\cdot B_i)\eta^k = 0. \tag{3.2}$$

In general ${}^*g_{i\bar{j}}(z,\eta)$ is not reducible to a complex Finsler metric. We found a case when ${}^*g_{i\bar{j}}$ is a complex Finsler metric as follows.

Proposition 3.3. *If $B_i = B_i(z)$ and $\sigma = \sigma(z) \geq -\frac{F^2}{|\beta|^2}$, then $(M,{}^*g_{i\bar{j}})$ becames a complex Finsler space, with*

$$ {}^*F^2 = F^2 + \sigma(z)|\beta|^2,\ \textit{where } \beta = B_i(z)\eta^i. \tag{3.3}$$

Remark 3.1. The condition $\beta = 0$ and $B_i = B_i(z)$ are incompatible, because they imply that $B = 0$.

Further on, we work under the assumptions that $B_i = B_i(z)$ and $\sigma = \sigma(z) \geq -\frac{F^2}{|\beta|^2}$. Then the complex Beil metric will take the following form:

$$ {}^*g_{i\bar{j}}(z,\eta) = g_{i\bar{j}}(z,\eta) + \sigma(z)B_i(z)B_{\bar{j}}(z). \tag{3.4}$$

Example 3.2. We set an example of complex Beil metric of complex dimension two. To avoid confusions, we rename the local coordinates $z^1,\ z^2,\ \eta^1,\eta^2$ as $z,\ w,\eta,\ \theta$, respectively. On the a complex domain $D = \left\{(z,w)\in\mathbb{C}^2|\ |w|<|z|\right\}$, let us define the purely Hermitian metric

$$g_{i\bar{j}} = \frac{\partial^2}{\partial z^i\bar{z}^j}\left(\log\frac{1}{|z|^2-|w^2|}\right),\quad L(z,w,\eta,\theta) = g_{i\bar{j}}\eta^i\bar{\eta}^j,$$

where $|z^i|^2 := z^i\bar{z}^i,\ z^i\in\{z,w\},\ \eta^i\in\{\eta,\theta\}$. After a direct computation, we obtain

$$g_{z\bar{z}} = 2|z|^2-|w|^2;\qquad g_{z\bar{w}} = -\bar{z}w;\quad g_{w\bar{w}} = 2|w|^2-|z|^2;$$
$$g^{\bar{z}z} = -\frac{2|w|^2-|z|^2}{2(|z|^2-|w|^2)^2};\qquad g^{\bar{z}w} = -\frac{\bar{z}w}{2(|z|^2-|w|^2)^2};\quad g^{\bar{w}w} = -\frac{2|z|^2-|w|^2}{2(|z|^2-|w|^2)^2}.$$

Choosing $B_z = w$, $B_w = z$ and $\sigma = 1$, we have $B^z = -\frac{\bar{w}|w|^2}{(|z|^2-|w|^2)^2}$, $B^w = -\frac{\bar{z}|z|^2}{(|z|^2-|w|^2)^2}$.

As a result of the above, we obtain a complex Finsler metric ${}^*g_{i\bar{j}} = g_{i\bar{j}} + \sigma B_iB_{\bar{j}}$, and the Lagrangian of the complex Beil metric ${}^*L = 2(|z|^2|\eta|^2 + |w|^2|\theta|^2)$, which, in turn, is the double of the *complex Euclidean metric.*

The non-linear connections play an important role in Finsler geometry. These connections allow us to work with $d-$tensors. It is very useful when the (*c.n.c.*) derives from the fundamental metric tensor of the space. This is an argument for which we try to express the Chern-Finsler (*c.n.c.*) of $(M,^* F)$.

The local coefficients of the Chern-Finsler (*c.n.c.*) associated the to complex Finsler space $(M,^* F)$, ${}^*N^i_j = {}^*g^{\bar{m}i}\frac{\partial\, {}^*g_{p\bar{m}}}{\partial z^j}\eta^p$, can be rewritten as

$$ {}^*N^i_j = N^i_j + A^i_j, \text{ where } A^i_j = {}^*g^{\bar{m}i}(\sigma B_p B_{\bar{m}})_{|j}\eta^p. \tag{3.5}$$

Note that A^i_j defined in (3.5) is a $d-$tensor, $(1,0)-$homogeneous in η.

In the complex Finsler space $(M,^* F)$ the adapted horizontal frame will be notated by ${}^*\delta_k := \partial_k - {}^*N^m_k\dot{\partial}_m = \delta_k - A^m_k\dot{\partial}_m$.

Now we are able to give the expressions of the Chern-Finsler (*c.l.c.*) ${}^*C\Gamma = ({}^*N^i_j, {}^* L^i_{jk}, {}^* C^i_{jk}, 0, 0)$.

Proposition 3.4. *In the complex Finsler space $(M,^* F)$, with ${}^*F$ given in (3.3), the local coefficients of the Chern-Finsler (c.l.c.) ${}^*C\Gamma$ are*

$$ {}^*L^i_{jk} = L^i_{jk} + \dot{\partial}_j A^i_k; \quad {}^*C^i_{jk} = C^i_{jk} -^* \sigma B^i B^{\bar{m}} C_{j\bar{m}k}. \tag{3.6}$$

The non-vanishing components of the torsions of the $N-$ (*c.l.c.*) ${}^*C\Gamma$ are the following

$$\begin{aligned} {}^*T^i_{jk} &= T^i_{jk} + \dot{\partial}_j A^i_k - \dot{\partial}_j A^i_k; \quad {}^*Q^i_{j\bar{k}} = C^i_{j\bar{k}} -^* \sigma B^{\bar{m}} B^i C_{j\bar{m}k}, \\ {}^*\Theta^i_{j\bar{k}} &= \Theta^i_{j\bar{k}} - \rho^i_{j\bar{p}} N^{\bar{p}}_{\bar{k}} +^* \delta_{\bar{k}} A^i_j; \quad {}^*\rho^i_{j\bar{k}} = \rho^i_{j\bar{k}} + \dot{\partial}_{\bar{k}} A^i_j, \end{aligned} \tag{3.7}$$

where T^i_{jk}, $\Theta^i_{j\bar{k}}$ and $\rho^i_{j\bar{k}}$ are the local torsion expressions of the Chern-Finsler (*c.l.c.*) on (M,F), [16].

In the following we compute the holomorphic curvature in direction of η with respect to ${}^*C\Gamma$. Transcribing (2.3), we obtain $\mathcal{K}_{^*F}(z,\eta) = \frac{2}{{}^*L^2}{}^*R_{\bar{j}k}\bar{\eta}^j\eta^k$, with ${}^*R_{\bar{j}k} = -{}^*g_{l\bar{j}}({}^*\delta_{\bar{p}}{}^*N^l_k)\bar{\eta}^p$:

$$\begin{aligned} \mathcal{K}_{^*F}(z,\eta) &= \left(1 - \frac{\sigma|\beta|^2}{{}^*L^2}\right)\mathcal{K}_F + \\ &+\frac{2}{L^2}\left(1 - \frac{\sigma|\beta|^2}{{}^*L^2}\right)\left[(\dot{\partial}_{\bar{p}}{}^*N^l_0)A^{\bar{p}}_{\bar{0}}\eta_l - (\delta_{\bar{0}}A^l_p)\eta^p\eta_l - \sigma\bar{\beta}B_l({}^*\delta^*_{\bar{0}}N^l_p)\bar{\eta}^p\right]. \end{aligned}$$

Subsequently, we emphasize other geometrical properties of the complex Finsler space with the complex Beil metric (3.4).

It is known that a complex Finsler metric is purely Hermitian if and only if the associated complex Cartan tensors are vanishing. For the complex Finsler metric given in (3.4), the conditions ${}^*C_{i\bar{j}k} = 0$ and ${}^*C_{i\bar{j}\bar{k}} = 0$ lead to:

Proposition 3.5. *The complex Finsler space $(M,^* F)$ with complex Beil metric is purely Hermitian if and only if (M,F) is purely Hermitian.*

Taking into account the first relation from (3.7), the weakly Kähler condition associated to (3.3) is

$$ {}^*g_{i\bar{l}}{}^*T^i_{jk}\eta^k\bar{\eta}^l = T^i_{jk}\eta^k(\eta_i + \sigma\bar{\beta}B_i) + [\partial_k(\sigma B_j B_{\bar{l}}) - \sigma B_{\bar{l}} B_p L^p_{jk}]\eta^k\bar{\eta}^l - {}^*g_{i\bar{l}}A^i_j\bar{\eta}^l = 0$$

Then, we have the following assertion:

Proposition 3.6. *Let (M, F) be a Kähler complex Finsler space. The complex Finsler space $(M,^* F)$, with $^*F$ from (3.3), is weakly Kähler if and only if*

$$\partial_j(\sigma|\beta|^2) - \partial_0(\sigma B_j B_{\bar{0}}) - {}^*C_{p\bar{0}j} A^p_0 = 0, \quad j = 1, \ldots, n.$$

A similar calculus leads us to determine the Kähler condition corresponding to the metric (3.3):

Proposition 3.7. *Let (M, F) a Kähler complex Finsler space. The complex Finsler space $(M,^* F)$, with $^*F$ from (3.3), is Kähler if and only if*

$$^*g^{\bar{m}i}[\partial_0(\sigma B_j B_{\bar{m}}) - \partial_j(\sigma B_0 B_{\bar{m}}) - {}^*C_{p\bar{m}j} A^p_0] = 0, \quad j = 1, \ldots, n.$$

Further on, by direct computations, we establish the necessary and sufficient conditions by which the metric (3.3) can be generalized Berwald and Berwald.

Proposition 3.8. *Let (M, F) be a generalized Berwald space. The complex Finsler space $(M,^* F)$ with $^*F$ from (3.3) is generalized Berwald if and only if*

$$\dot{\partial}_{\bar{h}} {}^*g^{\bar{m}i}(\sigma B_p B_{\bar{m}})_{|0}\eta^p = 0, \tag{3.8}$$

Corollary 3.9. *Let (M, F) be a complex Berwald space. $(M,^* F)$ with $^*F$ from (3.3) is a complex Berwald space if an only if the following conditions are satisfied*

i) $^*g^{\bar{m}i}[\partial_0(\sigma B_k B_{\bar{m}}) - \partial_k(\sigma B_0 B_{\bar{m}}) - {}^*C_{p\bar{m}k} A^p_0] = 0;$

ii) $^*C_{i\bar{m}\bar{p}} A^i_0 = 0.$

The next step in our study is centered on finding when the complex Finsler metrics L and $^*L$ are projectively related. The link between the complex spray G^i and $^*G^i = \frac{1}{2} {}^*N^i_j \eta^j$, corresponding to N^i_j and $^*N^i_j$ is below

$$^*G^i = \frac{1}{2} {}^*N^i_j \eta^j = G^i + \frac{1}{2} A^i_j \eta^j, \quad i = 1, \ldots, n. \tag{3.9}$$

Based on this, we prove the following main result of this section:

Theorem 3.10. *The complex Finsler metrics L and $^*L = L + \sigma|\beta|^2$, both defined on M, are projectively related, i.e.*

$$\begin{aligned} {}^*G^r &= G^r + Q^r + P\eta^r, \\ Q^r &= -\frac{1}{2^*L} g^{\bar{j}l} T^{\bar{k}}_{\bar{p}\bar{j}} (\dot{\partial}_l {}^*L) \bar{\eta}^p \bar{\eta}_k \eta^r, \; P = \frac{1}{2^*L} \left(A^i_j \eta^j + g^{\bar{j}i} T^{\bar{k}}_{\bar{p}\bar{j}} \bar{\eta}^p \bar{\eta}_k \right) (\dot{\partial}_i {}^*L), \end{aligned} \tag{3.10}$$

$(r = 1, \ldots, n)$, and the projective change is $^*G^r = G^r + \frac{1}{2} A^r_j \eta^j$.

Proof. A simple calculation shows that, $\theta^{*i} = g^{\bar{j}i} T^{\bar{k}}_{\bar{p}\bar{j}} \bar{\eta}^p \bar{\eta}_k$. By replacing this relation in (2.5), (3.10) becomes true. As a result, by using Theorem 2.1, the complex Finsler metrics are projectively related. □

Example 3.3. The complex version of the *Antonelli-Shimada metric*

$$L_{AS}(z,w,\eta,\theta) := e^{2f}\left(|\eta|^4+|\theta|^4\right)^{\frac{1}{2}}, \text{ with } \eta,\theta \neq 0,$$

is a generalized Berwald metric, defined on a domain D from $\widetilde{T'M}$, $\dim_C M = 2$, such that its metric tensor is non-degenerated, (for more details see [5]). The local coordinates z^1, z^2, η^1, η^2 are denoted by z, w, η, θ, respectively, and $f(z)$ is a real-valued function. Our aim is to find proper expressions for $\sigma(z)$ and $B_i(z)$, such that the complex Finsler space $(M, {}^*L)$ to be generalized Berwald. For this, we choose $\sigma(z) = e^{2f}$, and $\beta = \eta$. With these objects, we obtain a complex Beil metric

$${}^*L(z,w,\eta,\theta) := e^{2f}\left[\left(|\eta|^4+|\theta|^4\right)^{\frac{1}{2}}+|\eta|^2\right],$$

which is generalized Berwald.

4 The variational problem in a perturbed weakly gravitational space

Let (M,L) be a 2−dimensional complex Finsler space with

$$L = \left(1+\frac{2\Phi}{c^2}\right)|\eta^1|^2 - i\left(1-\frac{2\Phi}{c^2}\right)\eta^1\bar{\eta}^2 + i\left(1-\frac{2\Phi}{c^2}\right)\eta^2\bar{\eta}^1 - \left(1-\frac{2\Phi}{c^2}\right)|\eta^2|^2 \tag{4.1}$$

the weakly gravitational metric, studied in [17, 6]. It is a purely Hermitian metric with the fundamental metric tensor:

$$(g_{j\bar{k}}) := \begin{pmatrix} 1+\frac{2\Phi}{c^2} & -i\left(1-\frac{2\Phi}{c^2}\right) \\ i\left(1-\frac{2\Phi}{c^2}\right) & -\left(1-\frac{2\Phi}{c^2}\right) \end{pmatrix}, \tag{4.2}$$

$i := \sqrt{-1}$, $j,k = 1,2$, where Φ is a real valued smooth function on $T'M$, $\Phi > \frac{c^2}{2}$, where $c \in \mathbb{R}^*$. The inverse matrix of $(g_{j\bar{k}})$ is $\left(g^{\bar{k}j}(z,\eta)\right)_{j\bar{k}=1,2} = \begin{pmatrix} \frac{1}{2} & \frac{-i}{2} \\ \frac{i}{2} & -\frac{1+\frac{2\Phi}{c^2}}{2\left(1-\frac{2\Phi}{c^2}\right)} \end{pmatrix}$.

Also, from [6] we have the coefficients of the Chern-Finsler (*c.n.c.*) corresponding to (4.1): $N^1_k = 0$; $N^2_k = \frac{-2i}{c^2\left(1-\frac{2\Phi}{c^2}\right)}(\eta^1 - i\eta^2)\Phi_k$, where $\Phi_k := \frac{\partial\Phi}{\partial z^k}$, $k = 1,2$.

In this section, we perturb the weakly gravitational metric (4.1) to a complex Beil metric with an electromagnetic potential, $a|\beta|^2 = aB_j(z)B_{\bar{k}}(z)\eta^j\bar{\eta}^k$, where $a > 0$. And so, we obtain a complex Finsler metric which arises from the weakly gravitational metric ${}^*L = L + a|\beta|^2$, with the fundamental metric tensor

$$({}^*g_{j\bar{k}}) := \begin{pmatrix} 1+\frac{2\Phi}{c^2}+aB_1B_{\bar{1}} & -i\left(1-\frac{2\Phi}{c^2}\right)+aB_1B_{\bar{2}} \\ i\left(1-\frac{2\Phi}{c^2}\right)+aB_2B_{\bar{1}} & -\left(1-\frac{2\Phi}{c^2}\right)+aB_2B_{\bar{2}} \end{pmatrix}, \tag{4.3}$$

and its inverse ${}^*g^{\bar{k}j} = g^{\bar{k}j} - {}^*aB^{\bar{k}}B^j$, where ${}^*a = \frac{a}{a\mathbb{B}^2+1}$. This metric is called by us *weakly gravitational Beil metric.*

Using the general results form the previous sections we get the local coefficients of the Chern-Finsler (*c.n.c.*) of $(M,^* L)$:

$$^*N^j_k = N^j_k + a^* g^{\bar{m}j}\left[\partial_k(B_pB_{\bar{m}})\eta^p + \frac{2i}{c^2\left(1-\frac{2\Phi}{c^2}\right)}(\eta^1 - i\eta^2)\Phi_k B_{\bar{m}}B_2\right]. \tag{4.4}$$

Subsequently, we study the variational problem for the weakly gravitational Beil metric $^*L = L + a|\beta|^2$ in the canonical parametrization of a curve on the complex manifold M with respect to the purely Hermitian weakly gravitational metric (4.1).

Let us consider $c(t),\ c \in \mathbb{R}$ a C^∞ curve on complex manifold M, and $(z^k(t), \eta^k = \frac{\mathrm{d}z^k}{\mathrm{d}t})$ its extension to $T'M$. The Euler-Lagrange equations with respect to a complex Lagrangian $^*L$ are

$$E_k(^*L) := \frac{\partial^* L}{\partial z^k} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial^* L}{\partial \eta^k}\right) = 0, \quad k = 1, 2, \tag{4.5}$$

where $^*L$ is considered along the curve c on $T'M$. Generally, the solutions of the Euler-Lagrange equations are extremal curves with respect to arc length.

After we develop the calculus in (4.5), for $^*L = L + a|\beta|^2$, where $a > 0$, along the extremal curve c on $T'M$, we have achieved

Proposition 4.1. *The Euler-Lagrange equations with respect to* $^*L = L + a|\beta|^2$ *are*

$$E_k(^*L) = E_k(L) + aE_k(|\beta|^2) = 0, \quad k = 1, 2. \tag{4.6}$$

Now, we choose $s(t)$ the arc length of the curve c on $T'M$ with respect to the weakly gravitational metric F as a parametrization of the curve c on $T'M$. Since $\mathrm{d}s^2 = L(z, \frac{\mathrm{d}z}{\mathrm{d}t})\mathrm{d}t^2$ it yields $L(z, \frac{\mathrm{d}z}{\mathrm{d}s}) = 1$. In the following steps, we calculate $E_k(L)$ and $E_k(|\beta|^2)$, $k = 1, 2$, in the canonical parametrization.

$$\begin{aligned}
E_1(L) &= \frac{2}{c^2}(\bar\eta^1 + i\bar\eta^2)[-i(\Phi_1 - i\Phi_2)\eta^2 - \Phi_{\bar j}\bar\eta^j] - \\
&\quad -L\left[\left(1+\frac{2\Phi}{c^2}\right)\frac{\mathrm{d}^2\bar z^1}{\mathrm{d}s^2} - i\left(1-\frac{2\Phi}{c^2}\right)\frac{\mathrm{d}^2\bar z^2}{\mathrm{d}s^2}\right];\\
E_2(L) &= \frac{2}{c^2}(\bar\eta^1 + i\bar\eta^2)[i(\Phi_1 - i\Phi_2)\eta^1 + i\Phi_{\bar j}\bar\eta^j] - L\left(1-\frac{2\Phi}{c^2}\right)\left(i\frac{\mathrm{d}^2\bar z^1}{\mathrm{d}s^2} - \frac{\mathrm{d}^2\bar z^2}{\mathrm{d}s^2}\right);\\
E_k(|\beta|^2) &= [\partial_k(B_pB_{\bar q}) - \partial_p(B_kB_{\bar q})]\eta^p\bar\eta^q - \partial_{\bar p}(B_kB_{\bar q})\bar\eta^p\bar\eta^q - LB_kB_{\bar q}\frac{\mathrm{d}^2\bar z^q}{\mathrm{d}s^2}, k = 1, 2.
\end{aligned}$$

Substituting the formulas of $E_k(L)$ and $E_k(|\beta|^2)$ in (4.6), we obtain the Euler-Lagrange equations of $(M,^* L)$ in the canonical parametrization.

Following the same arguments as in [16, 3], the equations of a complex geodesics for $(M,^* L)$ are:

$$\frac{2i}{c^2}(\delta^i_k - \delta^2_k)(\bar\eta^1 + i\bar\eta^2)(\Phi_1 - i\Phi_2)\eta^k + a[\partial_k(B_pB_{\bar q}) - \partial_p(B_kB_{\bar q})]\eta^p\bar\eta^q = 0, \tag{4.7}$$

$$\frac{2}{c^2}(\delta^i_k - i\delta^2_k)(\bar\eta^1 + i\bar\eta^2)\Phi_{\bar j}\bar\eta^j + Lg_{k\bar q}\frac{\mathrm{d}^2\bar z^q}{\mathrm{d}s^2} + a\left[\partial_{\bar p}(B_kB_{\bar q})\bar\eta^p\bar\eta^q + LB_kB_{\bar q}\frac{\mathrm{d}^2\bar z^q}{\mathrm{d}s^2}\right] = 0, \tag{4.8}$$

for $k = 1, 2$. The conjugate of (4.8) contracted with ${}^*g^{\bar{k}m}$, it leads to:

$$\text{(4.9)}\quad \frac{\mathrm{d}^2 z^m}{\mathrm{d}s^2} + \frac{2}{c^2}({}^*g^{\bar{1}m} + i{}^*g^{\bar{2}m})\left(\frac{\mathrm{d}z^1}{\mathrm{d}s} - i\frac{\mathrm{d}z^2}{\mathrm{d}s}\right)\Phi_j\frac{\mathrm{d}z^j}{\mathrm{d}s} + a{}^*g^{\bar{k}m}\partial_p(B_{\bar{k}}B_q)\frac{\mathrm{d}z^p}{\mathrm{d}s}\frac{\mathrm{d}z^q}{\mathrm{d}s} = 0,$$

$m = 1, 2$.

We note that (4.9) can be rewritten in the form $\frac{\mathrm{d}^2 z^m}{\mathrm{d}t^2} + 2\tilde{G}^m(z(t), \eta(t)) = 0$, where $\tilde{G}^m = \frac{1}{c^2}({}^*g^{\bar{1}m} + i{}^*g^{\bar{2}m})\left(\eta^1 - i\eta^2\right)\Phi_j\eta^j + \frac{a}{2}{}^*g^{\bar{k}m}\partial_j(B_{\bar{k}}B_q)\eta^q\eta^j$. Using the changes of complex coordinates on $T'M$, we can prove by direct computation, that the functions $\tilde{G}^m$ are the coefficients of a complex spray on $T'M$. Keeping that a (*c.n.c.*) by contraction with η determines a complex spray, i.e. $N^i_j\eta^j = 2G^i$, from (4.9) we can conclude that the functions

$$\text{(4.10)}\qquad \widetilde{N}^m_j(z,\eta) := \frac{2}{c^2}({}^*g^{\bar{1}m} + i{}^*g^{\bar{2}m})\left(\eta^1 - i\eta^2\right)\Phi_j + a{}^*g^{\bar{k}m}\partial_j(B_{\bar{k}}B_q)\eta^q$$

are coefficients of a (*c.n.c.*). Upon closer inspection of (4.10), we can point out that:

Theorem 4.2. *The* (*c.n.c.*) $\widetilde{N}^j_k$ *and the Chern-Finsler* (*c.n.c.*) *associated to the complex Finsler space* $(M, L + a|\beta|^2)$ *coincide.*

Proof. Using the formulas (4.3) and (4.4), we obtain a relation between the local coefficients of the (*c.n.c.*) $\widetilde{N}^j_k$ and the Chern-Finsler (*c.n.c.*) ${}^*N^j_k$:

$$\text{(4.11)}\qquad {}^*N^j_k = \widetilde{N}^j_k - \frac{2}{c^2}{}^*aB^j\Phi_k(\eta^1 - i\eta^2)\left(B^{\bar{1}} + iB^{\bar{2}} - B_2\frac{i}{1 - \frac{2\Phi}{c^2}}\right).$$

In explicit form we have $B_2 = g_{2\bar{p}}B^{\bar{p}} = g_{2\bar{1}}B^{\bar{1}} + g_{2\bar{2}}B^{\bar{2}} = i\left(1 - \frac{2\Phi}{c^2}\right)(B^{\bar{1}} + iB^{\bar{2}})$. Replacing this result in (4.11), the statement is proven. □

To find the geodesics of the complex Finsler space with weakly gravitational Beil metric we must analyse the equation (4.7), related to the weakly Kähler condition, according to [16]. Indeed, the relation (4.7)

$$\frac{2i}{c^2}(\delta^i_k - \delta^2_k)(\bar{\eta}^1 + i\bar{\eta}^2)(\Phi_1 - i\Phi_2)\eta^k + a[\partial_k(B_pB_{\bar{q}}) - \partial_p(B_kB_{\bar{q}})]\eta^p\bar{\eta}^q = 0,$$

becomes true for the weakly Kähler conditions of the Chern-Finsler (*c.n.c.*) associated to the complex Beil metric, formulated in Proposition 3.6. But we have proved in Theorem 3.2 that the complex Finsler metrics L and ${}^*L$ are projectively related. This means that, as point sets, they have the same complex geodesics. So, if we find the geodesics of (M, L) with weakly gravitational metric, our goal will be achieved. For this we use an important result, namely, Theorem 3.6 from [6].

Using the arguments presented above, we can formulate the following result:

Theorem 4.3. *Let* F *be the purely Hermitian metric (4.1) on the manifold* M*. If* ${}^*F = \sqrt{F + a|\beta|^2}$ *is Kähler, then the geodesic curves of* $(M, {}^*F)$ *are the following:*

$$\gamma(s) = (\lambda_1 s + \mu_1, \lambda_2 s + \mu_2), \quad \lambda_k, \mu_k \in \ , \ \lambda_k \neq 0, \ k = 1, 2.$$

Example 4.1. Consider a charged particle moving along a weakly gravitational field. The path parameter is taken to be the proper time t. The position of the particle is given by $z^k(t)$, and the velocity and acceleration are $\eta^k = \frac{\mathrm{d}z^k}{\mathrm{d}t}$ and $a^k = \frac{\mathrm{d}\eta^k}{\mathrm{d}t}$, respectively.

Now we assume that B_j is the electromagnetic potential $A_j(z)$. Then we obtain a model, when the particle is acted upon only by an electromagnetic field from the potential A^j. As a result, if the condition $a|\beta|^2 = \textit{constant}$ is satisfied, then the equation of motion in a weakly gravitational field, which is given by the metric (4.2), is

$$a^k + \frac{-2i}{c^2\left(1-\frac{2\Phi}{c^2}\right)}\left[\delta_2^k - {}^*aA^kA_2\right](\eta^1 - i\eta^2)^* a\Phi_j\eta^j = 0, \quad k = 1,2.$$

Example 4.2. Let $\Phi(z) = \frac{c^2}{2}e^{z^2+\bar{z}^2+i(z^1-\bar{z}^1)}$ a real valued function on $\mathbb{C}^2$. Given the condition $\Phi > \frac{c^2}{2}$, we obtain the Hermitian complex Finsler metric

$$L = (1+e^Z)|\eta^1|^2 - i(1-e^Z)\eta^1\bar{\eta}^2 + i(1-e^Z)\eta^2\bar{\eta}^1 - (1-e^Z)|\eta^2|^2,$$

where $Z := z^2 + \bar{z}^2 + i(z^1 - \bar{z}^1)$, defined on $D := \left\{z \in \mathbb{C}^2 \mid \mathrm{Re}z^2 - \mathrm{Im}z^1 > 0\right\}$, (see [6]). For the function Φ, the relation $i\Phi_2 = \Phi_1$ is satisfied, and so the metric L is Kähler. If we add $a|\beta|^2 = a|z^1|^2|\eta^1|^2$ to this metric, we obtain a weakly gravitational Beil metric:

$$ {}^*L = (1+e^Z+a|z^1|^2)|\eta^1|^2 - i(1-e^Z)\eta^1\bar{\eta}^2 + i(1-e^Z)\eta^2\bar{\eta}^1 - (1-e^Z)|\eta^2|^2. \tag{4.12}$$

The associated Chern-Finsler (*c.n.c.*) has the following local expression:

$$ {}^*N_k^j = N_k^j + \frac{a\bar{z}^1}{a|z^1|^2+2}\eta^1(\delta_1^j - i\delta_2^j)\delta_k^1 \tag{4.13}$$

Due to the fact that the spray coefficients of the (*c.n.c.*) (4.13) are holomorphic in η, we deduce that (4.12) is a generalized Berwald metric. Moreover, because ${}^*L$ satisfies the conditions from Proposition 3.6, we deduce that it is weakly Kähler. In conclusion, from the above, we obtain that the (4.12) metric is a complex Berwald one.

The holomorphic curvature of the metric (4.12) is $\mathcal{K}_L = -\frac{e^Z}{L^2}|\eta^1 - i\eta^2|^2[2 + c^2(1 - e^Z)]$, and the holomorphic curvature of the weakly gravitational Beil metrics (4.12) is $\mathcal{K}_{^*L} = \frac{L^2}{^*L^2}\mathcal{K}_L - \frac{8a}{^*L^2(a|z^1|^2+2)}|\eta^1|^4$. Moreover, if $Z < \ln\left(1+\frac{2}{c^2}\right)$ then the holomorphic curvature of ${}^*L$ is negative for any $a \geq 0$.

Acknowledgements. This paper is supported by the Sectoral Operational Programme Human Resources Development (SOP HRD), ID134378 financed from the European Social Fund and by the Romanian Government.

References

[1] M. Abate, G. Patrizio, *Finsler Metrics - A Global Approach*, Springer-Verlag, Pisa 1994.

[2] N. Aldea, *Complex Finsler Spaces of Constant Holomorphic Curvature,* Diff. Geom. and its Appl., Proc. Conf. Prague 2004, (2005), 179-190.

[3] N. Aldea, G. Munteanu, *On Complex Finsler Spaces with Randers Metrics*, Journal Korean Math. Soc. 46, 5 (2009), 949-966.

[4] N. Aldea, G. Munteanu, *On Projective Invariants of the Complex Finsler Spaces,* Diff. Geom. Appl. 13, 5 (2012), 2178-2187.

[5] N. Aldea, G. Munteanu, *On Complex Landsberg and Berwald Spaces,* Journal of Geometry and Physics 62 (2012), 368-380.

[6] N. Aldea, G. Munteanu, *New Candidates for a Hermitian Approach of Gravity,* Int. J. Geom. Meth. Mod. Phys. 10, 9 (2013).

[7] M. Anastasiei, H. Shimada, *The Beil Metrics Associated to a Finsler Space*, Balkan J. Geom. Appl. 3, 2 (1998), 1-16.

[8] M. Anastasiei, H. Shimada, *Deformation of Finslerian Metrics,* Fund. Theories Phys. 109 (2000), 53-65.

[9] R.G. Beil, *Electrodynamics from a Metric*, Int. Journal of Theoretical Physics 26, 2 (1987), 189-197.

[10] R.G. Beil, *Finsler Geometry and a Unified Field Theory in Finsler Geometry*, Contemporary Math. 196 (1993), 265-272.

[11] V. Bălan, P. Stavrinos, K. Trenčevski, *Weak Gravitational Models Based on Beil Metrics*, Applied Differential Geometry - General Relativity and The Workshop on Global Analysis, Differential Geometry and Lie Algebras (2000), 7-22.

[12] B. Chen, Y. Shen, *Kähler Finsler Metrics are Actually Strongly Kähler* Chin. Ann. Math. Ser. B 30,2 (2009), 173-178.

[13] R. Miron, M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications,* Fundamental Theories of Physics, Vol. 59, Kluwer Academic Publishers 1994.

[14] R. Miron, *The Geometry of Higher Order Lagrange Spaces. Applications to Mechanics and Physics,* Kluwer Academic Publishers, Dordrecht 1997.

[15] R. Miron, *The Geometry of Ingarden Spaces,* Reports on Mathematical Physics 5, 2 (2004), 131-147.

[16] G. Munteanu, *Complex Spaces in Finsler, Lagrange and Hamilton Geometries,* Kluwer Acad. Publ. 2004.

[17] G. Munteanu, N. Aldea, *A Complex Finsler Approach of Gravity,* Int. J. Geom. Methods. Phys. 9, 7 (2012).

[18] A. Szász, *Generalized Complex Lagrange Spaces with Beil Metric,* Bull. of the *Transilvania* Univ. Braşov 56, 2 (2014).

[19] C. Udriste, *Completeness of Finsler Manifolds*, Publ. Math. Debrecen 42, 1-2(1993), 45-50.

Author's address:

Annamária Szász
Transilvania University, Faculty of Mathematics and Informatics,
Iuliu Maniu 50, Braşov 500091, Romania
E-mail: am.szasz@yahoo.com