Convex mappings between Riemannian manifolds

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Abstract. Starting with the second fundamental form of a differentiable mapping between arbitrary dimensioned Riemannian manifolds, this paper defines, in a natural way, its convexity. The classical concept of geodesic and the new concept of convex (concave) curve on a Riemannian manifold are expressed in relation to convex mappings. Some analytical and geometric descriptions are given in order to establish the position of convex mappings in the context of other remarkable applications, such as harmonic, subharmonic, superharmonic and totally geodesic ones. Also, some invariant convexity is defined and analyzed, based on Riemannian cone fields structures.

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1 Introduction

The theory of harmonicity in Riemannian context classifies harmonic morphisms and totally geodesic mappings as harmonic mappings, that is differentiable mappings of class C^∞ with null local tension field ([3]-[8], [12], [18], [24]). More precisely, recall that harmonic morphisms are semi-conformal harmonic mappings (see [2], [9]-[11], [21]), while totally geodesic mappings have null second fundamental form.

Inspired by the Riemannian convexity of functions analyzed in [22] and also by the subharmonicity of functions studied in [13], some new concepts came in order to be defined:
(1) subharmonic/ superharmonic morphisms (see [1]) as differentiable mappings pulling back germs of subharmonic/ superharmonic functions into germs of subharmonic/ superharmonic functions;
(2) subharmonic and superharmonic mappings (see [1]) as a class C^∞ mappings having positive/ negative tension field local components; from geometric point of view, the subharmonic mappings pull back germs of partial increasing convex functions into germs of subharmonic functions, while the superharmonic mappings pull back germs

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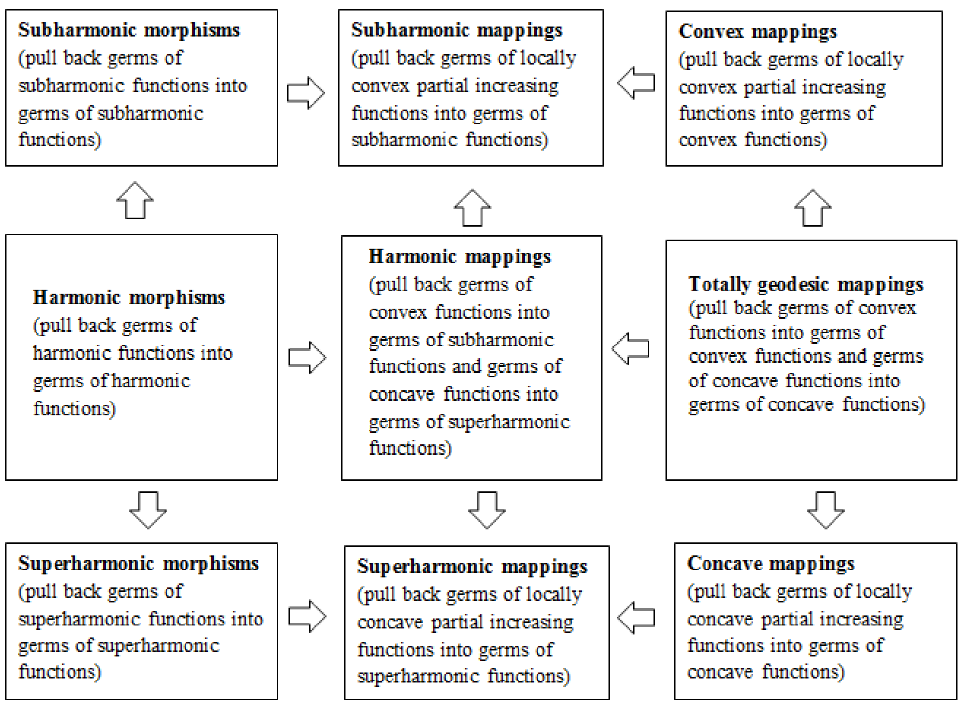
of partial increasing concave functions into germs of superharmonic functions;
(3) convex and concave mappings between Riemannian manifolds (the topic of the current paper), as a class C^∞ mappings with Hessian matrices fields containing the local components of the second fundamental form positive/ negative semidefinite. Convexity of mappings is studied through its analytical and geometrical features, and through its correlation with different aspects of harmonicity. More precisely, some equivalent geometric definition of convexity may be phrased; one of these, for example, states that convex mappings pull back germs of locally convex partial increasing functions into germs of convex functions.

The following diagram includes all these results and gives a complete perspective on harmonicity.



The midline of the array expresses a classical outcome, while the rest of it consists in original results (some of them partially published previously, others included in the present paper).

2 Geometric tools related to Riemannian differentiable mappings

2.1 The second fundamental form of C^∞ mappings

This section is dedicated entirely to recalling basic definitions and instruments related to differentiable mappings between Riemannian manifolds ([2], [15], [16], [25]). Let (M, g) and (N, h) be two Riemannian manifolds, and let $\varphi \in C^\infty(M, N)$ be a class

C^∞ differentiable mapping between them. The differential of φ at $x \in M$ is the homomorphism of tangent spaces $d\varphi_x : T_xM \to T_{\varphi(x)}N,\ d\varphi_x(X_x)(f) = X_x(f \circ \varphi),\ \forall f \in C^\infty(M)$. Moreover, if $\varphi^{-1}TN = \cup_{x\in M} T_{\varphi(x)}N$, then $d\varphi \in E = Hom(TM, \varphi^{-1}TN) = T^*M \otimes \varphi^{-1}TN \to M$. Since E is a fiber bundle over M, there exists an induced linear connection ${}^E\nabla$, called *pull-back connection*, generated by the Levi-Civita connection ∇^M on M and the pull-back connection ∇^φ of the inverse tangent bundle $\varphi^{-1}TN \to M$, generated itself by the Levi-Civita connection ∇^N on N. More precisely, if $X, Y \in C^\infty(TM)$, $F \in C^\infty(Hom(TM, \varphi^{-1}TN))$ and $Z \in C^\infty(\varphi^{-1}TN)$, then

$$\nabla^\varphi(X, Z) = \nabla^\varphi_X Z = \nabla^N_{d\varphi(X)} Z$$

and

$${}^E\nabla F(X, Y) = \left({}^E\nabla_X F\right) Y = \nabla^\varphi_X F(Y) - F(\nabla^M_X Y).$$

In particular, for $F = d\varphi$, it follows:

$${}^E\nabla d\varphi(X, Y) = \nabla^\varphi_X(d\varphi(Y)) - d\varphi(\nabla^M_X Y) = \nabla^N_{d\varphi(X)}(d\varphi(Y)) - d\varphi(\nabla^M_X Y).$$

Definition 2.1. If (M, g), (N, h) and φ are as above, then $\beta : C^\infty(M, N) \to C^\infty(T^*M \times T^*M \otimes \varphi^{-1}TN)$, defined by $\beta(\varphi) \overset{not}{=} {}^E\nabla d\varphi$ is called the *second fundamental form* of the differentiable mapping φ.

Since $C^\infty(T^*M \otimes T^*M \otimes \varphi^{-1}TN) = C^\infty(Hom(TM \times TM, \varphi^{-1}TN))$, it follows that $\beta(\varphi)$ is a 2-covariant tensor field on M, i.e., $\beta(\varphi) : C^\infty(TM) \times C^\infty(TM) \to C^\infty(\varphi^{-1}TN)$.

Proposition 2.1. *The second fundamental form of a differentiable mapping is a symmetric bilinear tensor field.*

Definition 2.2. If $\varphi \in C^\infty(M, N)$ is a differentiable mapping between Riemannian manifolds and $\beta(\varphi)$ denotes the second fundamental form, then $\tau(\varphi) = Tr_g \beta(\varphi)$ is called *the tension field of the mapping* φ.

Definition 2.3. A mapping $\varphi \in C^\infty(M, N)$ satisfying $\tau(\varphi) \equiv 0$ is called *harmonic mapping.*

Definition 2.4. A mapping $\varphi \in C^\infty(M, N)$ satisfying $\beta \equiv 0$ is called *totally geodesic mapping.*

Remark 2.5. Let $(x^1, ..., x^m)$ and $(y^1, ..., y^n)$ denote the local coordinates on M and N, and let $\{\frac{\partial}{\partial x^i}\}_{i=\overline{1,m}}$ and $\{\frac{\partial}{\partial y^\alpha}\}_{\alpha=\overline{1,n}}$ denote the corresponding local frame fields in $C^\infty(TM)$ and $C^\infty(\varphi^{-1}TN)$, respectively. In the following we use classical Eisenhart notations for local tensorial calculus: $\varphi^\alpha_i = \frac{\partial \varphi^\alpha}{\partial x^i}$, $\varphi^\gamma_{ij} = \frac{\partial^2 \varphi^\gamma}{\partial x^i \partial x^j} -^g \Gamma^k_{ij} \varphi^\gamma_k$ etc., where ${}^g\Gamma^k_{ij}$ denote the Christoffel symbols on M.

Since the range of the second fundamental form of some differentiable mapping consists in sections of the pull-back fiber bundle $\varphi^{-1}TN \to M$, denote by $\varphi^\gamma_{;ij}$ its local components with respect to the canonical local frame field, that is

$$\beta(\varphi)_{ij} = ({}^E\nabla d\varphi)_{ij} =^E \nabla d\varphi(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \varphi^\gamma_{;ij} \frac{\partial}{\partial y^\gamma}.$$

This suggests that it represents a second order partial covariant derivative. Indeed, according to the definition of β,

$$\beta(\varphi)_{ij} =^E \nabla d\varphi(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \nabla^N_{d\varphi(\frac{\partial}{\partial x^i})}(d\varphi(\frac{\partial}{\partial x^j})) - d\varphi(\nabla^M_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j})$$

$$= \nabla^N_{\varphi^\alpha_i \frac{\partial}{\partial y^\alpha}}(\varphi^\beta_j \frac{\partial}{\partial y^\beta}) - d\varphi({}^g\Gamma^k_{ij}\frac{\partial}{\partial x^k}) = \left[\frac{\partial^2 \varphi^\gamma}{\partial x^i \partial x^j} +^h \Gamma^\gamma_{\alpha\beta}\varphi^\alpha_i \varphi^\beta_j -^g \Gamma^k_{ij}\varphi^\gamma_k\right]\frac{\partial}{\partial y^\gamma}$$

$$= \left[\varphi^\gamma_{ij} +^h \Gamma^\gamma_{\alpha\beta}\varphi^\alpha_i\varphi^\beta_j\right]\frac{\partial}{\partial y^\gamma},$$

therefore,

$$\varphi^\gamma_{;ij} = \varphi^\gamma_{ij} +^h \Gamma^\gamma_{\alpha\beta}\varphi^\alpha_i\varphi^\beta_j.$$

This local expression of the second fundamental form also proves its symmetry.

Moreover, when choosing normal local coordinates with respect to $p \in M$ and $q = \varphi(p) \in N$, since ${}^g\Gamma^k_{ij}(p) = 0$ and ${}^h\Gamma^\gamma_{\alpha\beta}(q) = 0$, it follows

$$\beta(\varphi)_{ij}(p) = ({}^E\nabla d\varphi)_{ij}(p) = \frac{\partial^2\varphi^\sigma}{\partial x^i \partial x^j}(p)\frac{\partial}{\partial y^\sigma}\Big|_{\varphi(p)}.$$

Also, the tension field may be expressed through its local components $\tau^\gamma(\varphi) = g^{ij}\varphi^\gamma_{;ij}$.

Remark 2.6. In particular, if (M,g) is a Riemannian manifold and $f \in C^\infty(M)$, then df is a differentiable 1-form, and the corresponding second fundamental form is the Hessian tensor $\beta(f) = \text{Hess}f = \nabla df : C^\infty(TM) \times C^\infty(TM) \to C^\infty(M)$,

$$\beta(f)(X,Y) = X(Yf) - ({}^g\nabla_X Y)f,\ \forall X, Y \in C^\infty(TM),$$

or, equivalent,

$$\beta(f) = \left(\frac{\partial^2 f}{\partial x^i \partial x^j} -^g \Gamma^k_{ij}\frac{\partial f}{\partial x^k}\right)dx^i \otimes dx^j = f_{ij}dx^i \otimes dx^j.$$

The Hessian of f is the matrix field $(\beta(f)_{ij})_{i,j=\overline{1,m}}$.

Remark 2.7. Let (M,g) be a Riemannian manifold and $\gamma : \mathbb{R} \to M$ be a parametrized curve on M. If $\frac{d}{dt}$ denotes the coordinate vector field on $\mathbb{R}$, then $d\gamma(\frac{d}{dt}) = \dot\gamma$ and the second fundamental form of γ is

$$\beta(\gamma)(\frac{d}{dt}, \frac{d}{dt}) =^M \nabla_{\dot\gamma}\dot\gamma,$$

with local components

$$\beta(\gamma)^k = \gamma^k_{;} = \frac{d^2\gamma^k}{dt^2} +^g \Gamma^k_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt},\ \forall k = \overline{1,m}.$$

Since, for this particular situation, $\tau(f) = \beta(f)$, it follows that whenever the above components vanish, we are dealing with a geodesic of the manifold (M,g).

In the following, consider three Riemannian manifolds (M, g), (N, h) and (P, l), respectively and two C^∞ differentiable mappings $\varphi : M \to N$ and $\psi : N \to P$.

Proposition 2.2. *(Second fundamental form of composed mappings ([2]).) The behavior of the second fundamental form with respect to composition of mappings is described by*

$$\beta(\psi \circ \varphi) = \beta(\psi)(d\varphi \otimes d\varphi) + d\psi(\beta(\varphi)),$$

or, in local coordinates,

$$(\psi \circ \varphi)^r_{;ij} = \psi^r_{;\alpha\beta}\varphi^\alpha_i \varphi^\beta_j + \psi^r_\alpha \varphi^\alpha_{;ij}.$$

2.2 Convexity, subharmonicity and superharmonicity of functions

Convex and strictly convex functions [22], [23], subharmonic and superharmonic functions [13] are highly relevant elements in the context of partial differential equations, multivariable complex calculus and potential theory. Intuitively, subharmonic functions are related to one-variable convex functions as it follows: given a convex function, its graph is situated under each segment line connecting two of its points; similarly, if the values of a subharmonic function, restricted to an arbitrary sphere are smaller then the values of a harmonic function on that sphere, then same property is valid for the interior of the sphere, too. The complex analogue for convex functions are the plurisubharmonic functions ([13], [20]). They are relevant in complex analysis ([14], [19]), for defining Stein manifolds and also in the study of holomorphic and pseudoconvex domains.

Definition 2.8. ([22]) Let (M, g) be a complete Riemannian manifold and $U \subset M$ be an open totally convex subset. A function $f : U \to \mathbb{R}$ is called *(geodesic) convex (on U)* if its restriction to each geodesic segment is convex, i.e. for each geodesic $C : \mathbb{R} \to U$ and each $a, b \in \mathbb{R}$,

$$f(C(\lambda a + (1-\lambda)b)) \leq \lambda f(C(a)) + (1-\lambda) f(C(b)),\ \forall \lambda \in [0, 1].$$

When dealing with strict inequality, the function f is called strictly convex.

An important aspect related to convexity is the correlation with the second order covariant derivative (see [22]). If $f : M \to \mathbb{R}$ is a class C^2 convex function, then $Hess(f) = (f_{ij})_{i,j \in \overline{1,m}}$ is positive semidefinite all over M,

$$\beta(X, X) = ({}^M\nabla df)(X, X) \geq 0,\ \forall X \in C^\infty(TM).$$

If the previous inequality is strict for each nonzero vector field, that is the Hessian matrix field is positive definite, then f is strictly convex on M. Yet, the converse of this statement is not true. Moreover, the concavity of a function f is equivalent with the convexity of its opposite $-f$ and with the Hessian matrix field being negative semidefinite.

Definition 2.9. Let (M, g) be a Riemannian manifold and $U \subset M$ be an open subset. A class C^2 local function $f : U \to \mathbb{R}$ is called *subharmonic (superharmonic)* if its Laplacian is non-negative (non-positive), i.e. $\Delta^M f = Trace_g({}^g\nabla f) \geq (\leq) 0$.

If the above inequalities are strict, the f is called strictly subharmonic and strictly superharmonic, respectively. If , instead, $\Delta^M f = 0$, all over the open set U, then f is called harmonic function on U.

Remark 2.10. Each C^∞ geodesic convex function on a Riemannian manifold is a subharmonic function and also Lipschitz continuous on the compact subsets of manifold M (see [13]).

The following Lemma combines results from [2], [11] and [17] and provides an important tool in the study of convexity and harmonicity. Its proof is based on two important ideas: (1) locally, by choosing normal coordinates around an arbitrary fixed point, the Riemannian environment gains Euclidean behavior and (2) the C^∞ function subject to analyze may be replaced by a Taylor polynomial.

Lemma 2.3. *(Functions arising from given 2-jets.) Let (N, h) be an n-dimensional Riemannian manifold, $n \geq 2$ and c, C_α, $C_{\alpha\beta} \in \mathbb{R}$ given constants such that matrix $C = (C_{\alpha\beta})_{\alpha,\beta=\overline{1,n}}$ is symmetric. Then, for each point $q \in N$, there exists a C^∞ real valued function f defined on an open neighborhood of q, such that*

$$f(q) = c;\ f_\alpha(q) = C_\alpha;\ f_{\alpha\beta}(q) = C_{\alpha\beta} \tag{2.1}$$

and
A.

1. *if C is positive (negative) definite, then f is strictly convex (concave);*
2. *if C is positive (negative) semidefinite, then f is convex (concave) at q;*

B.

1. *if $\operatorname{Tr} C = \sum_{\alpha=1}^n C_{\alpha\alpha} > (<) 0$, then f is strict subharmonic (superharmonic);*
2. *if $\operatorname{Tr} C \geq (\leq) 0$, then f is subharmonic (superharmonic) at q;*
3. *if $\operatorname{Tr} C = 0$, then f is harmonic at q.*

Proof. Let $q \in N$ and $(V, (y^\alpha)_{\alpha=\overline{1,n}})$ be a normal local chart centered at q. Assume that f(q)=c=0. If $\Gamma^\gamma_{\alpha\beta}$ denote the Christoffel symbols with respect to the considered local chart, they may be rewritten

$$\Gamma^\gamma_{\alpha\beta} = K^\gamma_{\alpha\beta\sigma} y^\sigma + L^\gamma_{\alpha\beta\sigma\delta} y^\sigma y^\delta,$$

where $K^\gamma_{\alpha\beta\sigma} \in \mathbb{R}$ and $L^\gamma_{\alpha\beta\sigma\delta} \in C^\infty(V)$. Define $f : V \to \mathbb{R}$,

$$f = C_\alpha y^\alpha + \frac{1}{2} C_{\alpha\beta} y^\alpha y^\beta.$$

Since $(y^\alpha)_{\alpha=\overline{1,n}}$ are normal local coordinates centered at q, it follows that

$$f_\alpha(q) = C_\alpha,\ f_{\alpha\beta}(q) = C_{\alpha\beta}.$$

Moreover, $f_{\alpha\beta} = \frac{\partial^2 f}{\partial y^\alpha \partial y^\beta} - \Gamma^\gamma_{\alpha\beta} f_\gamma = C_{\alpha\beta} - \Gamma^\gamma_{\alpha\beta}(C_\gamma + C_{\gamma\sigma} y^\sigma)$ and it follows

$$f_{\alpha\beta} = C_{\alpha\beta} - K^\gamma_{\alpha\beta\sigma} C_\gamma y^\sigma - P_{\alpha\beta\sigma\delta} y^\sigma y^\delta,$$

where $P_{\alpha\beta\sigma\delta} = K^{\gamma}_{\alpha\beta\sigma} C_{\gamma\delta} + C_{\gamma} L^{\gamma}_{\alpha\beta\sigma\delta} + C_{\gamma\mu} L^{\gamma}_{\alpha\beta\sigma\delta} y^{\mu} \in C^{\infty}(V)$ and $C_{\alpha\beta},\ C_{\alpha},\ K^{\gamma}_{\alpha\beta\sigma} \in \mathbb{R}$.
A. If $C = (C_{\alpha\beta})_{\alpha,\beta=\overline{1,n}}$ is positive definite, then $(f_{\alpha\beta})_{\alpha,\beta=\overline{1,n}}$ is also positive definite in a neighborhood of q, therefore f is strictly convex. Moreover, statement $A2$ is a direct consequence of (2.1).
B. The normality of the local chart around q leads to $\Delta f(q) = \delta^{\alpha\beta} \frac{\partial^2 f}{\partial y^{\alpha} \partial y^{\beta}}(q) = \mathrm{Tr}\, C$, and similar arguments as above lead to the announced conclusion. □

3 Convex mappings between Riemannian manifolds

3.1 Definition of convexity

Recently, the subharmonicity and superharmonicity were extended from differentiable functions to differentiable mappings between Riemannian manifolds. Similar ideas are used in the following in order to introduce and analyze the new concept of convex Riemannian mapping.

Let (M, g) and (N, h) be two Riemannian manifolds and $\varphi : M \to N$ a C^{∞} differentiable mapping. If $(U, (x^1, ..., x^m))$ and $(V, (y^1, ..., y^n))$ are local charts around $p \in M$ and $q = \varphi(p)$, respectively, we denoted by $\varphi^{\sigma}_{;ij} = \varphi^{\sigma}_{ij} +^{h} \Gamma^{\sigma}_{\alpha\beta} \varphi^{\alpha}_{i} \varphi^{\beta}_{j}$ the components of the second fundamental form of φ with respect to the chosen coordinates. The Hessians of the mapping φ at p are the symmetric matrix fields $Hess^{\sigma} f(p) = (\beta(\varphi)^{\sigma}_{ij}(p))_{i,j=\overline{1,m}} = (\varphi^{\sigma}_{;ij}(p))_{i,j=\overline{1,m}}$, $\sigma = \overline{1,n}$ and the Laplacians of the mapping φ are the traces of these Hessians, with respect to metric g, i.e. $(\Delta f)^{\sigma}(p) = g^{ij}(p) \varphi^{\sigma}_{;ij}(p)$, that is the components of the tension field with respect to the chosen coordinate frame field.

Definition 3.1. A C^{∞} differentiable mapping $\varphi : M \to N$ is called *subharmonic (superharmonic)* if the corresponding Laplacians are positive (negative) on M, i.e.

$$g^{ij}(p) \varphi^{\sigma}_{;ij}(p) \geq 0\, (\leq 0),\ \forall p \in M,\ \forall \sigma = \overline{1,n}.$$

Definition 3.2. A C^{∞} differentiable mapping $\varphi : M \to N$ is called convex (concave) if the corresponding Hessian matrices are positive (negative) semidefinite on M, i.e.

$$\varphi^{\sigma}_{;ij}(p) \xi^i \xi^j \geq 0,\ \forall p \in M,\ \forall \xi = (\xi^i)_{i=\overline{1,m}} \in \mathbb{R}^m \backslash \{0\}.$$

Whenever the above inequalities are strict, we speak about strict convexity, concavity, subharmonicity and superharmonicity. Moreover, in this context, totally geodesic mappings appear as both convex and concave mappings.

3.2 Properties of convex mappings

The harmonic and the totally geodesic mappings have been characterized by T. Ishihara (see [17]), in relation to the pull-back transport property of convex germs as it follows: (i) φ is a harmonic mapping iff pulls back germs of convex functions into germs of subharmonic functions; (ii) φ is a totally geodesic mapping iff pulls back convex germs into convex germs; (iii) If $m = dimM \leq n = dimN$, a C^{∞} differentiable mapping $\varphi : (M, g) \to (N, h)$ pulls back strictly convex germs into strictly convex germs if and only if it is a totally geodesic immersion. When $m > n$, there are no differentiable mappings to return strictly convex germs into strictly convex germs. Inspired by these, the following theorem analyzes the pull-back transport properties of convex mappings. Let us start with a definition.

Definition 3.3. A C^∞ differentiable function $f : N \to \mathbb{R}$ is called *partial locally increasing* if there exists an open subset $V \subset N$, such that $f_\sigma(q) \geq 0,\ \forall q \in V,\ \forall \sigma = \overline{1,n}$.

Theorem 3.1. *(Pull-back transport properties) A C^∞ differentiable mapping $\varphi : M \to N$ is convex if and only if φ pulls back locally convex partial increasing functions on N into locally convex functions on M.*

Proof. The arguments here develop some ideas from [17].
"⇒" Let $f : N \to \mathbb{R}$ be a locally convex and increasing function on N. Since φ is a convex mapping, it follows that $df(\beta(\varphi)) = (f_\sigma \varphi^\sigma_{;ij})_{i,j=\overline{1,m}}$ is positive semidefinite on some open subset $U \subset M$. Also, since f is convex, it follows that $\beta(f)(d\varphi, d\varphi) = (f_{\alpha\beta}\varphi^\alpha_i \varphi^\beta_j)_{i,j=\overline{1,m}}$ is also positive semidefinite on U and, consequence of Proposition 2.2, $f \circ \varphi$ is convex on U.
"⇐" Suppose φ is not convex, therefore there exist a point $p \in M$ and an index $\tau \in \{1, ..., n\}$ such that $Hess^\tau f(p) = (\beta(\varphi)^\tau_{ij}(p))_{i,j=\overline{1,m}}$ fails from being positive semidefinite. More precisely, there exists $\xi = (\xi^1, ..., \xi^m) \in \mathbb{R}^m \backslash \{0\}$, such that $\beta(\varphi)^\tau_{ij}(p)\xi^i\xi^j = \varphi^\tau_{;ij}(p)\xi^i\xi^j < 0$. Let

$$\lambda = \varphi^\tau_{;ij}(p)\xi^i\xi^j < 0 \text{ and } \mu = \delta_{\alpha\beta}\varphi^\alpha_i(p)\varphi^\beta_j(p)\xi^i\xi^j.$$

Denote $C = (\delta_{\alpha\beta})_{\alpha,\beta=\overline{1,n}}$ and $C_\tau = -(\mu+1)/\lambda$, $C_\sigma = 0$, $\forall \sigma \neq \tau$. Applying the technical Lemma 2.3 for the positive definite matrix C, it follows that there exists a strictly convex function $f : V \to \mathbb{R}$, defined on an open neighborhood V of $q = \varphi(p)$, such that

$$f_\sigma(q) = C_\sigma = \begin{cases} -(\mu+1)/\lambda, & \text{if } \sigma = \tau \\ 0, & \text{if } \sigma \neq \tau \end{cases},\ f_{\alpha\beta}(q) = \delta_{\alpha\beta},\ \forall \alpha, \beta = \overline{1,n}.$$

Then,

$$\begin{aligned}(f \circ \varphi)_{ij}(p) &= f_{\alpha\beta}(q)\varphi^\alpha_i(p)\varphi^\beta_j(p) + f_\sigma(q)\varphi^\sigma_{;ij}(p) \\ &= \delta_{\alpha\beta}\varphi^\alpha_i(p)\varphi^\beta_j(p) - \frac{\mu+1}{\lambda}\varphi^\tau_{;ij}(p).\end{aligned}$$

It follows

$$\text{Hess}(f \circ \varphi)(\xi, \xi) = \delta_{\alpha\beta}\varphi^\alpha_i(p)\varphi^\beta_j(p)\xi^i\xi^j - \frac{\mu+1}{\lambda}\varphi^\tau_{;ij}(p)\xi^i\xi^j = \mu - \frac{\mu+1}{\lambda} \cdot \lambda = -1,$$

therefore $f \circ \varphi$ is not convex, contrary to the hypotheses. Therefore, the assumption about φ not being convex fails from being valid. □

Corollary 3.2. *A C^∞ differentiable mapping $\varphi : M \to N$ is convex if and only φ pulls back locally concave partial decreasing functions on N into locally concave functions on M.*

Corollary 3.3. *A C^∞ differentiable mapping $\varphi : M \to N$ is concave if and only φ pulls back locally convex partial decreasing functions on N into locally concave functions on M.*

Corollary 3.4. *A C^∞ differentiable mapping $\varphi : M \to N$ is concave if and only φ pulls back locally convex partial increasing functions on N into locally concave functions on M.*

Remark 3.4. To resume the results above, φ is a convex mapping iff it pulls back locally increasing germs of convex functions into germs of convex functions.

Some important concepts in the theory of Riemannian submanifolds refers to minimal and totally geodesic submanifolds. Recall that a minimal submanifold is the range of a harmonic isometric embedding, while totally geodesic submanifolds are ranges of totally geodesic isometric embeddings. Similarly, we may define subminimal, superminimal, convex, respectively concave submanifolds.

Theorem 3.5. *(Push-forward transport properties)*
(i) Any convex (concave) isometric embedding carries totally geodesic submanifolds into convex (concave) submanifolds.
(ii) Any convex (concave) isometric embedding carries minimal submanifolds into subminimal (superminimal) submanifolds.

Proof. Let $\varphi : (M, g) \to (N, h)$ be a differentiable convex mapping and $\mu : (P, l) \to (M, g)$ an isometric embedding, that is $\mu^* g = l$. Using the composition law developed in Proposition 2.2, we have

$$(\varphi \circ \mu)^\gamma_{uv} = \varphi^\gamma_{;ij}\mu^i_u\mu^j_v + \varphi^\gamma_k\mu^k_{;uv}$$

and

$$\tau^\gamma(\varphi \circ \mu) = l^{uv}(\varphi \circ \mu)^\gamma_{uv} = l^{uv}\varphi^\gamma_{;ij}\mu^i_u\mu^j_v + l^{uv}\varphi^\gamma_k\mu^k_{;uv}.$$

(i) If (P, l) is a geodesic submanifold, it follows that $\mu^k_{;uv}(\mu) = 0,\ \forall k = \overline{1, m},\ \forall u, v = \overline{1, p}$ and, since φ is convex mapping,

$$(\varphi \circ \mu)^\gamma_{uv}\xi^u\xi^v = \varphi^\gamma_{;ij}\mu^i_u\mu^j_v\xi^u\xi^v \geq 0,\ \forall\gamma = \overline{1, n},\ \forall\xi \in \mathbb{R}^p\backslash\{0\},$$

that is $\varphi \circ \mu$ is a convex mapping and its range is a convex submanifold.
(ii) If (P, l) is a minimal submanifold, it follows that $\tau(\mu)^k = l^{uv}\mu^k_{;uv} = 0,\ \forall k = \overline{1, m}$ and, since φ is convex mapping,

$$\tau^\gamma(\varphi \circ \mu) = l^{uv}\varphi^\gamma_{;ij}\mu^i_u\mu^j_v \geq 0,\ \forall\gamma = \overline{1, n}$$

that is $\varphi \circ \mu$ is a subharmonic mapping and its range is a subminimal submanifold. Moreover, the proof stands similarly if φ is concave. □

Corollary 3.6. *Convex (concave) mappings carry geodesic curves into convex (concave) curves.*

Remark 3.5. Since totally geodesic mappings are both convex and concave, the transport theorems formulated above confirm some basic properties of totally geodesic mappings. Indeed, theorem 3.5 confirms that the images of geodesics from M through φ are geodesics in N. The most elementary examples of totally geodesic mappings are the euclidean immersions $\mathbb{R}^n \subset \mathbb{R}^m \,(1 \le n \le m),\ (x^1, ..., x^n) \to (x^1, ..., x^n, 0, ..., 0)$ and their restrictions to unit spheres $S^{n-1} \subset S^{m-1}$.

Moreover, the composition of totally geodesic mappings is also a totally geodesic mapping, while the composition of harmonic mappings usually fails from staying harmonic and the inverse of a totally geodesic diffeomorphism is also a totally geodesic mapping.

4 Invariant Riemannian convexity of mappings

The convexity concept described above is highly dependent on the chosen coordinate frame. In order to overcome this limitation, this section introduces an invariant type of convexity.

4.1 Cone structures on Euclidean spaces

Definition 4.1. A subset $C \subset \mathbb{R}^n$ is called *convex and pointed cone* if $\mathbb{R}_+ C \subset C$, $C + C \subset C$ and $C \cap (-C) = \{0\}$.

Definition 4.2. Let C denote a closed and convex pointed cone, with non-void interior. The following partial order relations may be defined:

$$x \preceq_C y \Leftrightarrow y - x \in C;$$

$$x \prec_C y \Leftrightarrow y - x \in int(C);$$

$$x \overset{\preceq}{=}_C y \Leftrightarrow y - x \in C - \{0\};$$

4.2 Cone fields on Riemannian manifolds

Definition 4.3. Let (N, h) be a complete Riemannian manifold. A mapping $C : N \to \mathcal{P}(TN)$ is called a cone field on manifold N if, for each $q \in N$, $C(q)$ is a convex and pointed cone on $T_q N$.

Definition 4.4. Let C denote a cone field on a complete Riemannian manifold (N, h). The following cone field associated partial order relations may be defined:

$$q_1 \preceq_C q_2 \Leftrightarrow exp_{q_1}^{-1}(q_2) \in C(q_1);$$

$$q_1 \prec_C q_2 \Leftrightarrow exp_{q_1}^{-1}(q_2) \in int(C(q_1));$$

$$q_1 \overset{\preceq}{=}_C q_2 \Leftrightarrow exp_{q_1}^{-1}(q_2) \in C(q_1) - \{0\};$$

4.3 Riemannian C-convex mappings

Based on the geometric elements defined above, a natural convexity concept emerges. Let (M, g) and (N, h) be two complete Riemannian manifolds, $U \subset M$ be a totally geodesic open subset and let C be a fixed cone field on N.

Definition 4.5. A differentiable mapping $\varphi : U \subset M \to N$ is called C-convex if, for each two points $p_1,\ p_2$ in U and each geodesic $\gamma : [0, 1] \to U$ connecting them, the following relation stands

$$\varphi(\gamma(t)) \preceq_C \delta(t),\ \forall t \in [0, 1],$$

where $\delta : [0, 1] \to N$ denotes the minimal geodesic between $q_1 = \varphi(p_1)$ and $q_2 = \varphi(p_2)$.

Remark 4.6. According to the definition of the cone field associated partial order relation, the previous relation may be rewritten:

$$exp^{-1}_{\varphi(\gamma(t))}(\delta(t)) \in C(\varphi(\gamma(t)),\ \forall t \in [0, 1].$$

In the following, we develop some properties of cone convex mappings.

Theorem 4.1. *If $U \subset M$ is a totally geodesic open subset and $\varphi : U \subset (M, g) \to (N, h)$ is a C-convex mapping of class C^∞, then, for each two points $p_1,\ p_2$ in U and each geodesic $\gamma : [0, 1] \to M$ connecting them,*

$$exp^{-1}_{\varphi(p_1)}(\varphi(p_2)) - d\varphi_{p_1}(\dot{\gamma}(0)) \in C(\varphi(p_1)).$$

In particular, if $M = N = \mathbb{R}$, $C = [0, \infty)$ and $\varphi : U \subset \mathbb{R} \to \mathbb{R}$ is a convex function, we find a classical property of convex functions:

$$\varphi(p_2) - \varphi(p_1) \geq \varphi'(p_1)(p_2 - p_1),\ \forall p_1,\ p_2 \in U.$$

Proof. Since $exp^{-1}_{\varphi(\gamma(t))}(\delta(t)) \in C(\varphi(\gamma(t))$, $exp^{-1}_{q_1}(q_1) = 0 \in C(\varphi(\gamma(t))$ and $t > 0$, it follows, based on the properties of a cone field,

$$\frac{exp^{-1}_{\varphi(\gamma(t))}(\delta(t)) - exp^{-1}_{q_1}(q_1)}{t} \in C(\varphi(\gamma(t)).$$

Letting $t \to \infty$, we obtain

$$\frac{d}{dt}\left[exp^{-1}_{\varphi(\gamma(t))}(\delta(t))\right]_{t=0} \in C(q_1)$$

and, using the properties of the exponential mapping, we compute

$$\frac{d}{dt}\left[exp^{-1}_{\varphi(\gamma(t))}(\delta(t))\right]_{t=0} = \dot{\delta}(0) - \left.\frac{d\varphi(\gamma(t))}{dt}\right|_{t=0}.$$

Finally, we obtain

$$exp^{-1}_{\varphi(p_1)}(\varphi(p_2)) - d\varphi_{p_1}(\dot{\gamma}(0)) \in C(\varphi(p_1)).$$

□

Theorem 4.2. *If $U \subset M$ is a totally geodesic open subset and $\varphi : U \subset (M, g) \to (N, h)$ is a C-convex mapping of class C^∞, then, for each vector field $X \in C^\infty(TU)$,*

$$\beta(\varphi)(X, X) \in C.$$

In particular, if $M = N = \mathbb{R}$, $C = [0, \infty)$ and $\varphi : U \subset \mathbb{R} \to \mathbb{R}$ is a convex function, then we find a classical outcome:

$$f''(p) \geq 0, \ \forall p \in U.$$

Also, if $f : U \subset (M, g) \to \mathbb{R}$ is a convex function on a complete Riemannian manifold and C is as above, then the Hessian matrix field is positive semidefinite.

Proof. Let $p \in M$ be an arbitrary fixed point and $X \in T_x M$. Consider $\gamma : I \to U$ a C^2 geodesic, where I is a real interval, such that $0 \in I$, $\gamma(0) = p$ and $\dot{\gamma}(0) = X$. Applying Theorem 4.1, for $p_1 = p$, $q = \varphi(p)$ and $p_2 = \gamma(t)$, we obtain

$$exp_q^{-1}(\varphi(\gamma(t))) - t d\varphi_p(X) \in C(q). \tag{4.1}$$

On the other hand, by writing the Taylor formula associated to the differentiable function $t \to exp_q^{-1}(\varphi(\gamma(t)))$, we obtain

$$exp_q^{-1}(\varphi(\gamma(t))) = exp_q^{-1}(q) + t\, d\varphi_p(X) + \frac{t^2}{2} \frac{d}{dt^2}\bigg|_{t=0} \left[exp_q^{-1}(\varphi(\gamma(t)))\right] + \theta(t) t^3. \tag{4.2}$$

Combining relations (4.1) and (4.2) and using the properties of the convex pointed cone structures, it follows

$$\frac{d}{dt^2}\bigg|_{t=0} \left[exp_q^{-1}(\varphi(\gamma(t)))\right] \in C(q).$$

The computations lead to

$$d^2(exp_q^{-1})_q(d\varphi_p(X), d\varphi_p(X)) + d(exp_q^{-1})_q \left(\frac{d}{dt^2}\bigg|_{t=0} \varphi(\gamma(t)) \right) \in C(q),$$

that is

$$\frac{d}{dt^2}\bigg|_{t=0} (\varphi(\gamma(t))) = \nabla^N_{d\varphi_p(X)} d\varphi_p(X) \in C(q).$$

Moreover, since $\gamma(\cdot)$ is a geodesic on M, we may add the therm $d\varphi(\nabla^M_X X) = 0$ and we obtain $\beta(\varphi)(p)(X, X) \in C(q), \ \forall p \in U, \ \forall X \in T_p M$. □

Remark 4.7. Let (M, g) and (N, h) be two Riemannian manifolds and $\varphi : M \to N$ be a C^∞ differentiable mapping. Let $(U, (x^1, ..., x^m))$ and $(V, (y^1, ..., y^n))$ be some fixed local charts around $p \in M$ and $q = \varphi(p)$, respectively. We define the local cone field

$$C = \{Y = (Y^\alpha) \in C^\infty(TN) \,|\, Y^\alpha \geq 0, \ \forall \alpha = \overline{1, n}\}.$$

According to Theorem 4.2, if φ is C-convex, then the components of the second fundamental form are positive semidefinite. Therefore, the convexity of mappings introduced and analyzed in Section 3 is a local particular example of C-convexity. By defining other types of cones, we may derive some quite exotic convexities.

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