The eigenvalue problem in Finsler geometry

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Abstract. One of the fundamental problems is to study the eigenvalue problem for the differential operator in geometric analysis. In this article, we introduce the recent developments of the eigenvalue problem for the Finsler Laplacian.

M.S.C. 2010: 53C60; 35P30; 35J60.
Key words: Finsler manifold; Finsler Laplacian; eigenvalue problem.

1 Introduction

In Riemannian geometry, the study of the eigenvalue problem for the Laplace operator on Riemannian manifolds has a long history. For an overview, the reader is referred to the book ([2]) and Chapter 3 in book ([14], [1] and references therein). As a generalization of Riemannian geometry, Finsler geometry has been received more and more attentions recently since it has more and more applications in natural science. A Finsler manifold (M^n, F) means an n-dimensional smooth differential manifold equipped with a Finsler metric $F : TM \setminus \{0\} \to [0; +\infty)$ (see details in Section 2 below). On a Finsler manifold (M, F), the Laplace operator (often called the *Finsler Laplacian*) was introduced by Z. Shen via a variation of the energy functional (cf. [12],[13]). If F is Riemannian, then the Finsler Laplacian is exactly the usual Laplacian. Unlike the usual Laplacian, the Finsler Laplacian is a nonlinear elliptic operator. The standard linear elliptic theory can not be directly applied to the Finsler Laplacian. In spite of this, some progress has been made on the global analysis on Finsler manifolds in recent yeas (cf. [10], [11], [18], [22] and references therein). In this survey article, we focus on the recent developments of the eigenvalue problem for the Finsler Laplacian.

2 Preliminaries

2.1 Finsler manifold and Finsler Laplacian

Let M be an n-dimensional smooth manifold. A *Finsler metric* F on M means a function $F : TM \to [0, \infty)$ with the following properties: (1) F is C^∞ on $TM \setminus$

Balkan Journal of Geometry and Its Applications, Vol.20, No.2, 2015, pp. 92-102.

$\{0\}$; (2) $F(x,\lambda y) = \lambda F(x,y)$ for any $(x,y) \in TM$ and all $\lambda > 0$; (3) the matrix $(g_{ij}) = (\frac{\partial^2 F(x,y)}{\partial y^i \partial y^j})$ is positive. Such a pair (M,F) is called a *Finsler manifold* and $g(x,y) = g_{ij} y^i y^j$ is called the *Fundamental tensor* of F, where $y \in T_x M$. Given a smooth measure m, the triple (M,F,m) is called a *Finsler measure space.*

A Finsler metric F on M is said to be *reversible* if $F(x,-y) = F(x,y)$ for all $x \in M$ and $y \in T_x M$. Otherwise, F is said to be *nonreversible.* In this case, we can define the reverse Finsler metric $\overleftarrow{F}(x,y)$ by $\overleftarrow{F}(x,y) := F(x,-y)$. To consider the global analysis on a Finsler manifold M, we always assume that (M,F) is orientable throughout the paper.

For $x_1, x_2 \in M$, the *distance function* from x_1 to x_2 is defined by

$$d(x_1,x_2) := \inf_{\gamma} \int_0^1 F(\dot\gamma(t))dt,$$

where the infimum is taken over all C^1 curves $\gamma : [0,1] \to M$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$. Note that the distance function may not be symmetric unless F is reversible. The *diameter* of M is defined by $d := \sup_{x,y\in M} d(x,y)$.

Given a Finsler metric F on a manifold M, there is a dual Finsler metric F^* on the cotangent bundle T^*M given by

$$F^*(x,\xi_x) := \sup_{y\in T_xM\setminus\{0\}} \frac{\xi(y)}{F(x,y)}, \quad \forall \xi \in T_x^*M.$$

The *Legendre transformation* $\mathcal{L} : TM \to T^*M$ is defined by

$$\mathcal{L}(y) := \begin{cases} g_y(y,\cdot) & y \neq 0, \\ 0 & y = 0. \end{cases}$$

One can check that it is a diffeomorphism from $TM\setminus\{0\}$ onto $T^*M\setminus\{0\}$, and norm-preserving, namely, $F(y) = F^*(\mathcal{L}(y)), \forall y \in TM$. Consequently, $g^{ij}(y) = g^{*ij}(\mathcal{L}(y))$ (see §3.1 in [12]).

For a smooth function $u : M \to R$, we define the *gradient vector* $\nabla u(x)$ of u at x by $\nabla u(x) := \mathcal{L}^{-1}(du(x)) \in T_x M$. In a local coordinate system, we can reexpress ∇u as

$$\nabla u(x) = \begin{cases} g^{ij}(x,\nabla u)\frac{\partial u}{\partial x^i}\frac{\partial}{\partial x^j} & x \in M_u, \\ 0 & x \in M \setminus M_u, \end{cases}$$

where $M_u = \{x \in M | du(x) \neq 0\}$. Obviously, $\nabla u = 0$ if $du = 0$. In general, ∇u is only continuous on M, but smooth on M_u.

Given a smooth measure $dm = \sigma(x)dx$ on (M,F), for a weakly differentiable vector field $V : M \to TM$, we define its *divergence* $\mathrm{div}_m V : M \to R$ through the identity

$$\int_M \varphi \mathrm{div}_m V dm = -\int_M d\varphi(V) dm,$$

where $\varphi \in C_0^\infty(M)$ (i.e., the set of smooth functions on M with a compact support).

The *Finsler Laplacian* Δ_m of u is formally defined by $\Delta_m u := \mathrm{div}_m(\nabla u)$, which is a nonlinear elliptic differential operator of second order. To be more precise, $\Delta_m u$

is defined in a distributional sense through the identity

$$\int_M \varphi \Delta_m u dm = -\int_M d\varphi(\nabla u) dm,$$

for all $\varphi \in C_0^\infty(M)$. It is locally expressed by

$$\Delta_m u(x) = \mathrm{div}_m(\nabla u)(x) = \frac{1}{\sigma(x)} \frac{\partial}{\partial x^i}\left(\sigma(x) g^{ij}(x, \nabla u) \frac{\partial u}{\partial x^j}\right),$$

where $x \in M_u := \{x \in M | du(x) \neq 0\}$ ([12]). For the sake of simplicity, we denote Δ_m as Δ in the following.

Similarly, we can define the reverse gradient $\overleftarrow{\nabla}$ and the reverse Laplacian $\overleftarrow{\Delta}$ for the reverse Finsler metric $\overleftarrow{F}$ etc.. In fact, we have $\overleftarrow{g}(x, y) = g(x, -y)$, $\overleftarrow{\nabla} u = -\nabla(-u)$ and $\overleftarrow{\Delta}(u) = -\Delta(-u)$. Note that $\nabla(-u)$ and $-\nabla(u)$ are different in general.

Assume that (M, F, m) is a Finsler manifold with a nonempty boundary ∂M. Then ∂M is also a Finsler manifold with a Finsler structure $F_{\partial M}$ induced by F. For any $x \in \partial M$, there exist exactly two unit normal vectors ν such that

$$T_x(\partial M) = \{V \in T_x M | g_\nu(\nu, V) = 0, g_\nu(\nu, \nu) = 1\}.$$

Note that, if ν is a normal vector, then $-\nu$ may be not a normal vector unless F is reversible. Throughout this paper, we choose the normal vector that points outward M.

The *normal curvature* $\Lambda_\nu(V)$ at $x \in \partial M$ in a direction $V \in T_x(\partial M)$ is defined by $\Lambda_\nu(V) = g_\nu(\nu, D_{\dot\gamma}^{\dot\gamma} \dot\gamma(0))$, where γ is the unique geodesic for the Finsler structure $F_{\partial M}$ on ∂M induced by F with the initial data $\gamma(0) = x$ and $\dot\gamma(0) = V$. M is said to has *convex boundary* if, for any $x \in \partial M$, the normal curvature Λ_ν at x is nonpositive in any directions $V \in T_x(\partial M)$. We remark that the convexity of M means that $D_{\dot\gamma}^{\dot\gamma} \dot\gamma(0)$ lies at the same side of $T_x M$ as M. Hence the choice of the normal vector is not essential for the definition of convexity (see [12]).

2.2 Chern connection and Weighted Ricci Curvature

Let $\pi : TM \setminus \{0\} \to M$ be the projective map. The pull-back $\pi^* TM$ admits a unique linear connection, which is called the *Chern connection.* The Chern connection D is determined by the following equations

$$D_X^V Y - D_Y^V X = [X, Y],$$
$$Z g_V(X, Y) = g_V(D_Z^V X, Y) + g_V(X, D_Z^V Y) + C_V(D_Z^V V, X, Y)$$

for $V \in TM \setminus \{0\}$ and $X, Y, Z \in TM$, where

$$C_V(X, Y, Z) := C_{ijk}(V) X^i Y^j Z^k = \frac{1}{4} \frac{\partial^3 F^2(x, V)}{\partial V^i \partial V^j \partial V^K} X^i Y^j Z^k$$

is the *Cartan tensor* of F and $D_X^V Y$ is the *covariant derivative* with respect to the reference vector V. Note that $C_V(V, X, Y) = 0$ from the homogeneity of F. In terms of the Chern connection, a geodesic $\gamma = \gamma(t)$ satisfies $D_{\dot\gamma}^{\dot\gamma} \dot\gamma = 0$.

Given two linear independent vectors $V, W \in T_xM \setminus \{0\}$, the *flag curvature* is defined by

$$K^V(V,W) = \frac{g_V(R^V(V,W)W,V)}{g_V(V,V)g_V(W,W) - g_V(V,W)^2},$$

where R^V is the *Riemannian curvature* defined by

$$R^V(X,Y)Z := D^V_X D^V_Y Z - D^V_Y D^V_X Z - D^V_{[X,Y]}Z.$$

Then the *Ricci curvature* is given by

$$\mathrm{Ric}(V) := \sum_{i=1}^{n-1} K^V(V, e_i),$$

where $\{e_1, \cdots, e_{n-1}, \frac{V}{F(V)}\}$ is the orthonomal basis of T_xM with respect to g_V.

Motivated by the work of Lott-Villani ([8]) and Sturm ([15], [16]) on metric measure space, Ohta introduced the weighted Ricci curvature on Finsler manifolds in [9].

Definition 2.1. ([9]) Given a vector $V \in T_xM$, let $\eta : (-\varepsilon, \varepsilon) \longrightarrow M$ be the geodesic such that $\eta(0) = x$ and $\dot\eta(0) = V$. We set $dm = e^{-\Psi}\mathrm{vol}_{\dot\eta}$ along η, where $\mathrm{vol}_{\dot\eta}$ is the volume form of $g_{\dot\eta}$. Define the weighted Ricci curvature involving a parameter $N \in [n, \infty]$ by

(1) $\mathrm{Ric}_n(V) := \begin{cases} \mathrm{Ric}(V) + (\Psi\circ\eta)''(0) & \text{if } (\Psi\circ\eta)'(0) = 0, \\ -\infty & \text{if } (\Psi\circ\eta)'(0) \neq 0, \end{cases}$

(2) $\mathrm{Ric}_N(V) := \mathrm{Ric}(V) + (\Psi\circ\eta)''(0) - \frac{(\Psi\circ\eta)'(0)^2}{N-n}$ for $N \in (n, \infty)$,

(3) $\mathrm{Ric}_\infty(V) := \mathrm{Ric}(V) + (\Psi\circ\eta)''(0)$

For $c \geq 0$ and $N \in [n, \infty]$,define $\mathrm{Ric}_N(cV) := c^2\mathrm{Ric}(V)$.

We say that $\mathrm{Ric}_N \geq K$ for some $K \in R$ if $\mathrm{Ric}_N(V) \geq KF(V)^2$ for all $V \in TM$. Ohta proved in [9] that the bound $\mathrm{Ric}_N(V) \geq KF(V)^2$ is equivalent to Lott-Villani and Sturm's weak curvature-dimension condition.

3 Eigenvalue and eigenfunctions

Let (M, F, m) be an n-dimensional Finsler measure space. Recall that the class $L^2(M)$ is defined in terms of the manifold structure of M (i.e. independent of the choice of F) and the Lebesgue space $(L^2(M), \|\cdot\|)$ is a Banach space, where $\|u\|_2 = \left(\int_M |u|^2 dm\right)^{1/2}$ is the norm of $L^2(M)$. For any open set $\Omega \subset M$, let

$$W^{1,2}(\Omega) := \left\{ u \in L^2(\Omega) \mid \int_\Omega [F^*(du)]^2 dm < \infty \right\}$$

and

$$\|u\|_{\Omega,1,2} := \left(\int_\Omega |u|^2 dm\right)^{1/2} + \left(\int_\Omega [F^*(du)]^2 dm\right)^{1/2} \tag{3.1}$$

for $u \in W^{1,2}(\Omega)$. Then $\|\cdot\|_{\Omega,1,2}$ is a (positively homogeneous) norm by the inequality $F^*(\xi+\eta) \le F^*(\xi) + F^*(\eta)$ for any $\xi, \eta \in T^*(M)$ (Lemma 1.2.3 in [12]). We will suppress Ω in (3.1) if $\Omega = M$, e.g., $\|u\|_{1,2} = \|u\|_{M,1,2}$.

In general, $(W^{1,2}(\Omega), \|u\|_{\Omega,1,2})$ is not a linear space over $\mathbb{R}$. For example, let $\mathbb{B}^n(1) := \{x \in \mathbb{R}^n | |x| < 1\}$, $u : \mathbb{B}^n(1) \to \mathbb{R}$ be defined by $u(x) = -\sqrt{1-|x|}$ and

$$F(x,y) = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - <x,y>^2)}}{1-|x|^2} + \frac{<x,y>}{1-|x|^2}, \quad x \in \mathbb{B}^n(1), \quad y \in \mathbb{R}^n$$

be the Funk metric. Then $u \in W^{1,2}(\mathbb{B}^n(1))$, but $-u \notin W^{1,2}(\mathbb{B}^n(1))$ with respect to the Busemann-Hausdorff measure (see details in [6]). However, if M is compact, then $(W^{1,2}(M), \|\cdot\|_{1,2})$ is a normed linear space with respect to the (positive homogeneous) norm $\|\cdot\|_{1,2}$.

In the following, we always assume that (M, F, m) is a compact Finsler manifold (M, F) without or with a smooth boundary equipped with a smooth measure m. Let H_0^1 be a space of functions $u \in W^{1,2}(M)$ with $\int udm = 0$ if $\partial M = \varnothing$ and $u|_{\partial M} = 0$ if $\partial M \ne \varnothing$. For any nonzero function $u \in H_0^1 \setminus \{0\}$, we define the energy of u by

$$E(u) := \frac{\int_M [F^*(x,du)]^2 dm}{\int_M |u|^2 dm}.$$

Note that E is C^1 on $H_0^1 \setminus \{0\}$. We say λ is an *eigenvalue* of (M, F, m) if there is a function $u \in H_0^1 \setminus \{0\}$ such that $d_u E = 0$ with $\lambda = E(u)$. In this case, u is called an *eigenfunction* corresponding to λ.

Theorem 3.1. *([3]) There is a function $u \in H_0^1 \setminus \{0\}$ with $\|u\|_2 = 1$, which minimizes the canonical energy functional E. Thus $\lambda_1 := \inf_{u \in H_0^1} E(u)$ is a critical value of E and u is a critical point of E corresponding to λ_1.*

By a direct calculation, $d_u E = 0$ if and only if

$$\Delta u = -\lambda u \tag{3.2}$$

in a weak sense. Theorem 3.1 implies that there is a weak solution of the equation (3.2) with $\lambda = \inf_{u \in H_0^1} E(u)$ in H_0^1. In this case, λ is called the *first eigenvalue* of the Finsler Laplacian Δ, denoted by λ_1, and u is called the *first eigenfunction* of Δ corresponding to λ_1. If F is a Riemannian metric, then Δ is reduced to the usual Laplace operator and all eigenfunctions of Δ are smooth on a Riemannian manifold M. A natural question arises: is every eigenfunction on Finsler manifolds smooth? The following example shows the answer is negative.

Example 3.1. ([3], [13]) Let $u_0 = \chi(|x|)$ be the eigenfunction corresponding to the first eigenvalue $\lambda_1(\mathbb{B}^n(1))$ of the standard ball $\mathbb{B}^n(1)$ in the Euclidean space $\mathbb{R}^n$, where $|\cdot|$ is the standard Euclidean norm in $\mathbb{R}^n$ and χ is a smooth function on $\mathbb{R}$ satisfying

$$\chi''(t) + \frac{n-1}{t}\chi'(t) + \lambda_1 \chi(t) = 0, \qquad \chi(1) = 0.$$

Let $(\mathbb{R}^n, F)$ be an arbitrary Minkowski space and $B^n(r) := \{x \in \mathbb{R}^n | F(x) < 1\}$. Define

$$u(x) := \chi(F(x).$$

It can be shown that $\lambda_1(\mathbb{B}^n(r)) = \lambda_1(B^n(1))$ and u is an eigenfunction corresponding to $\lambda_1(B^n(1))$. Clearly, u is C^∞ at $x \neq 0$ and only $C^{1,1}$ at $x = 0$.

The above example also shows that one can not expect a better regularity of eigenfunctions than $C^{1,1}$. In fact, it was proved that the following

Theorem 3.2. *([3]) Let (M, F, m) be a compact Finsler measure space without or with boundary and $u \in H_0^1$ be a weak solution of (3.2). Then $u \in C^{1,\alpha}(M)$ for some $0 < \alpha < 1$ and $u \in C^\infty(M_u)$, where $M_u = \{x \in M | du(x) \neq 0\}$.*

4 Eigencone corresponding to the first eigenvalue

Let (M, F, m) be a compact Finsler measure space without boundary or with a smooth boundary. If $\partial M = \varnothing$, then (3.2) is called the *closed eigenvalue problem.* Theorem 3.1 implies that the smallest eigenvalue of the closed eigenvalue problem is given by

$$\lambda_1^C = \inf \left\{ E(u) | 0 \neq u \in W^{1,2}(M) \text{ and } \int_M u dm = 0 \right\},$$

which is called the *first closed eigenvalue.* If $\partial M \neq \varnothing$, then (3.2) with the Dirichlet boundary condition $u|_{\partial M} = 0$ (resp. with the Neumann boundary condition $\nabla u \in T(\partial M)$) is called the *Dirichlet (resp. Neumann) eigenvalue problem.* By Theorem 3.1, the smallest eigenvalues of the Dirichlet and Neumann eigenvalue problem are given by

$$\begin{aligned} \lambda_1^D &= \inf \left\{ E(u) | 0 \neq u \in W^{1,2}(M) \text{ and } u|_{\partial M} = 0 \right\}, \\ \lambda_1^N &= \inf \left\{ E(u) | 0 \neq u \in W^{1,2}(M) \text{ and } \int_M u dm = 0 \right\}, \end{aligned}$$

which are called the *first Dirichlet and Neumann eigenvalue* respectively. The weak solution u of the closed (resp. Dirichlet or Neumann) eigenvalue problem with $\lambda = \lambda_1^C$ (resp. λ_1^D or λ_1^N) is called the *first closed (resp. Dirichlet or Neumann) eigenfunction.*

Denote by V_{λ_1} the union of the zero function and the set of all eigenfunctions corresponding to the first eigenvalue λ_1. If F is Riemannian, it is well known that each eigenspace V_λ is a linear subspace of H_0^1. However, for a Finsler metric F, V_{λ_1} is only a cone, not a subspace in H_0^1, which is called the *eigencone* corresponding to λ_1. In fact, if u is a weak solution of $\Delta u = -\lambda_1 u$, then $-u$ is a weak solution of $\overleftarrow{\Delta} u = -\lambda_1 u$.

We say two functions $f(x)$ and $h(x)$ on M are *linearly dependent* if there is a nonzero constant c such that $f = ch$ or $h = cf$. Otherwise, we say $f(x)$ and $h(x)$ are *linearly independent.* A nonzero function is always linearly independent.

Example 4.1. Let $F(y) = |y| + < c, y >$ be a Minkowski norm on $\mathbb{R}^n$, where c is a constant vector, $<,>$ is a usual Euclidean inner product and $|\cdot|$ is an Euclidean norm. With respect to the Busemann Hausdorff measure, the volume form of F is given by

$$dV = (1 - |c|^2)^{(n+1)/2} dx.$$

Let $u(y) = f(F(y))$ for some nondecreasing C^2 function on $\mathbb{R}^+$. Then

$$\Delta u = \frac{(n-1)f'(F(y))}{F(y)} + f''(F(y)). \tag{4.1}$$

If f is nonincreasing, then an analogous expression holds for $v(x) = f(F(-y))$, i.e.,

$$\Delta u = \frac{(n-1)f'(F(-y))}{F(y)} + f''(F(-y)). \tag{4.2}$$

For nonincreasing f, the right hand of (4.1) coincides with $\overleftarrow{\Delta} u$. Similarly, the right hand of (4.2) coincides with $\overleftarrow{\Delta}$ for nondecreasing f.

Assume that λ_1 is the first Dirichlet eigenvalue of Δ and f is a strictly increasing function satisfying the following ODE on the ball $B^n(1) := \{y \in \mathbb{R}^n | F(y) < 1\}$:

$$f''(t) + \frac{n-1}{t} f'(t) + \lambda_1 f(t) = 0, \qquad f(1) = 0.$$

Then $u(y) = f(F(y))$ and $v(y) = -f(F(-y))$ are the first Dirichlet eigenfunctions of Δ corresponding to λ_1. In fact, $\lambda_1(B^n(1)) = \lambda_1(\mathbb{B}^n(1))$, where $\lambda_1(\mathbb{B}^n(1))$ is the first eigenvalue of the Euclidean Laplacian Δ_0 with the first eigenfunction $u(y) = f(|y|)$ defined on an Euclidean ball $\mathbb{B}^n(1)$ in $\mathbb{R}^n$ (see Example 3.1). A direct calculation shows that $u(y)$ and $v(y)$ are linearly independent unless F is reversible.

The above example shows that the first Dirichlet eigencone for the Finsler Laplacian may not one-dimensional space. Denote $V^D_{\lambda_1}$ as an eigencone consisting of the first Dirichlet eigenfunctions corresponding to λ^D_1 and

$$M^u_{\geq} := \{x \in M | u(x) \geq 0\}, \qquad M^u_{\leq} := \{x \in M | u(x) \leq 0\}.$$

In fact, we can prove the following

Theorem 4.1. *Let (M, F, m) be an n-dimensional compact Finsler measure space with a smooth boundary. Assume λ^D_1 is the first Dirichlet eigenvalue for the Finsler Laplacian and $u \in V^D_{\lambda_1}$ is an eigenfunction corresponding to λ^D_1. Then one of the followings holds.*

(1) All the first Dirichlet eigenfunctions in $V^D_{\lambda_1}$ are positive or negative on $M^u_{\geq} \setminus \partial M$ and on $M^u_{\leq} \setminus \partial M$ respectively. Further, λ_1 is simple, i.e., for any $0 \neq v \in V^D_{\lambda_1}$, v and u are linearly dependent on $M^u_{\geq}$ and $M^u_{\leq}$ respectively.

(2) There exists a function $v \in V^D_{\lambda_1}$ such that u and v are linearly independent. Moreover, all the other first Dirichlet eigenfunctions are positive on either $M^u_{\geq} \setminus \partial M$ or $M^v_{\geq} \setminus \partial M$ and negative on either $M^u_{\leq} \setminus \partial M$ or $M^v_{\leq} \setminus \partial M$ respectively. Further, all the other first Dirichlet eigenfunctions are merely positive constant multiples of each other on either $M^u_{\geq}$ or $M^v_{\geq}$ and on either $M^u_{\leq}$ or $M^v_{\leq}$ respectively.

Theorem 4.1 implies that the first Dirichlet eigencone is at most two-dimensional space. In particular, if F is a reversible Finsler metric and u is the first Dirichlet eigenfunction on (M, F), then $|u|$ is also a nonnegative first Dirichlet eigenfunction. In this case, only the case (1) of Theorem 4.1 occurs. Thus we have

Corollary 4.2. *([3]) Let (M, F, m) be a compact reversible Finsler measure space with a smooth boundary. Suppose that $u \in H_0^1$ is a Dirichlet eigenfunction corresponding to λ_1^D. Then either $u > 0$ or $u < 0$ in $M \setminus \partial M$. Furthermore, the eigencone $V_{\lambda_1}^D$ corresponding to λ_1^D is a one-dimensional space.*

In particular, if F is a Riemannian metric, then Corollary 4.2 is reduced to the classical result (cf. Theorem 8.38 in [4] or Theorem 1.3 in [7]).

5 Lower bound estimates of the first eigenvalue

In this section, we introduce some recent developments on lower bound estimates for the first eigenvalue of the Finsler Laplacian. The first result is due to Y.Ge and Z.Shen, who gave the Faber-Krahn type and Cheng type lower bound estimates for the first eigenvalue respectively ([3], [13]). Recently, B. Wu and Y. Xin gave some Mckean type lower bound estimates for the first eigenvalue in [19] under different assumptions. In the following, we only concentrate on the lower bound estimates of the first eigenvalue on a Finsler manifold whose weighted Ricci curvature is bounded from below. On this, an important progress was made by G.Wang and C.Xia, inspired by Bakry-Qian's work (cf.[1]).

Theorem 5.1. *([18]) Let (M^n, F, m) be an n-dimensional compact Finsler manifold without boundary or with a convex boundary. Assume that $Ric_N \geq K$ for some real numbers $N \in [n, \infty]$ and $K \in \mathbb{R}$. Let λ_1 be the first (nonzero) closed or Neumann eigenvalue for the Finsler Laplacian, i.e.,*

$$\Delta_m u = -\lambda_1 u, \quad \textit{in} \ \ M$$

with the Neumann boundary condition

$$\nabla u \in T_x(\partial M),$$

if ∂M is not empty. Then

$$\lambda_1 \geq \lambda_1(K, N, d),$$

where d is the diameter of M and $\lambda_1(K, N, d)$ is the first (nonzero) eigenvalue of the 1-dimensional problem

$$(5.1)\ \ v'' - T(t)v' = -\lambda_1(K, N, d)v, \quad \textit{in} \ \ (-\frac{d}{2}, \frac{d}{2}), \qquad v'(-\frac{d}{2}) = v'(\frac{d}{2}) = 0,$$

with $T(t)$ defined by

$$(5.2)\quad T(t) = \begin{cases} \sqrt{(N-1)K}\tan\left(\sqrt{\frac{K}{N-1}}\right)t, & \textit{for} \ \ K > 0, 1 < N < \infty, \\ -\sqrt{-(N-1)K}\tanh\left(\sqrt{-\frac{K}{N-1}}\right)t, & \textit{for} \ \ K < 0, 1 < N < \infty, \\ -\frac{N-1}{t}, & \textit{for} \ \ K = 0, 1 < N < \infty, \\ Kt, & \textit{for} \ \ N = \infty. \end{cases}$$

In particular, if F is Riemannian, Theorem 5.1 is reduced to Theorem 3 in [1], which improved all known lower bound estimates for the first eigenvalue of the Laplacian (see the introduction in [1]). Note that $T(t)$ in (5.2) satisfies $T' = K + \frac{T^2}{N-1}$. From this and (5.1), one obtains a unified lower bound for the first eigenvalue of the Finsler Laplacian via integrating by parts.

Theorem 5.2. *Let (M^n, F, m) be an n-dimensional compact Finsler manifold without boundary or with a convex boundary. Assume that $Ric_N \geq K$ for some real numbers $N \in [n, \infty]$ and $K \in \mathbb{R}$. Let λ_1 be the first (nonzero) closed or Neumann eigenvalue for the Finsler Laplacian, i.e.,*

$$\Delta_m u = -\lambda_1 u, \quad \text{in } M$$

with the Neumann boundary condition

$$\nabla u \in T_x(\partial M),$$

if ∂M is not empty. Then

$$\lambda_1 \geq \sup_{s\in(0,1)} \left\{ 4s(1-s)\frac{\pi^2}{d^2} + sK \right\}, \tag{5.3}$$

where d is the diameter of M.

Note that the diameter $d < \pi\sqrt{(N-1)/K}$ if $K > 0$ (see Theorem 7.3 in [9]). By a direct calculation, it is easy to see that

$$\sup_{s\in(0,1)} \left\{ 4s(1-s)\frac{\pi^2}{d^2} + sK \right\} = \begin{cases} 0, & \text{if } Kd^2 < -4\pi^2, \\ (\frac{\pi}{d} + \frac{Kd}{4\pi})^2, & \text{if } Kd^2 \in [-4\pi^2, 4\pi^2], \\ K, & \text{if } Kd^2 \in (4\pi^2, \frac{N-1}{K}\pi^2]. \end{cases}$$

In particular, if $K = 0$, then the right side in (5.3) arrives the maximum $\frac{\pi^2}{d^2}$. In this case, $\lambda_1 \geq \frac{\pi^2}{d^2}$, which is optimal in the sense of Theorem 5.3 below.

Theorem 5.3. *Let (M^n, F, m), λ_1 and d be as in Theorem 5.2. Assume the weighted Ricci curvature Ric_N of M is nonnegative for $N \in [n, \infty]$. Then $\lambda_1 \geq \frac{\pi^2}{d^2}$ and the equality holds if and only if M is a 1-dimensional segment or 1-dimensional circle.*

Proof. We only give the sketch of the proof here. See [21] for more details. The proof is divided into four steps as follows.

Step I. Prove that a necessary (but not necessarily sufficient) condition for $\lambda_1 = \frac{\pi^2}{d^2}$ is that $\max u = -\min(u)$ for any eigenfunction $u(x)$.

Step II. We always may assume $\min(u) = -1$. From Step I, we can prove if $\lambda_1 = \frac{\pi^2}{d^2}$, then

(1) the function $P(x) := F^2(x, \nabla u) + \lambda u(x)^2 = \lambda_1$ on M. Moreover, $M_u = \{x \in M \mid u(x) \neq \pm 1\}$.

(2) the vector field $X := \frac{\nabla u}{F(\nabla u)}$ is a geodesic field of F on M_u.

Let $M_0 := u^{-1}(0)$, which is a hypersurface of M. Define a map

$$\varphi : I \times M_0 \to M_u, \quad \varphi(s, x) = \varphi_s(x),$$

where $\varphi_s(x) = \exp_x(sX)$ be the one-parameter transformation group generated by X on M_u.

Step III. Based on Step II, we prove $I = (-d/2, d/2)$ and φ is a diffeomorphism. Furthermore, we prove the weighted Riemannian manifold (M_u, g_X, m) admitting an isometric splitting $((-d/2, d/2) \times M_0, dt^2 \otimes h_X, m)$ with $M_0 = u^{-1}(0)$, where h_X is the induced Riemannian metric on M_0 from g_X defined by $g_X(x) = g(x, X)$.

Step IV. Prove $\dim M = 1$. This will be completed by the arguments of the following two cases.

Case I. If M_0 has more than one connected component, then the connected components of M_0 are discrete, which implies that $\dim M = 1$.

Case II. If M_0 has only one connected component, then we define a map $\Phi : [-\frac{d}{2}, \frac{d}{2}] \times M_0 \to M$ by

$$\Phi(s, x) = \exp_x(sX), \quad \text{for } x \in M_0, \quad X \in T_x M_0.$$

Claim: Φ is a unique differential extension of φ and Φ is a diffeomorphism. Finally, we prove the maximum and minimum point are unique respectively. Hence $\dim M = 1$. □

In particular, if (M^n, F) is a Riemannian manifold without boundary or with a convex boundary, then Theorem 5.3 is reduced to Zhong-Yang and Hang-Wang's results, which asserted that $\lambda_1(M^n) \geq \frac{\pi^2}{d^2}$ if M has nonnegative Ricci curvature and the equality holds if and only if M is a 1-dimensional circle or 1-dimensional segment ([23], [5]). In Finslerian case, the authors considered the optimal lower bound for the first eigenvalue on compact Finsler manifolds with nonnegative ∞-weighted Ricci curvature Ricci_∞ under some extra assumptions in [21],. Thus, Theorem 5.3 extends Zhong-Yang's well known sharp estimate in Riemannian case and Yin-He-Shen's result in Finslerian case. It is worth mentioning that the proof of Theorem 5.3 is not based on the gradient estimate of the eigenfunctions, which was used in [21] and [23], but on a comparison theorem on the gradient of the first eigenfunction with that of a one dimensional model function, which was given in [18].

Acknowledgements. This work is supported by NNSFC (No.11171297) and Zhejiang Provincial Natural Science Foundation of China (No. LY15A010002).

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