The role of Frenet motion in Pappus type theorems

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Abstract. Let $c(t)$ be a curve in a space form M^n_λ of sectional curvature λ. Let P_0 be a totally geodesic hypersurface of M^n_λ through $c(0)$ and orthogonal to $c(t)$. Let $\mathcal{D}_0$ and $\mathcal{C}_0$ be a domain and a hypersurface, respectively, of P_0. Let $\mathcal{D}$ and $\mathcal{C}$ be, respectively, the domain and the hypersurface of M^n_λ obtained by a motion along $c(t)$. We show that, after some rotation of $\mathcal{D}_0$ and $\mathcal{C}_0$, the Frenet motion gives the supremum and the infimum value of $\mathrm{vol}(\mathcal{D})$ and a lower bound of $\mathrm{vol}(\mathcal{C})$, when the centres of mass of $\mathcal{D}_0$ and $\mathcal{C}_0$ are not at $c(0)$ (and, in the case of $\mathrm{vol}(\mathcal{C})$, $c(t)$ is a plane curve).

M.S.C. 2010: 53C40, 52C21.
Key words: Pappus theorems on volumes; tubes; Frenet motion.

1 Introduction

In [10], H. Weyl gave some nice formulae for the volumes of a tube and a tubular hypersurface around a submanifold P of the Euclidean space and the sphere. A consequence of these formulae is that these volumes depend only on the intrinsic geometry of P and the radius of the tube (this last quantity encodes all the information on the geometry of the section of the tube). See [7] for a modern approach and further references.

In [8], A. Gray and the second author initiated a way, via Pappus type theorems, to get a deeper understanding of these formulae. The starting point was the computations by W. Goodman and G. Goodman in [6] (completed by L. E. Pursell and H. Flanders in [9] and [5]) generalizing Pappus formulae for the volume of a domain (or a surface) in $\mathbb{R}^3$ obtained by the motion of a plain domain (or a plain curve) along a curve in $\mathbb{R}^3$. In [8], all these formulae where generalized to simply connected space forms M^n_λ of constant sectional curvature λ and arbitrary dimension n. Given a curve $c(t)$ in M^n_λ, let P_0 be the totally geodesic hypersurface of M^n_λ through $c(0)$ and orthogonal to $c(t)$, let $\mathcal{D}_0$ be a domain of P_0 and let $\mathcal{C}_0$ be a hypersurface of P_0, and let $\mathcal{D}$ and $\mathcal{C}$ be, respectively, the domain and the hypersurface of M^n_λ obtained by a motion along $c(t)$ of $\mathcal{D}_0$ and $\mathcal{C}_0$ respectively. In [8] it is shown that:

(a) $\mathrm{vol}(\mathcal{D})$ depends only on the geometry of $\mathcal{D}_0$, the length and the first curvature of $c(t)$, and not on the other i-th curvatures; but, generally, $\mathrm{vol}(\mathcal{D})$ depends on the motion along $c(t)$.

Balkan Journal of Geometry and Its Applications, Vol.21, No.2, 2016, pp. 27-34.

(b) if the centre of mass of $\mathcal{D}_0$ is on the curve, then vol($\mathcal{D}$) does not depend on the motion nor on the curvature of $c(t)$, only on the length of $c(t)$ and the geometry of $\mathcal{D}_0$;

(c) for parallel motions, it is still true for vol($\mathcal{C}$) that it depends only on the length of $c(t)$ and the geometry of $\mathcal{C}_0$ when $c(0)$ is the centre of mass of $\mathcal{C}_0$.

In [1], X. Gual and the authors studied vol($\mathcal{C}$) in more detail, and showed, among others, that,

(d) For generic $\mathcal{C}_0$, but with the centre of mass on the curve, vol($\mathcal{C}$) depends on the motion, but *the parallel motion gives the minimum value of* vol($\mathcal{C}$).

In this paper we shall investigate a bit more the case when the centre of mass is not on the curve. In this situation, as noted above in b), even vol($\mathcal{D}$) depends on the motion. Then it is natural to ask if (like for case (d)) there is some motion where vol($\mathcal{D}$) attains its minimum, and a similar question can be stated for vol($\mathcal{C}$).

We shall show that, in general, vol($\mathcal{D}$) does not attain its minimum value, but we shall obtain sharp lower and upper bounds of vol($\mathcal{D}$) which are given by the volume of a domain obtained by a Frenet motion of a domain $R\mathcal{D}_0$ obtained by a rotation R of $\mathcal{D}_0$ in P_0 (see Theorem 3.1). Then, Frenet motion plays for $\mathcal{D}$ (when the centre of mass is not at $c(t)$) a role similar to parallel motion for $\mathcal{C}$ when the centre of mass is on $c(t)$.

For vol($\mathcal{C}$), when the centre of mass of $\mathcal{C}_0$ is not on $c(t)$, we shall get a lower bound only for plane curves (Theorem 3.2). The restriction to plane curves is because, in order to get lower bounds, the parallel motion still has its role, like when the center of mass was on $c(t)$, and, moreover, we have to mix it with the role of Frenet motion, and only on plane curves a motion can be parallel and Frenet at the same time. It remains open to see if this lower bound is sharp (see the remark after Theorem 3.2) and to find upper and lower bounds for vol($\mathcal{C}$) for a generic curve $c(t)$.

The corresponding results for the complex case were published in [3]. Later we realized that properties on the volume of a tube have some analog on the first eigenvalue of the laplacian (cf. [2] and [4]). Then we have written this paper with the hope of obtaining also analog results on the first eigenvalue in a future work.

2 Preliminaries

First, we shall establish some notation and definitions, partially taken from the papers by Gray-Miquel [8] and by Domingo-Gual-Miquel [1].

We shall consider C^∞ curves $c : I = [0, L] \longrightarrow \mathrm{M}^n_\lambda$ parametrized by their arc-length t. We shall suppose that c is an embedding from $[0, L]$ into M^n_λ if $c(0) \neq c(L)$ or induces an embedding from S^1 into M^n_λ if $c(0) = c(L)$. By $\mathcal{N}c(I)$, we shall denote the normal bundle of $c(I)$ in M^n_λ, and P_t will denote the totally geodesic hypersurface of M^n_λ tangent to $\{c'(t)\}^\perp$.

Given a smooth orthonormal frame $\{E_2(t), ..., E_n(t)\}$ of the normal bundle of $c(t)$, the *motion along c associated to this frame* is the smooth map $\varphi : \{c'(0)\}^\perp \times I \to \mathcal{N}c(I)$ defined by

$$\varphi\left(\sum_{i=2}^n \mu^i E_i(0),\ t\right) = \sum_{i=2}^n \mu^i E_i(t), \tag{2.1}$$

or, equivalently, the smooth map $\phi : P_0 \times I \longrightarrow M$, defined by

$$\phi(\exp_{c(0)} \mu, t) = \exp_{c(t)} \varphi(\mu, t) \quad \text{for every} \quad \mu \in \{c'(0)\}^{\perp}. \tag{2.2}$$

Such a motion, denoted by φ or ϕ indistinctly, defines two families of isometries

$$\varphi_t : T_{c(0)}P_0 \longrightarrow T_{c(t)}P_t, \quad \varphi_t(\mu) = \varphi(\mu, t) \quad \text{and} \quad \phi_t : P_0 \longrightarrow P_t, \quad \phi_t(x) = \phi(x, t).$$

They are related by

$$\varphi_t = \phi_{t*c(0)} \quad \text{and} \quad \phi_t(\exp_{c(0)} \mu) = \exp_{c(t)} \varphi_t(\mu). \tag{2.3}$$

A smooth orthonormal frame $\{E_i(t)\}_{i=2}^n$ on $c(t)$ is called a *weak Frenet frame* if $E_2(t) = f_2(t)$, the standard normal vector of $c(t)$. That is, a weak Frenet frame is a orthonormal C^∞ frame $\{f_2(t), \ldots, f_n(t)\}$ satisfying $\nabla_{c'(t)}c'(t) = k_1(t)f_2(t)$, where $k_1(t)$ is the first curvature of the curve.

A *weak Frenet motion* is a motion associated to a weak Frenet frame. It will be denoted by ϕ^F or φ^F.

Let D be the connection induced on the normal bundle $\mathcal{N}c(I)$ by the Levi-Civita connection on $M^n(\lambda)$. A *parallel motion* is a motion associated to a D-parallel frame along c; it is unique along any given curve.

Let us denote by $\mathcal{B}_0$ a domain $\mathcal{D}_0$ or a hypersurface $\mathcal{C}_0$ contained in P_0 and such that the exponential map restricted to $\varphi(\exp_{c(0)}^{-1}(\mathcal{B}_0) \times I)$ is a diffeomorphism. The set $\mathcal{B} = \phi(\mathcal{B}_0 \times I)$ (domain $\mathcal{D}$ or hypersurface $\mathcal{C}$) is called the set *obtained by the motion* ϕ *of* $\mathcal{B}_0$ *along* $c(t)$, and we denote $\mathcal{B}_t = \phi_t(\mathcal{B})$, whereas ω_t will be the volume element of $\mathcal{B}_t$.

For every $\lambda \in \mathbb{R}$, $s_\lambda : \mathbb{R} \to \mathbb{R}$ will denote the solution of the equation $s'' + \lambda s = 0$ with the initial conditions $s(0) = 0$ and $s'(0) = 1$; and $c_\lambda = s'_\lambda$.

For every $x \in P_0$, $N_x(t)$ will denote the unit vector tangent at $c(t)$ to the minimizing geodesic γ_{xt} from $c(t)$ to $\phi_t(x)$.

$r : P_t \longrightarrow \mathbb{R}$ will denote the function defined by $r(\phi_t(x)) = \text{dist}(c(t), \phi_t(x)) = \text{dist}(c(0), x) = r(x)$.

τ_t^x will denote the parallel transport in P_t from $c(t)$ to $\phi_t(x)$ along γ_{xt}.

In Gray-Miquel [8], the following formula has been proved

$$\text{vol}(\mathcal{D}) = L \int_{\mathcal{D}_0} c_\lambda(r)\sigma_0 - \int_0^L k_1(t) \left(\int_{\mathcal{D}_t} s_\lambda(r) N_2(t) \sigma_t \right) dt, \tag{2.4}$$

where σ_t is the volume element of $\mathcal{D}_t$, and $N_2(t)(\phi_t(x)) = \langle N_x(t), f_2(t) \rangle$.

And, in Domingo-Gual-Miquel [1], has been obtained that

$$\text{vol}(\mathcal{C}) = \int_0^L \left(\int_{\mathcal{C}_t} \sqrt{\left\langle \tau_t^x \frac{DN_x}{dt}(t), \xi_t \right\rangle^2 s_\lambda(r)^2 + (c_\lambda(r) - s_\lambda(r) N_2(t) k_1(t))^2} \;\; \eta_t \right) dt, \tag{2.5}$$

where η_t is the volume element of $\mathcal{C}_t$, and ξ_t is the outer unit normal vector field of $\mathcal{C}_t$.

As a consequence of (2.5) we have the inequality

$$\text{vol}(\mathcal{C}) \geq L \int_{\mathcal{C}_0} c_\lambda(r)\eta_0 - \int_0^L k_1(t) \left(\int_{\mathcal{C}_t} s_\lambda(r) N_2(t) \;\; \eta_t \right) dt, \tag{2.6}$$

and the equality holds in (2.6) if the motion φ is parallel.

3 The theorems.

Theorem 3.1. . *Let $c(t)$ be a curve in M^n_λ having a weak Frenet frame, and let $\mathcal{D}_0$ be a domain of P_0. There are two isometries R_m and R_M of P_0 with $c(0)$ as a fixed point such that, for every motion ϕ along $c(t)$,*

$$\text{vol}((R_M\mathcal{D})^F) \geq \text{vol}(\mathcal{D}) \geq \text{vol}((R_m\mathcal{D})^F), \tag{3.1}$$

where $(R_M\mathcal{D})^F = \phi^F((R_M\mathcal{D}) \times I)$, $a \in \{m, M\}$. Moreover, these bounds are sharp (that is, they give the supremum and the infimum for vol$(\mathcal{D})$ *among all the $\mathcal{D}$ obtained by a motion of $\mathcal{D}_0$ along $c(t)$).*

Proof. First, let us remark that, as a consequence of formula (2.4), all the domains $(R_a\mathcal{D})^F$ obtained from a given domain $R_a\mathcal{D}_0$ by a weak Frenet motion along a given curve $c(t)$ have the same volume, then the bounds in Theorem 1 are well defined.

Let us define the function of $\mathcal{D}_0$

$$Mom(\mathcal{D}_0) = \int_{\mathcal{D}_0} s_\lambda(r) N_2(0)\sigma_0. \tag{3.2}$$

Let us denote by $Is_{c(0)}$ the connected component containing he identity of the group of isometries of P_0 with $c(0)$ as a fixed point. We shall identify the group $SO(n-1)$ with $Is_{c(0)}$ in the usual way $R \in SO(n-1) \mapsto R \in Is_{c(0)}$ defined by $R \exp_{c(0)} X = \exp_{c(0)} RX$. Having account that $r \circ R = r$, we have

$$\begin{aligned} Mom(R\mathcal{D}_0) &= \textstyle\int_{R\mathcal{D}_0} s_\lambda(r) \langle N_x(0), f_2(0)\rangle \;\; R^{-1*}\sigma_0 \\ &= \textstyle\int_{\mathcal{D}_0} s_\lambda(r) \circ R \;\;\; \langle N_x(0), f_2(0)\rangle \circ R \; \sigma_0 = \int_{\mathcal{D}_0} s_\lambda(r) \;\;\; \langle RN_x(0), f_2(0)\rangle \;\; \sigma_0. \end{aligned}$$

It follows from this expression that the map

$$\mathcal{F} : Is_{c(0)} \longrightarrow \mathbb{R} \quad \text{defined by} \quad F(R) = Mom(R\mathcal{D}_0) \tag{3.3}$$

is continuous, then, since $SO(n-1)$ is compact, $\mathcal{F}$ attains its maximum at some $R_m \in SO(n-1) \equiv Is_{c(0)}$ and its minimum at some $R_M \in SO(n-1)$.

Let $R(t) \in SO(n-1)$ be the isometry of $T_{c(0)}P_0$ satisfying $R(t)^{-1}f_2(0) = \phi^{-1}_{t*}f_2(t)$. Now, let's compute

$$\begin{aligned} \int_{\mathcal{D}_t} s_\lambda(r) \;\;\; \langle N_x(t), f_2(t)\rangle \;\; \sigma_t \quad &= \textstyle\int_{\mathcal{D}_0} s_\lambda(r) \;\;\; \left\langle N_x(0), \phi^{-1}_{t*} f_2(t)\right\rangle \;\; \sigma_0 \\ &= \int_{\mathcal{D}_0} s_\lambda(r) \quad \langle R(t)N_x(0), f_2(0)\rangle \;\; \sigma_0 = Mom(R(t)\mathcal{D}_0). \end{aligned} \tag{3.4}$$

But, since R_m and R_M are the maximum and the minimum, respectively, for $\mathcal{F}$, we have

$$Mom(R_M\mathcal{D}_0) \leq Mom(R(t)\mathcal{D}_0) \leq Mom(R_m\mathcal{D}_0). \tag{3.5}$$

From (2.4), (3.4) and (3.5), it follows that

$$\begin{aligned} \mathrm{vol}(\mathcal{D}) &= L\int_{\mathcal{D}_0} c_\lambda(r)\sigma_0 - \int_0^L k_1(t)Mom(R(t)\mathcal{D}_0)dt \\ &\geq L\int_{\mathcal{D}_0} c_\lambda(r)\sigma_0 - \int_0^L k_1(t)Mom(R_m\mathcal{D}_0). \end{aligned} \tag{3.6}$$

But, under the conditions of Theorem 1, the last expression is just $\mathrm{vol}(R_m\mathcal{D})^F)$, because, in a Frenet motion, $\int_{\mathcal{D}_0} s_\lambda(r)N_2(0)\sigma_0 = \int_{\mathcal{D}_t} s_\lambda(r)N_2(t)\sigma_t$. This finishes the proof of the inequality in the right side in (3.1). The proof of the inequality in the left side is similar, using R_M instead of R_m.

Now, let us prove that the bounds are sharp. We want to show that, for every $\varepsilon > 0$ there is a motion ϕ^ε satisfying

$$|\mathrm{vol}(\phi^\varepsilon(\mathcal{D}_0 \times I)) - \mathrm{vol}(R_a\mathcal{D}_0)^F| < \varepsilon, \quad a \in \{m, M\}. \tag{3.7}$$

Let $0 < t_0 < t_1 < L$. Let us consider a C^∞ map $\mathcal{R} : [0, L] \longrightarrow SO(n-1)$ satisfying $\mathcal{R}(t) = R_a^{-1}$ for $t \in [t_0, t_1]$ and $\mathcal{R}(0) = Id = \mathcal{R}(L)$. This map $\mathcal{R}(t)$ can be constructed as follows. Under the action of R_a, $\mathbb{R}^{n-1}$ decomposes as the direct sum of planes H_i such that R_a restricted to H_i is a rotation of angle α_i and a subspace H on which R_a is the identity. For every i we can construct, by the standard procedure, a C^∞ real function θ_i satisfying $\theta_i(t) = \alpha_i$ for $t \in [t_0, t_1]$ and $\theta_i(0) = 0 = \theta_i(L)$. Then , we define $\mathcal{R}(t)$ equal to the rotation of angle $\theta_i(t)$ when restricted to each plane H_i and equal to the identity when restricted to H.

Now, choose a weak Frenet frame $\{f_1(t), ..., f_n(t)\}$, and define ϕ^ε as the motion associated to the frame $E_1(t) = f_1(t)$ and $E_i(t) = \mathcal{R}(t)f_i(t)$. Then, for $t \in [t_0, t_1]$,

$$\begin{aligned} \varphi^\varepsilon\left(\sum_{i=2}^n \mu^i f_i(0), t\right) &= \sum_{i=2}^n \mu^i E_i(t) \\ &= \mathcal{R}(t)\sum_{i=2}^n \mu^i f_i(t) = \varphi^F(\mathcal{R}(t)\sum_{i=2}^n \mu^i f_i(0), t), \end{aligned}$$

that is,

$$\varphi_t^\varepsilon(\mu) = \varphi_t^F(R_a(\mu)) \quad \text{for} \quad t \in]t_0, t_1[. \tag{3.8}$$

Then, using the upper and lower bounds just proved of Theorem 1,

$$\begin{aligned} &|\mathrm{vol}(\phi^\varepsilon(\mathcal{D}_0 \times I)) - \mathrm{vol}(R_a\mathcal{D}_0)^F| \\ &\quad = |\mathrm{vol}(\phi^\varepsilon(\mathcal{D}_0 \times [0, t_0])) - \mathrm{vol}(\phi^F((R_a\mathcal{D}_0) \times [0, t_0]))| \\ &\qquad + |\mathrm{vol}(\phi^\varepsilon(\mathcal{D}_0 \times [t_1, L])) - \mathrm{vol}(\phi^F((R_a\mathcal{D}_0) \times [t_1, L]))| \\ &\quad \leq |\mathrm{vol}(\phi^F((R_M\mathcal{D}_0) \times [0, t_0])) - \mathrm{vol}(\phi^F((R_m\mathcal{D}_0) \times [0, t_0]))| \\ &\qquad + |\mathrm{vol}(\phi^F((R_M\mathcal{D}_0) \times [t_1, L])) - \mathrm{vol}(\phi^F((R_m\mathcal{D}_0) \times [t_1, L]))| \end{aligned}$$

Since

$$\begin{aligned}&\mathrm{vol}(\phi^F((R_a\mathcal{D}_0)\times([0,t_0]\cup[t_1,L]))\\&=(t_0+L-t_1)\textstyle\int_{\mathcal{D}_0}c_\lambda(r)\sigma_0-Mom(R_a\mathcal{D}_0)\left(\int_0^{t_0}k_1(t)dt+\int_{t_1}^L k_1(t)dt\right)\end{aligned}$$

is a continuous function on t_0 and t_1, we may choose t_0 and t_1 small enough to have (3.7), as wanted. □

Remark 3.1. (1) It is obvious from the proof that the isometries R_m and R_M depend only on $\mathcal{D}_0$ and its position respect to $c(0)$ in P_0.

(2) We can ask if the sharp lower bound given by Theorem 1 is a minimum. This would require that the inequality (3.6) be an equality. If $c(t)$ is not a geodesic, $k_1(t)\neq 0$. Since $k_1(t)\geq 0$ and $Mom(R(t)\mathcal{D}_\prime)\geq Mom(R_m\mathcal{D}_\prime)$ for every t, the equality in (3.6) implies the existence of an interval I such that $Mom(R(t)\mathcal{D}_\prime)=Mom(R_m\mathcal{D}_\prime)$ for every $t\in I$, which requires some symmetries on $\mathcal{D}_\prime$ if $R(t)\neq R_m$. Then, *in general, the infimum given by (3.1) is not a minimum.* The same argument works for the supremum.

Theorem 3.2. . *Let $c(t)$ be a plane curve (i.e., a curve in M^n_λ contained in a geodesic plane) having a weak Frenet frame, and let $\mathcal{C}_0$ be a hypersurface of P_0. There is an isometry R of P_0 with $c(0)$ as a fixed point such that, for every motion ϕ along $c(t)$,*

$$\mathrm{vol}(\mathcal{C})\geq\mathrm{vol}((R\mathcal{C})^F). \tag{3.9}$$

Proof. First we remark that, although, for a general curve, $\mathrm{vol}((R\mathcal{C})^F)$ depends on the weak Frenet motion chosen, the bound of this theorem is again well defined because, for a plane curve, a weak Frenet motion is a parallel motion, and this is unique on a given curve.

From (2.6), to find a lower bound for $\mathrm{vol}(\mathcal{C})$ it is enough to obtain a lower bound of

$$LBV(\mathcal{C}):=L\int_{\mathcal{C}_0}c_\lambda(r)\eta_0-\int_0^L k_1(t))\ \left(\int_{\mathcal{C}_t}s_\lambda(r)N_2(t)\ \ \eta_t\right)\ dt. \tag{3.10}$$

The same arguments given in the proof of Theorem 1, changing $\mathcal{D}$ by $\mathcal{C}$ and σ by η everywhere give that there is some $R\in Is_{c(0)}$ satisfying that

$$LBV(\mathcal{C})\geq L\int_{\mathcal{C}_0}c_\lambda(r)\sigma_0-\int_0^L k_1(t)Mom(R\mathcal{C}_0). \tag{3.11}$$

In general, (3.11) gives an universal lower bound for any motion and any curve $c(t)$. But, when the curve $c(t)$ is plane, Frenet and parallel motions coincide, and the right side of (3.11) is the volume of the hypersurface obtained by the Frenet motion of $R\mathcal{C}_0$ along $c(t)$, as follows from (2.5) and the argument at the end of the proof of the inequality on the right of Theorem 1. This finishes the proof of inequality (3.9). □

Remark 3.2. (1) Like in Theorem 1, the isometry R depends only on $\mathcal{D}_0$ and its location respect to $c(0)$ in P_0.

(2) The condition that $c(t)$ has a weak Frenet frame cannot be dropped out, as can be shown by considering a motion along a plane curve $c : I \longrightarrow \mathbb{R}^3$ strictly convex in $[0, t[$ and strictly concave in $]t, L]$, $t \neq 0$.

(3) The proof of the sharpness of inequalities given in Theorem 1 does not work to prove that inequality (3.9) is sharp. An analytic reason is that the idea of the proof of Theorem 1 is to approximate the hypersurface $(R\mathcal{C})^F$ by a hypersurface $\mathcal{C}^\varepsilon$ obtained from $\mathcal{C}_0$ by a motion ϕ^ε associated to a frame obtained from a weak Frenet frame by isometries $\mathcal{R}(t)$ constructed using some C^∞ functions θ_i. Then, if we want that $\mathrm{vol}(\mathcal{C}^\varepsilon)$ be near to $\mathrm{vol}((R\mathcal{C})^F)$, we need that $\frac{DN_x}{dt}(t)$ be near to 0, but $\frac{DN_x}{dt}(t)$ involves the derivatives of the functions θ_i which can go to ∞ faster than $1/t$ when t goes to 0. Geometrically, when t_0 goes to 0 and t_1 goes to L, ϕ^ε goes to a motion which takes $\mathcal{C}_0$ onto $R\mathcal{C}_0$ on time 0, then follows the Frenet motion ϕ^F and, just on time L, takes $R\mathcal{C}_0$ onto $\mathcal{C}_0$. The resulting hypersurface is the union of $(R\mathcal{C})^F$ and the domain of P_0 obtained by the action on $\mathcal{C}_0$ of all the isometries S which, restricted to H are the identity and, restricted to H_i are rotations of angles β_i, with $0 \leq \beta_i \leq \alpha_i$; then $\lim_{(t_0,t_1)\to(0,L)} \mathrm{vol}(\phi^\varepsilon(\mathcal{C}_0 \times [0, L])) \neq \mathrm{vol}((R\mathcal{C})^F)$.

Then the question arises: "find a sharp bound for $\mathrm{vol}(\mathcal{C})$". By the reasons given above we think that, even for plane curves, the bound given by (3.9) is not the best one.

Acknowledgements. Research partially supported by the DGI (Spain) project MTM2013-46961-P, and the Generalitat Valenciana Project PROMETEOII/2014/064.

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