On warped product generalized Roter type manifolds

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Dedicated to Professor Lajos Tamássy on his ninety-first birthday

Abstract. Generalized Roter type manifolds form an extended class of Roter type manifolds, which gives rise the form of the curvature tensor in terms of algebraic combinations of the fundamental metric tensor and Ricci tensors upto level 2. The object of the present paper is to investigate the characterization of a warped product manifold to be a generalized Roter-type (and hence as a special case for Roter type and conformally flat) manifold. We also present an example by a metric which ensures the existence of a warped product generalized Roter type manifold but is not Roter type manifold.

M.S.C. 2010: 53C15, 53C25, 53C35.
Key words: Roter type manifold; generalized Roter type manifold; conformally flat manifold; Ricci tensors of higher levels; warped product manifold

1 Introduction

Let M be an n (≥ 3)-dimensional connected semi-Riemannian smooth manifold equipped with a semi-Riemannian metric g. We denote the Levi-Civita connection, the Riemann-Christoffel curvature tensor, Ricci tensor, scalar curvature and the space of all smooth functions on M by ∇, R, S, κ and $C^\infty(M)$ respectively. The manifold M is flat if $R = 0$ and M is of constant curvature if R is a constant multiple of the Gaussian curvature tensor. For a conformally flat manifold M, R can be expressed as

$$R = J_1 g \wedge g + J_2 g \wedge S,$$

where J_1, $J_2 \in C^\infty(M)$. Especially, M is flat (resp., constant curvature and conformally flat) if $J_1 = J_2 = 0$ (resp., $J_1 = \frac{\kappa}{n(n-1)}$, $J_2 = 0$ and $J_1 = -\frac{\kappa}{2(n-1)(n-2)}$, $J_2 = \frac{1}{n-2}$). The manifold M is Roter type (briefly, RT_n; see [4, 17]) if R can be expressed as a linear combination of $g \wedge g$, $g \wedge S$ and $S \wedge S$. Very recently, Shaikh et al. [25] introduced the notion of generalized Roter type manifold. A manifold is said to be generalized Roter type (briefly, GRT_n) if its curvature tensor can be expressed as a linear combination of $g \wedge g$, $g \wedge S$, $S \wedge S$, $g \wedge S^2$, $S \wedge S^2$ and $S^2 \wedge S^2$. We note

Balkan Journal of Geometry and Its Applications, Vol.21, No.2, 2016, pp. 82-95.

that the name “generalized Roter type” was first used in [25]. For general properties of GRT_n and its proper existence we refer the readers to see [28] and also references therein.

The paper is organized as follows. Section 2 is concerned with preliminaries. Section 3 deals with warped product manifolds and various curvature relations. Section 4 is devoted to the study of warped products GRT_n and obtained the characterization of such manifolds (see Theorem 4.1). We obtain the characterization of a warped product manifold to be RT_n and conformally flat. The last section deals with the proper existence of such notion by an example with a suitable metric.

2 Preliminaries

Let M be an n (≥ 3)-dimensional semi-Riemannian manifold and S^2 be its level 2 Ricci tensor of type (0,2). In terms of local coordinates, the tensor S^2 can be expressed as

$$S^2_{ij} = g^{kl} S_{ik} S_{jl}.$$

Similarly the Ricci tensors of level 3 and 4, S^3 and S^4 are respectively defined as

$$S^3_{ij} = g^{kl} S^2_{ik} S_{jl} \quad \text{and} \quad S^4_{ij} = g^{kl} S^3_{ik} S_{jl}.$$

Now for two $(0,2)$ tensors A and E, their Kulkarni-Nomizu product ([5], [7], [11], [18]) $A \wedge E$ is given by

$$(A \wedge E)_{ijkl} = A_{il}E_{jk} + A_{jk}E_{il} - A_{ik}E_{jl} - A_{jl}E_{ik}.$$

In particular, we can define $g \wedge g$, $g \wedge S$, $S \wedge S$, $g \wedge S^2$, $S \wedge S^2$ and $S^2 \wedge S^2$ as follows:

$$(g \wedge g)_{ijkl} = 2(g_{il}g_{jk} - g_{ik}g_{jl}), \quad (g \wedge S)_{ijkl} = g_{il}S_{jk} + S_{il}g_{jk} - g_{ik}S_{jl} - S_{ik}g_{jl},$$

$$(S \wedge S)_{ijkl} = 2(S_{il}S_{jk} - S_{ik}S_{jl}), \quad (g \wedge S^2)_{ijkl} = g_{il}S^2_{jk} + S^2_{il}g_{jk} - g_{ik}S^2_{jl} - S^2_{ik}g_{jl},$$

$$(S \wedge S^2)_{ijkl} = S_{il}S^2_{jk} + S^2_{il}S_{jk} - S_{ik}S^2_{jl} - S^2_{ik}S_{jl}, \quad (S^2 \wedge S^2)_{ijkl} = 2(S^2_{il}S^2_{jk} - S^2_{ik}S^2_{jl}).$$

We note that the tensor $\frac{1}{2}(g \wedge g)$ is known as Gaussian curvature tensor and is denoted by G. A tensor D of type (0,4) on M is said to be generalized curvature tensor ([5], [7], [11]), if

$$(i) D_{ijkl} + D_{jikl} = 0, \quad (ii) D_{ijkl} = D_{klij}, \quad (iii) D_{ijkl} + D_{jkil} + D_{kijl} = 0.$$

Moreover if D satisfies the second Bianchi identity, i.e.,

$$D_{ijkl,m} + D_{jmkl,i} + D_{mikl,j} = 0,$$

then D is called a proper generalized curvature tensor, where ‘coma’ denotes the covariant derivative. If A and B are two symmetric $(0,2)$ tensors, then $A \wedge B$ is obviously a generalized curvature tensor.

We mention that there are various generalized curvature tensors which are linear combination of Riemann-Christoffel curvature tensor with Kulkarni-Nomizu products of some tensors. One such important curvature tensor is the conformal curvature tensor C, and is given by

$$C = R - \frac{1}{n-2} g \wedge S + \frac{\kappa}{2(n-1)(n-2)} g \wedge g.$$

We refer the readers to see [27] for details about the various curvature tensors and geometric structures along with their equivalency.

Definition 2.1. Let M be a semi-Riemannian manifold satisfying the following condition

$$R = N_1 g \wedge g + N_2 g \wedge S + N_3 S \wedge S, \tag{2.1}$$

for some N_1, N_2 and $N_3 \in C^\infty(M)$. The above condition is called a Roter type condition and M is called a Roter type manifold ([**?**, 6, 13, 15, 19, 21]) with N_1, N_2 and N_3 as the associated scalars.

It may be mentioned that every conformally flat manifold of dimension ≥ 4, as well as every 3-dimensional manifold are Roter type.

Definition 2.2. Let M be a semi-Riemannian manifold satisfying the following condition

$$R = L_1 g \wedge g + L_2 g \wedge S + L_3 S \wedge S + L_4 g \wedge S^2 + L_5 S \wedge S^2 + L_6 S^2 \wedge S^2, \tag{2.2}$$

for some $L_i \in C^\infty(M)$, $1 \leqslant i \leqslant 6$. The above condition is called a generalized Roter type condition and M is called a generalized Roter type manifold ([25], [28]) with L_i's as the associated scalars.

For details about the geometric properties of generalized Roter type manifold we refer the readers to see [28]. We mention that such decompositions of R were already investigated in [8], [12], [23] and very recently in [9], [10], [24]. Throughout this paper by a proper GRT_n we mean a GRT_n which is not a RT_n, and by a proper RT_n we mean a RT_n which is not conformally flat. A GRT_n or a RT_n is said to be special if one or more of their associated scalars are identically zero or assume some particular values.

Again contracting the Roter type and generalized Roter type conditions Shaikh and Kundu [28] presented some generalizations of Einstein metric conditions.

Definition 2.3. [1] If in a semi-Riemannian manifold M, S and g (resp., S^2, S and g; S^3, S^2, S and g; S^4, S^3, S^2, S and g) are linearly dependent then it is called $Ein(1)$ (resp., $Ein(2)$; $Ein(3)$; $Ein(4)$) manifold. The $Ein(1)$ manifold is known as Einstein manifold and in this case we have $S = \frac{\kappa}{n} g$.

We note that every $Ein(i)$ manifold is $Ein(i+1)$ for $i = 1, 2, 3$ but not conversely [25]. It is well known that every manifold of constant curvature is always Einstein. Again a RT_n is $Ein(2)$ except $N_1 = -\frac{\kappa}{2(n-1)(n-2)}$, $N_2 = \frac{1}{n-2}$, $N_3 = 0$; and a GRT_n is $Ein(4)$ except $L_1 = \frac{1}{2}\left(\frac{L_4\left(\kappa^2 - \kappa^{(2)}\right)}{n-1} - \frac{\kappa}{(n-1)(n-2)}\right)$, $L_2 = \frac{1}{n-2} - L_4\kappa$, $L_3 = \frac{1}{2}L_4(n-2)$, $L_5 = 0$, $L_6 = 0$, where $\kappa^{(2)} = tr(S^2)$.

3 Warped product manifolds

Let $(\overline{M}, \overline{g})$ and $(\widetilde{M}, \widetilde{g})$ be two semi-Riemannian manifolds of dimension p and $(n-p)$ respectively $(1 \leq p \leq n-1)$. The product metric $\mathring{g}$ on $M = \overline{M} \times \widetilde{M}$ is defined as

$$\mathring{g} = \pi^*(\overline{g}) + \sigma^*(\widetilde{g}),$$

where $\pi : M \to \overline{M}$ and $\sigma : M \to \widetilde{M}$ are the natural projections. Generalizing this notion of product metric, Kručkovič [22] introduced the notion of semi-decomposable metric g on M as

$$g = \pi^*(\overline{g}) + (f \circ \pi)\sigma^*(\widetilde{g}),$$

where f is a positive smooth function on $\overline{M}$. Again to construct a large class of complete manifolds of negative curvature, Bishop and O'Neill [2] obtained the same notion and named as warped product manifold. We mention that in the literature of differential geometry the name warped product is more widely used and here we also use the name 'warped product manifold'.

Let M be the warped product manifold equipped with the warped product metric g. If we consider a product chart

$$\left(U \times V; x^1, x^2, ..., x^p, x^{p+1} = y^1, x^{p+2} = y^2, ..., x^n = y^{n-p}\right)$$

on M, then in terms of local coordinates, g can be expressed as

$$g_{ij} = \begin{cases} \overline{g}_{ij} & \text{for } i = a \text{ and } j = b, \\ f\widetilde{g}_{ij} & \text{for } i = \alpha \text{ and } j = \beta, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

where $a, b \in \{1, 2, ..., p\}$ and $\alpha, \beta \in \{p+1, p+2, ..., n\}$. We note that throughout the paper we consider $a, b, c, ... \in \{1, 2, ..., p\}$ and $\alpha, \beta, \gamma, ... \in \{p+1, p+2, ..., n\}$ and $i, j, k, ... \in \{1, 2, ..., n\}$. Here $\overline{M}$ is called the base, $\widetilde{M}$ is called the fiber and f is called the warping function of M. If $f = 1$, then the warped product reduces to the product manifold. Moreover, when Ω is a quantity formed with respect to g, we denote by $\overline{\Omega}$ and $\widetilde{\Omega}$, the similar quantities formed with respect to $\overline{g}$ and $\widetilde{g}$ respectively.

The non-zero local components R_{hijk} of the Riemann-Christoffel curvature tensor R, S_{jk} of the Ricci tensor S and the scalar curvature κ of M are given by

$$R_{abcd} = \overline{R}_{abcd}, \quad R_{a\alpha b\beta} = fT_{ab}\widetilde{g}_{\alpha\beta}, \quad R_{\alpha\beta\gamma\delta} = f\widetilde{R}_{\alpha\beta\gamma\delta} - f^2 P\widetilde{G}_{\alpha\beta\gamma\delta}, \tag{3.2}$$

$$S_{ab} = \overline{S}_{ab} - (n-p)T_{ab}, \quad S_{\alpha\beta} = \widetilde{S}_{\alpha\beta} + Q\widetilde{g}_{\alpha\beta}, \quad \text{and} \tag{3.3}$$

$$\kappa = \overline{\kappa} + \frac{\widetilde{\kappa}}{f} - (n-p)[(n-p-1)P - 2\, tr(T)], \tag{3.4}$$

where $G_{ijkl} = g_{il}g_{jk} - g_{ik}g_{jl}$ are the components of Gaussian curvature and

$$T_{ab} = -\frac{1}{2f}(f_{a,b} - \frac{1}{2f}f_a f_b), \qquad tr(T) = g^{ab}T_{ab},$$

$$P = \frac{1}{4f^2}g^{ab}f_a f_b, \qquad Q = -f((n-p-1)P + tr(T)), \quad f_a = \partial_a f = \frac{\partial f}{\partial x^a}.$$

For more detail about warped product components of basic tensors we refer the readers to see [20], [26] and also references therein.

Now from above results we can easily calculate the local components of various necessary tensors. The non-zero local components of S^2, $(g \wedge g)$, $(g \wedge S)$, $(S \wedge S)$, $(g \wedge S^2)$, $(S \wedge S^2)$ and $(S^2 \wedge S^2)$ are given as follows:

$$(3.5) \quad \begin{cases} (i) S^2_{ab} = \overline{S}^2_{ab} + (n-p)(\overline{S} \cdot T)_{ab} + (n-p)^2 T^2_{ab}, \\ (ii) S^2_{\alpha\beta} = \frac{1}{f}[\widetilde{S}^2_{\alpha\beta} + 2Q\widetilde{S}_{\alpha\beta} + Q^2 \widetilde{g}_{\alpha\beta}]. \end{cases}$$

$$(3.6) \quad \begin{cases} (i)(g \wedge g)_{abcd} = (\overline{g} \wedge \overline{g})_{abcd}, \\ (ii)(g \wedge g)_{a\alpha b\beta} = -2f\overline{g}_{ab}\widetilde{g}_{\alpha\beta}, \\ (iii)(g \wedge g)_{\alpha\beta\gamma\delta} = f^2(\widetilde{g} \wedge \widetilde{g})_{\alpha\beta\gamma\delta}. \end{cases}$$

$$(3.7) \quad \begin{cases} (i)(g \wedge S)_{abcd} = (\overline{g} \wedge \overline{S})_{abcd} - (n-p)(\overline{g} \wedge T)_{abcd}, \\ (ii)(g \wedge S)_{a\alpha b\beta} = -\overline{g}_{ab}(\widetilde{S}_{\alpha\beta} + Q\widetilde{g}_{\alpha\beta}) - f\widetilde{g}_{\alpha\beta}(\overline{S}_{ab} - (n-p)T_{ab}), \\ (iii)(g \wedge S)_{\alpha\beta\gamma\delta} = f(\widetilde{g} \wedge \widetilde{S})_{\alpha\beta\gamma\delta} + 2fQ\widetilde{G}_{\alpha\beta\gamma\delta}. \end{cases}$$

$$(3.8) \quad \begin{cases} (i)(S \wedge S)_{abcd} = (\overline{S} \wedge \overline{S})_{abcd} - 2(n-p)(\overline{S} \wedge T)_{abcd} \\ \qquad\qquad + (n-p)^2(T \wedge T)_{abcd}, \\ (ii)(S \wedge S)_{a\alpha b\beta} = -2(\widetilde{S}_{\alpha\beta} + Q\widetilde{g}_{\alpha\beta})(\overline{S}_{ab} - (n-p)T_{ab}), \\ (iii)(S \wedge S)_{\alpha\beta\gamma\delta} = (\widetilde{S} \wedge \widetilde{S})_{\alpha\beta\gamma\delta} + 2Q(\widetilde{S} \wedge \widetilde{g})_{\alpha\beta\gamma\delta} + Q^2(\widetilde{g} \wedge \widetilde{g})_{\alpha\beta\gamma\delta}. \end{cases}$$

$$(3.9) \quad \begin{cases} (i)(g \wedge S^2)_{abcd} = (\overline{g} \wedge \overline{S}^2)_{abcd} + (n-p)(\overline{g} \wedge (\overline{S} \cdot T))_{abcd} \\ \qquad\qquad + (n-p)^2(\overline{g} \wedge T^2)_{abcd}, \\ (ii)(g \wedge S^2)_{a\alpha b\beta} = -\frac{1}{f}\overline{g}_{ab}(\widetilde{S}^2_{\alpha\beta} + 2Q\widetilde{S}_{\alpha\beta} + Q^2\widetilde{g}_{\alpha\beta}) \\ \qquad\qquad - f\widetilde{g}_{\alpha\beta}(\overline{S}^2_{ab} + (n-p)\overline{S} \cdot T_{ab} + (n-p)^2 T^2_{ab}), \\ (iii)(g \wedge S^2)_{\alpha\beta\gamma\delta} = (\widetilde{g} \wedge \widetilde{S}^2)_{\alpha\beta\gamma\delta} + 2Q(\widetilde{g} \wedge \widetilde{S})_{\alpha\beta\gamma\delta} + Q^2(\widetilde{g} \wedge \widetilde{g})_{\alpha\beta\gamma\delta}. \end{cases}$$

$$(3.10) \quad \begin{cases} (i)(S \wedge S^2)_{abcd} = (\overline{S} \wedge \overline{S}^2)_{abcd} + (n-p)(\overline{S} \wedge (\overline{S} \cdot T))_{abcd} \\ \qquad\qquad + (n-p)^2(\overline{S} \wedge T^2)_{abcd} - (n-p)(\overline{S}^2 \wedge T)_{abcd} \\ \qquad\qquad - (n-p)^2(T \wedge (\overline{S} \cdot T))_{abcd} \\ \qquad\qquad + (n-p)^3(T \wedge T^2)_{abcd}, \\ (ii)(S \wedge S^2)_{a\alpha b\beta} = -\frac{1}{f}(\overline{S}_{ab} - (n-p)T_{ab})(\widetilde{S}^2_{\alpha\beta} + 2Q\widetilde{S}_{\alpha\beta} + Q^2\widetilde{g}_{\alpha\beta}) \\ \qquad\qquad - (\widetilde{S}_{\alpha\beta} + Q\widetilde{g}_{\alpha\beta}) \\ \qquad\qquad\quad (\overline{S}^2_{ab} + (n-p)(\overline{S} \cdot T)_{ab} + (n-p)^2 T^2_{ab}), \\ (iii)(S \wedge S^2)_{\alpha\beta\gamma\delta} = \frac{1}{f}[(\widetilde{S} \wedge \widetilde{S}^2)_{\alpha\beta\gamma\delta} + 4Q(\widetilde{S} \wedge \widetilde{S})_{\alpha\beta\gamma\delta} \\ \qquad\qquad + Q^2(\widetilde{S} \wedge \widetilde{g})_{\alpha\beta\gamma\delta} + Q(\widetilde{g} \wedge \widetilde{S}^2)_{\alpha\beta\gamma\delta} \\ \qquad\qquad + 2Q^2(\widetilde{g} \wedge \widetilde{S})_{\alpha\beta\gamma\delta} + 2Q^3(\widetilde{g} \wedge \widetilde{g})_{\alpha\beta\gamma\delta}]. \end{cases}$$

$$
(3.11)\quad \begin{cases}
(i)(S^2\wedge S^2)_{abcd} = (\overline{S}^2\wedge\overline{S}^2)_{abcd} + (n-p)^2((\overline{S}\cdot T)\wedge(\overline{S}\cdot T))_{abcd} \\
\qquad +(n-p)^2(T^2\wedge T^2)_{abcd} + 2(n-p)^3(\overline{S}^2\wedge T^2)_{abcd} \\
\qquad +2(n-p)^3((\overline{S}\cdot T^2)\wedge T^2)_{abcd} \\
\qquad +2(n-p)(\overline{S}^2\wedge(\overline{S}\cdot T))_{abcd}, \\
(ii)(S^2\wedge S^2)_{a\alpha b\beta} = -\frac{2}{f}(\overline{S}^2_{ab} + (n-p)(\overline{S}\cdot T)_{ab} + (n-p)^2T^2_{ab}) \\
\qquad (\widetilde{S}^2_{\alpha\beta} + 2Q\widetilde{S}_{\alpha\beta} + Q^2\widetilde{g}_{\alpha\beta}), \\
(iii)(S^2\wedge S^2)_{\alpha\beta\gamma\delta} = \frac{1}{f^2}[(\widetilde{S}^2\wedge\widetilde{S}^2)_{\alpha\beta\gamma\delta} + 4Q^2(\widetilde{S}\wedge\widetilde{S})_{\alpha\beta\gamma\delta} \\
\qquad +Q^4(\widetilde{g}\wedge\widetilde{g})_{\alpha\beta\gamma\delta} + 4Q(\widetilde{S}^2\wedge\widetilde{S})_{\alpha\beta\gamma\delta} \\
\qquad +2Q^2(\widetilde{g}\wedge\widetilde{S}^2)_{\alpha\beta\gamma\delta} + 4Q^3(\widetilde{g}\wedge\widetilde{S})_{\alpha\beta\gamma\delta}].
\end{cases}
$$

From above it follows that the components of $g\wedge g$, $g\wedge S$, $S\wedge S$, $g\wedge S^2$, $S\wedge S^2$ and $S^2\wedge S^2$ are given in a quadratic form of Kulkarni-Nomizu product for base and fiber part, and quadratic form of the product for the mixed part. So each of them can be expressed by a matrix. For example, $(g\wedge S)_{abcd}$, $(g\wedge S)_{a\alpha b\beta}$ and $(g\wedge S)_{\alpha\beta\gamma\delta}$ can respectively be expressed as

$\wedge$	$\overline{g}$	$\overline{S}$	$\overline{S}^2$	T	T^2	$\overline{S}\cdot T$
$\overline{g}$	0	$\frac{1}{2}$	0	$\frac{p-n}{2}$	0	0
$\overline{S}$	$\frac{1}{2}$	0	0	0	0	0
$\overline{S}^2$	0	0	0	0	0	0
T	$\frac{p-n}{2}$	0	0	0	0	0
T^2	0	0	0	0	0	0
$\overline{S}\cdot T$	0	0	0	0	0	0

,

	$\widetilde{g}$	$\widetilde{S}$	$\widetilde{S}^2$
$\overline{g}$	$-Q$	-1	0
$\overline{S}$	$-f$	0	0
$\overline{S}^2$	0	0	0
T	$f(p-n)$	0	0
T^2	0	0	0
$\overline{S}\cdot T$	0	0	0

and

$\wedge$	$\widetilde{g}$	$\widetilde{S}$	$\widetilde{S}^2$
$\widetilde{g}$	fQ	$\frac{f}{2}$	0
$\widetilde{S}$	$\frac{f}{2}$	0	0
$\widetilde{S}^2$	0	0	0

.

Similarly, we can get the matrix representations of the components for the other tensors $g\wedge g$, $S\wedge S$, $g\wedge S^2$, $S\wedge S^2$ and $S^2\wedge S^2$.

4 Warped product generalized Roter-type manifolds

Theorem 4.1. *If $M^n = \overline{M}^p\times_f\widetilde{M}^{n-p}$ is a warped product manifold, then M satisfies the generalized Roter type condition*

$$
(4.1)\qquad R = L_1 g\wedge g + L_2 g\wedge S + L_3 S\wedge S + L_4 g\wedge S^2 + L_5 S\wedge S^2 + L_6 S^2\wedge S^2
$$

if and only if

(i) *the Riemann-Christoffel curvature tensor $\overline{R}$ of $\overline{M}$ can be expressed as*

$\wedge$	$\overline{g}$	$\overline{S}$	$\overline{S}^2$	T	T^2	$\overline{S}\cdot T$
$\overline{g}$	L_1	$\frac{L_2}{2}$	$\frac{L_4}{2}$	$\frac{1}{2}L_2(p-n)$	$\frac{1}{2}L_4(n-p)^2$	$\frac{1}{2}L_4(n-p)$
$\overline{S}$	$\frac{L_2}{2}$	L_4	$\frac{L_5}{2}$	$L_4(p-n)$	$\frac{1}{2}L_5(n-p)^2$	$\frac{1}{2}L_5(n-p)$
$\overline{S}^2$	$\frac{L_4}{2}$	$\frac{L_5}{2}$	L_6	$\frac{1}{2}L_5(p-n)$	$L_6(n-p)^2$	$L_6(n-p)$
T	$\frac{1}{2}L_2(p-n)$	$L_3(p-n)$	$\frac{1}{2}L_5(p-n)$	$L_3(n-p)^2$	$-\frac{1}{2}L_5(n-p)^3$	$-\frac{1}{2}L_5(n-p)^2$
T^2	$\frac{1}{2}L_4(n-p)^2$	$\frac{1}{2}L_5(n-p)^2$	$L_6(n-p)^2$	$-\frac{1}{2}L_5(n-p)^3$	$L_6(n-p)^4$	$L_6(n-p)^3$
$\overline{S}\cdot T$	$\frac{1}{2}L_4(n-p)$	$\frac{1}{2}L_5(n-p)$	$L_6(n-p)$	$-\frac{1}{2}L_5(n-p)^2$	$L_6(n-p)^3$	$L_6(n-p)^2$

(ii) *$f\widetilde{R}$, $\widetilde{R}$ being the Riemann-Christoffel curvature tensor of $\widetilde{M}$, can be expressed as*

$\wedge$	$\widetilde{g}$	$\widetilde{S}$	$\widetilde{S}^2$
$\widetilde{g}$	$\frac{L_6Q^4}{f^2} + \frac{L_5Q^3}{f} + (L_3+L_4)Q^2 +$ $fL_2Q + f^2L_1 - \frac{fP}{2}$	$(L_3+L_4)Q +$ $\frac{L_2f^3+3L_5Q^2f+4L_6Q^3}{2f^2}$	$\frac{1}{2}\left(L_4 + \frac{Q(fL_5+2L_6Q)}{f^2}\right)$
$\widetilde{S}$	$(L_3+L_4)Q + \frac{L_2f^3+3L_5Q^2f+4L_6Q^3}{2f^2}$	$L_3 + \frac{2Q(fL_5+2L_6Q)}{f^2}$	$\frac{fL_5+4L_6Q}{2f^2}$
$\widetilde{S}^2$	$\frac{1}{2}\left(L_4 + \frac{Q(fL_5+2L_6Q)}{f^2}\right)$	$\frac{fL_5+4L_6Q}{2f^2}$	$\frac{L_6}{f^2}$

(iii) the following expression vanishes identically on M:

	$\widetilde{g}$	$\widetilde{S}$	$\widetilde{S}^2$
$\overline{g}$	$-\frac{L_4Q^2}{f} - L_2Q - 2fL_1$	$-L_2 - \frac{2L_4Q}{f}$	$-\frac{L_4}{f}$
$\overline{S}$	$-\frac{L_5Q^2}{f} - 2L_3Q - fL_2$	$-\frac{2(fL_3+L_5Q)}{f}$	$-\frac{L_5}{f}$
$\overline{S}^2$	$-\frac{2L_6Q^2}{f} - L_5Q - fL_4$	$-L_5 - \frac{4L_6Q}{f}$	$-\frac{2L_6}{f}$
T	$f(L_2(p-n)-1) + \frac{L_5Q^2(p-n)}{f} + 2L_3Q(p-n)$	$-\frac{2(n-p)(fL_3+L_5Q)}{f}$	$\frac{L_5(p-n)}{f}$
T^2	$-\frac{(n-p)^2\left(L_4f^2+L_5Qf+2L_6Q^2\right)}{f}$	$-\frac{(n-p)^2(fL_5+4L_6Q)}{f}$	$-\frac{2L_6(n-p)^2}{f}$
$\overline{S}\cdot T$	$-\frac{(n-p)\left(L_4f^2+L_5Qf+2L_6Q^2\right)}{f}$	$-\frac{(n-p)(fL_5+4L_6Q)}{f}$	$\frac{2L_6(p-n)}{f}$

.

Proof. In terms of local coordinates, (4.1) can be expressed as

$$\begin{aligned} R_{ijkl} &= L_1(g\wedge g)_{ijkl} + L_2(g\wedge S)_{ijkl} + L_3(S\wedge S)_{ijkl} \\ &+ L_4(g\wedge S^2)_{ijkl} + L_5(S\wedge S^2)_{ijkl} + L_6(S^2\wedge S^2)_{ijkl}. \end{aligned} \tag{4.2}$$

From (4.2) it follows that we can consider the following three cases:

$$\begin{gathered} \text{(I) } i=a, j=b, k=c, l=d; \\ \text{(II) } i=\alpha, j=\beta, k=\gamma, l=\delta; \\ \text{(III) } i=a, j=\alpha, k=b, l=\beta. \end{gathered}$$

Consider the case I: $i=a, j=b, k=c, l=d$ in (4.2) and using (3.2)-(3.11), we get

$$\begin{aligned} \overline{R}_{abcd} = {} & L_1(\overline{g}\wedge\overline{g})_{abcd} + L_3((\overline{S}-(n-p)T)\wedge(\overline{S}-(n-p)T))_{abcd} \\ & +L_2(\overline{g}\wedge(\overline{S}-(n-p)T))_{abcd} + L_4(\overline{g}\wedge(\overline{S}-(n-p)\overline{S}\cdot T+(n-p)^2T^2))_{abcd} \\ & +L_5(S\wedge(\overline{S}-(n-p)\overline{S}\cdot T+(n-p)^2T^2))_{abcd} \\ & +L_6((\overline{S}-(n-p)\overline{S}\cdot T+(n-p)^2T^2)\wedge(\overline{S}-(n-p)\overline{S}\cdot T+(n-p)^2T^2))_{abcd}. \end{aligned}$$

Now expressing the above in matrix form, we get (i). Similarly setting $i=\alpha, j=\beta, k=\gamma, l=\delta$ in (4.2), we get (ii).
Again, by putting $i=a, j=\alpha, k=b, l=\beta$ in (4.2) and using (3.2)-(3.11), we obtain

$$\begin{aligned} fT_{ab}\widetilde{g}_{\alpha\beta} = {} & -2L_1f\overline{g}_{ab}\widetilde{g}_{\alpha\beta} - L_2\left[\overline{g}_{ab}(\widetilde{S}_{\alpha\beta}+Q\widetilde{g}_{\alpha\beta}) + f\widetilde{g}_{\alpha\beta}(\overline{S}_{ab}-(n-p)T_{ab})\right] \\ & -(\overline{S}_{ab}-(n-p)T_{ab})\left[2L_3(\widetilde{S}_{\alpha\beta}+Q\widetilde{g}_{\alpha\beta}) + \frac{L_5}{f}(\widetilde{S}^2_{\alpha\beta}+2Q\widetilde{S}_{\alpha\beta}+Q^2\widetilde{g}_{\alpha\beta})\right] \\ & -\frac{L_4}{f}\overline{g}_{ab}(\widetilde{S}^2_{\alpha\beta}+2Q\widetilde{S}_{\alpha\beta}+Q^2\widetilde{g}_{\alpha\beta}) \\ & -L_4f\widetilde{g}_{\alpha\beta}(\overline{S}^2_{ab}+(n-p)\overline{S}\cdot T_{ab}+(n-p)^2T^2_{ab}) \\ & -L_5(\overline{S}^2_{ab}+(n-p)(\overline{S}\cdot T)_{ab}+(n-p)^2T^2_{ab})(\widetilde{S}_{\alpha\beta}+Q\widetilde{g}_{\alpha\beta}), \\ & -\frac{2L_6}{f}(\overline{S}^2_{ab}+(n-p)(\overline{S}\cdot T)_{ab}+(n-p)^2T^2_{ab})(\widetilde{S}^2_{\alpha\beta}+2Q\widetilde{S}_{\alpha\beta}+Q^2\widetilde{g}_{\alpha\beta}). \end{aligned}$$

Now simplifying above and expressing in matrix form, we obtain (iii). This completes the proof. □

The above theorem yields the following:

Corollary 4.2. *If $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ is a warped product manifold with $(n-p) \geq 3$ satisfying the generalized Roter-type condition* (4.1)*, then*

(i) the fiber $\widetilde{M}$ is generalized Roter type.

(ii) the fiber $\widetilde{M}$ is Roter type if $J_1 \neq 0$, where

$$\begin{aligned} J_1 &= -\frac{1}{f}\Big[L_4 p + L_5(tr(T)(n-p) + \overline{\kappa}) \\ &\quad +2L_6\left(tr(T^2)(n-p)^2 + (n-p)tr(\overline{S}\cdot T) + \overline{\kappa^{(2)}}\right)\Big]. \end{aligned}$$

(iii) the fiber $\widetilde{M}$ is of vanishing conformal curvature tensor if $J_1 \neq 0$ and

$$\frac{(J_2)^2 L_6}{f^2(J_1)^2} + \frac{J_2(fL_5 + 4L_6Q)}{f^2 J_1} + \frac{2Q(fL_5 + 2L_6Q)}{f^2} + L_3 = 0,$$

$$\begin{aligned} \textit{where } J_2 = &- \frac{1}{f}\Big[(fL_5 + 4L_6Q)\left((n-p)\left(tr(T^2)(n-p) + tr(\overline{S}\cdot T)\right) + \overline{\kappa^{(2)}}\right) \\ &+ p(fL_2 + 2L_4Q) + 2tr(T)(n-p)(fL_3 + L_5Q) + 2\overline{\kappa}(fL_3 + L_5Q)\Big]. \end{aligned}$$

(iv) the fiber $\widetilde{M}$ is of constant curvature if $J_1 = 0$ and $J_2 \neq 0$.

Corollary 4.3. *If $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ is a warped product manifold with $p \geq 3$ satisfying generalized Roter-type condition* (4.1)*, then the base $\overline{M}$ is generalized Roter type if T can be expressed as a linear combination of $\overline{g}$ and $\overline{S}$.*

From Theorem 4.1 we can easily get the necessary and sufficient condition for a warped product manifold to be Roter type.

Corollary 4.4. *If $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ is a non-flat warped product manifold, then M satisfies the Roter type condition*

$$R = N_1 g \wedge g + N_2 g \wedge S + N_3 S \wedge S \tag{4.3}$$

if and only if

(i) $\overline{R} =$

$\wedge$	$\overline{g}$	$\overline{S}$	T
$\overline{g}$	N_1	$\frac{N_2}{2}$	$\frac{1}{2}N_2(p-n)$
$\overline{S}$	$\frac{N_2}{2}$	N_3	$N_3(p-n)$
T	$\frac{1}{2}N_2(p-n)$	$N_3(p-n)$	$N_3(n-p)^2$

,

(ii) $f\widetilde{R} =$

$\wedge$	$\widetilde{g}$	$\widetilde{S}$
$\widetilde{g}$	$N_1f^2 + N_2Qf + N_3Q^2 - \frac{fP}{2}$	$\frac{fN_2}{2} + N_3Q$
$\widetilde{S}$	$\frac{fN_2}{2} + N_3Q$	N_3

,

(iii)

	$\overline{g}$	$\overline{S}$	T
$\widetilde{g}$	$-2fN_1 - N_2Q$	$-fN_2 - 2N_3Q$	$-f(1 + N_2(n-p)) - 2N_3Q(n-p)$
$\widetilde{S}$	$-N_2$	$-2N_3$	$2N_3(p-n)$

$= 0.$

Proof. The result follows from Theorem 4.1, by setting $L_1 = N_1, L_2 = N_2, L_3 = N_3$ and $L_4 = L_5 = L_6 = 0$. □

Corollary 4.5. *If $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ is a non-flat warped product manifold with $(n-p) \geq 3$ satisfying the Roter type condition* (4.3)*, then*
(i) the fiber is of Roter type.
(ii) the fiber is of vanishing conformal curvature tensor if M is conformally flat.
(iii) the fiber is of constant curvature if $-2(n-p)N_3 tr(T) - N_2 p - 2N_3\kappa \neq 0$.

Corollary 4.6. *If $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ is a non-flat warped product manifold with $p \geq 3$ satisfying the Roter type condition* (4.3)*, then the base is of Roter type if T, $\overline{g}$ and $\overline{S}$ are linearly dependent with non-zero coefficient of T.*

Recently, Deszcz et al. [14] studied the warped product Roter type manifold with 1-dimensional fiber and showed the following:

Corollary 4.7. *[14] If $M^n = \overline{M}^{n-1} \times_f \widetilde{M}^1$ is a non-flat warped product manifold satisfying the Roter type condition* (4.3)*, then $\overline{M}$ realizes a Roter type condition at those points, where it does not satisfy the Einstein metric condition.*

Proof. Since the dimension of $\widetilde{M}$ is $n-p=1$, from the condition (iii) of Corollary 4.4, we get

$$(2fN_1 + N_2 Q)\overline{g} + (fN_2 + 2N_3 Q)\overline{S} + (f(1+N_2) + 2N_3 Q)T = 0.$$

If at $x \in M$, $\overline{S} \neq \frac{\bar{\kappa}}{n-1}\overline{g}$, then $(f(1+N_2) + 2N_3 Q) \neq 0$ at x and T can be expressed as linear combination of $\overline{S}$ and $\overline{g}$ and hence by Corollary 4.6, $\overline{M}$ satisfies a Roter type condition at x. This completes the proof. □

Now we can easily deduce the necessary and sufficient condition for a warped product manifold to be conformally flat manifold, as follows:

Corollary 4.8. *If $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$, $1 \leq p \leq n-1$ is a non-flat warped product manifold, then M is conformally flat if and only if*
(i) $\overline{R} = \frac{\kappa}{2(n-1)(n-2)}\overline{g} \wedge \overline{g} + \frac{1}{n-2}\overline{g} \wedge \overline{S} - \frac{n-p}{n-2}\overline{g} \wedge T,$
(ii) $\widetilde{R} = \left[\frac{f\kappa}{2(n-1)(n-2)} + \frac{Q}{n-2} - \frac{1}{2}P\right]\widetilde{g} \wedge \widetilde{g} + \frac{1}{(n-2)}\widetilde{g} \wedge \widetilde{S},$
(iii) $\left[\frac{2f\kappa}{2(n-1)(n-2)} + \frac{Q}{n-2}\right]\overline{g}_{ab}\widetilde{g}_{\alpha\beta} + \frac{1}{n-2}\overline{g}_{ab}\widetilde{S}_{\alpha\beta} + \frac{f}{n-2}\overline{S}_{ab}\widetilde{g}_{\alpha\beta} + f\left(\frac{n-p}{n-2} + 1\right)T_{ab}\widetilde{g}_{\alpha\beta} = 0.$

Proof. The result follows from Corollary 4.4 by taking $N_1 = \frac{\kappa}{2(n-1)(n-2)}$, $N_2 = \frac{1}{n-2}$ and $N_3 = 0$. □

From above we can state the following:

Corollary 4.9. *[3] If $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ is a conformally flat warped product manifold, then*
(i) for $(n-p) \geq 2$, the fiber is of constant curvature.
(ii) for $p \geq 2$, the base is of vanishing conformal curvature tensor.

Proof. By contracting the condition (iii) of Corollary 4.8, it follows that T is a linear combination of $\overline{g}$ and $\overline{S}$, and $\widetilde{S}$ is a scalar multiple of $\widetilde{g}$. Putting these in the condition (i) and (ii) of Corollary 4.8, we get the results. □

Since for the decomposable manifold the warping function f is 1, we have $T = 0$, $P = 0$ and $Q = 0$. Thus applying these values in (3.2) to (3.11) we get the non-zero components of R, S, κ, S^2, $g\wedge g$, $g\wedge S$, $S\wedge S$, $g\wedge S^2$, $S\wedge S^2$ and $S^2\wedge S^2$. Consequently, from Theorem 4.1 we can state the following:

Corollary 4.10. *If $M^n = \overline{M}^p \times \widetilde{M}^{n-p}$ is a decomposable manifold, then M satisfies the generalized Roter-type condition* (4.1) *if and only if*

$$(i)\ \overline{R} = \begin{array}{|c|c|c|c|}\hline \wedge & \overline{g} & \overline{S} & \overline{S}^2 \\ \hline \overline{g} & L_1 & \frac{L_2}{2} & \frac{L_4}{2} \\ \hline \overline{S} & \frac{L_2}{2} & L_4 & \frac{L_5}{2} \\ \hline \overline{S}^2 & \frac{L_4}{2} & \frac{L_5}{2} & L_6 \\ \hline \end{array}\ ,\quad (ii)\ \widetilde{R} = \begin{array}{|c|c|c|c|}\hline \wedge & \widetilde{g} & \widetilde{S} & \widetilde{S}^2 \\ \hline \widetilde{g} & L_1 & \frac{L_2}{2} & \frac{L_4}{2} \\ \hline \widetilde{S} & \frac{L_2}{2} & L_3 & \frac{L_5}{2} \\ \hline \widetilde{S}^2 & \frac{L_4}{2} & \frac{L_5}{2} & L_6 \\ \hline \end{array}\ ,\quad (iii) \begin{array}{|c|c|c|c|}\hline & \widetilde{g} & \widetilde{S} & \widetilde{S}^2 \\ \hline \overline{g} & 2L_1 & L_2 & L_4 \\ \hline \overline{S} & L_2 & 2L_3 & L_5 \\ \hline \overline{S}^2 & L_4 & L_5 & 2L_6 \\ \hline \end{array} = 0.$$

Note. From the above corollary we can get the necessary and sufficient conditions for a decomposable manifold to be Roter type by taking $L_4 = L_5 = L_6 = 0$, and conformally flat by taking $L_3 = L_4 = L_5 = L_6 = 0$, $L_1 = \frac{\kappa}{2(n-1)(n-2)}$ and $L_2 = \frac{1}{n-2}$. Again from the above results we see that the decompositions of a semi-Riemannian product generalized Roter type manifold are also generalized Roter type manifold but the converse is not necessarily true, in general. We also note that the same situations arise for Roter type and conformally flat manifolds.

Remark 4.1. In this context we state the necessary and sufficient conditions of a warped product manifold to be Einstein. Let $M^n = \overline{M}^p \times_f \widetilde{M}^{n-p}$ be a warped product manifold ([1], see also [16]). Then M is Einstein if and only if

$(i)\ \overline{S} - (n-p)T = \frac{\kappa}{n}\overline{g}$ and $(ii)\ \widetilde{S} = \left(\frac{f\kappa}{n} - Q\right)\widetilde{g}$.

5 Examples

Example 5.1: Consider the warped product $M = \overline{M} \times_f \widetilde{M}$, where $\overline{M}$ is an open interval of $\mathbb{R}$ with usual metric $\overline{g} = (dx^1)^2$ in local coordinate x^1 and $\widetilde{M}$ is a 4-dimensional manifold equipped with a semi-Riemannian metric

$$\widetilde{g} = (dx^2)^2 + h(dx^3)^2 + h(dx^4)^2 + h\psi(dx^5)^2$$

in local coordinates (x^2, x^3, x^4, x^5), where the warping function f is a function of x^1, and h and ψ are everywhere non-zero functions of x^2 and x^3 respectively. We can easily evaluate the local components of necessary tensors of $\widetilde{M}$. The non-zero components of the Riemann-Christoffel curvature tensor $\widetilde{R}$ and the Ricci tensor $\widetilde{S}$ of $\widetilde{M}$ upto symmetry are

$$\psi\widetilde{R}_{1212} = \psi\widetilde{R}_{1313} = \widetilde{R}_{1414} = \psi\frac{\left((h')^2 - 2hh''\right)}{4h}, \quad \psi\widetilde{R}_{2323} = \widetilde{R}_{3434} = -\frac{\psi}{4}(h')^2,$$
$$\widetilde{R}_{2424} = \frac{1}{4}\left(-\psi(h')^2 - 2h\psi'' + \frac{h(\psi')^2}{\psi}\right),$$

$$\widetilde{S}_{11} = \frac{3\left(2hh'' - (h')^2\right)}{4h^2}, \quad \widetilde{S}_{22} = \frac{1}{4}\left(2h'' + \frac{(h')^2}{h} - \frac{(\psi')^2 - 2\psi\psi''}{\psi^2}\right),$$
$$\widetilde{S}_{33} = \frac{2hh'' + (h')^2}{4h}, \quad \widetilde{S}_{44} = \frac{1}{4}\left(2(\psi h'' + \psi'') + \frac{\psi(h')^2}{h} - \frac{(\psi')^2}{\psi}\right).$$

We can easily check that this manifold is generalized Roter type and $Ein(3)$ manifold. Moreover:

(i) if $(h')^2 - hh'' = 0$, i.e., $h = c_1 e^{c_2 x^2}$, then it is a $Ein(2)$ manifold, hence Roter type;

(ii) if $(\psi')^2 - 2\psi\psi'' = 0$, i.e., $\psi = \frac{(c_1 x^3 + 2c_2)^2}{4c_2}$, then it is a manifold of constant curvature, where c_1 and c_2 are arbitrary constants.

By straightforward calculation, we can evaluate the components of various necessary tensors of M. The non-zero components of the Riemann-Christoffel curvature tensor R and the Ricci tensor S of M upto symmetry are

$$\begin{aligned}
&h\psi R_{1212} = \psi R_{1313} = \psi R_{1414} = R_{1515} = h\psi \tfrac{(f')^2 - 2ff''}{4f},\\
&\psi R_{2323} = \psi R_{2424} = R_{2525} = \tfrac{\psi}{4}\left(-h\left(f'\right)^2 - 2fh'' + \tfrac{f\left(h'\right)^2}{h}\right),\\
&\psi R_{3434} = R_{4545} = -\tfrac{\psi}{4}\left(h^2\left(f'\right)^2 + f\left(h'\right)^2\right),\\
&R_{3535} = \tfrac{1}{4}\left[f\left(-\psi\left(h'\right)^2 - 2h\psi'' + \tfrac{h\left(\psi'\right)^2}{\psi}\right) - h^2\psi\left(f'\right)^2\right],\\
&S_{11} = -\tfrac{\left(f'\right)^2 - 2ff''}{f^2}, \quad S_{22} = \tfrac{1}{4}\left(2f'' + \tfrac{2\left(f'\right)^2}{f} + \tfrac{6hh'' - 3\left(h'\right)^2}{h^2}\right),\\
&\psi S_{33} = S_{55} = \tfrac{1}{4}\left(2h\psi f'' + \tfrac{2h\psi\left(f'\right)^2}{f} + 2\psi h'' + \tfrac{\psi\left(h'\right)^2}{h} + 2\psi'' - \tfrac{\left(\psi'\right)^2}{\psi}\right),\\
&S_{44} = \tfrac{1}{4}\left(2\left(hf'' + h''\right) + \tfrac{2h\left(f'\right)^2}{f} + \tfrac{\left(h'\right)^2}{h}\right).
\end{aligned}$$

From these we can easily calculate the components of S^2, S^3, S^4 and also the components of G, $g \wedge S$, $S \wedge S$, $g \wedge S^2$, $S \wedge S^2$ and $S^2 \wedge S^2$. We observe that for every f, h and ψ, the manifold is $Ein(4)$ but not of generalized Roter type. We now discuss the results for particular value of the functions f, h and ψ step by step as follows:

Step I: If $(h')^2 - hh'' = 0$, i.e., $h = c_1 e^{c_2 x^2}$, then M is generalized Roter type and $Ein(3)$. We note that in this case fiber $\widetilde{M}$ is proper Roter type and hence M is a proper generalized Roter type warped product manifold with proper Roter type fiber.

Step II: Again consider

$$-2(f')^2 + f(2f'' - 1) = 0, \text{ i.e., } f = \frac{1}{16c_1^2} e^{-\sqrt{c_1}(x^1 + c_2)}\left(e^{\pm\sqrt{c_1}(x^1 + c_2)} + 4c_1\right)^2,$$

where c_1 and c_2 are arbitrary non-zero constants. Then the manifold is a $Ein(2)$ manifold and hence the manifold is proper Roter type. In this case fiber remains also Roter type. So M is a warped product proper Roter type manifold with proper Roter type fiber.

Step III: Finally, consider $(\psi')^2 - 2\psi\psi'' = 0$, i.e., $\psi = \frac{(c_1 x^3 + 2c_2)^2}{4c_2}$. Then M is of constant curvature and in this case fiber is also of constant curvature.

If $f = (x^1)^2$, $h = c_2 \cos^2(x^2 - 2c_1)$ and $\psi = e^{x^3}$, then the manifold M is a special generalized Roter type and is $Ein(3)$. In this case the fiber $\widetilde{M}$ is proper generalized Roter type and is $Ein(3)$. Hence M is a warped product generalized Roter type manifold with proper generalized Roter type fiber.

6 Conclusions

The present paper is devoted to the study of warped product generalized Roter type manifolds and obtained the necessary and sufficient conditions for a warped product manifold to be of generalized Roter type manifold. As a particular case we obtain the characterization of a warped product Roter type manifold. It is shown that the fiber of a warped product GRT_n (resp., RT_n, conformally flat) is also generalized Roter type (resp., Roter type, manifold of constant curvature). We find out the conditions for which the fiber of a warped product GRT_n is Roter type, conformally flat or a manifold of constant curvature and the base is a generalized Roter type. It is also shown that the two decompositions of a GRT_n product manifold are both generalized Roter type but the product of two generalized Roter type manifolds is not always generalized Roter type. Finally by suitable metric, the existence of such case is ensured by an example.

Acknowledgements. The second named author gratefully acknowledges to CSIR, New Delhi (File No. 09/025 (0194)/2010-EMR-I) for the financial assistance. All the algebraic computations of Section 5 are calculated with help of Wolfram Mathematica.

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