New aspects of Ionescu–Weitzenböck's inequality

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Abstract. The focus of this article is *Ionescu-Weitzenböck' s inequality* using *the circumcircle mid-arc triangle.* The original results include: (i) an improvement of the *Finsler-Hadwiger's* inequality; (ii) several refinements and some applications of this inequality; (iii) a new version of the Ionescu-Weitzenböck inequality, in an inner product space, with applications in differential geometry.

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1 History of Ionescu-Weitzenböck's inequality

Given a triangle ABC, denote a, b, c the side lengths, s the semiperimeter, R the circumradius, r the inradius, m_a and h_a the lengths of median, respectively of altitude containing the vertex A, and Δ the area of ABC. In this paper we give a similar approach related to Ionescu-Weitzenböck's inequality and we obtain several refinements and some applications of this inequality.

R. Weitzenböck [12] showed that: *In any triangle* ABC, *the following inequality holds*:

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}\Delta. \tag{1.1}$$

In the theory of geometric inequalities Weitzenböck's inequality plays an important role, its applications are very interesting and useful. This inequality was given to solve at third International Mathematical Olympiad, Veszprém, Ungaria, 8-15 iulie 1961.

I. Ionescu (*Problem 273*, Romanian Mathematical Gazette (in Romanian), 3, 2 (1897); 52), the founder of Romanian Mathematical Gazette, published in 1897 the problem: *Prove that there is no triangle for which the inequality*

$$4\Delta\sqrt{3} > a^2 + b^2 + c^2$$

can be satisfied. We observe that the inequality of Ionescu is the same with the inequality of Weitzenböck. D. M. Bătineţu-Giurgiu and N. Stanciu (*Ionescu-Weitzenböck's*

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type inequalities (in Romanian), Gazeta Matematică Seria B, 118, 1 (2013), 1–10) suggested that the inequality (1.1) must be named the inequality of Ionescu-Weitzenböck.

A very important result is given in (*On Weitzenböck inequality and its generalizations*, 2003, [on-line at http://rgmia.org/v6n4.php]), where S.-H. Wu, Z.-H. Zhang and Z.-G. Xiao proved that Ionescu-Weitzenböck's inequality and Finsler-Hadwiger's inequality

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}\Delta + (a-b)^2 + (b-c)^2 + (c-a)^2 , \tag{1.2}$$

are equivalent. In fact, applying Ionescu-Weitzenböck's inequality in a special triangle, we deduce Finsler-Hadwiger's inequality. C. Lupu, R. Marinescu and S. Monea (*Geometrical proof of some inequalities* Gazeta Matematică Seria B (in Romanian), 116, 12 (2011), 257–263) treated this inequality and A. Cipu (*Optimal reverse Finsler-Hadwiger inequalities*, Gazeta Matematică Seria A (in Romanian), 3-4/2012, 61–68) shows optimal reverse of Finsler-Hadwiger inequalities.

The more general form

$$\Delta \leq \frac{\sqrt{3}}{4}\left(\frac{a^k + b^k + c^k}{3}\right)^{\frac{2}{k}}, k > 0,$$

of Ionescu-Weitzenböck's inequality appeared in a problem of C. N. Mills, O. Dunkel (*Problem 3207*, Amer. Math. Monthly, 34 (1927), 382–384). A number of eleven proofs of the Weitzenböck's inequality were presented by A. Engel [4]. N. Minculete and I. Bursuc (*Several proofs of the Weitzenböck Inequality*, Octogon Mathematical Magazine, **16**, 1 (2008)) also presented several proofs of the Ionescu-Weitzenböck inequality. In (N. Minculete, *Problema 26132*, Gazeta Matematică Seria B (in Romanian), 4 (2009)) is given the following inequality

$$a^2 + b^2 + c^2 \geq 4\Delta\left(\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}\right),$$

which implies the inequality of Ionescu – Weitzenböck, because $\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} \geq \sqrt{3}$.

The papers [1]-[12] provides sufficient mathematical updates for obtaining the original results included in the following sections.

2 Refinement of Ionescu-Weitzenböck's inequality

Let us present several improvements of Ionescu – Weitzenböck's inequality.

Theorem 2.1. *Any triangle satisfies the following inequality*

$$a^2 + b^2 + c^2 - 4\sqrt{3}\Delta \geq 2\left(m_a^2 - h_a^2\right). \tag{2.1}$$

Proof. Using the relation $4m_a^2 = 2\left(b^2 + c^2\right) - a^2$, the inequality (2.1) is equivalent with

$$a^2 + b^2 + c^2 - 4\sqrt{3}\Delta \geq 2\frac{2\left(b^2 + c^2\right) - a^2}{4} - 2h_a^2 = b^2 + c^2 - \frac{a^2}{2} - 2h_a^2,$$

i.e. $\frac{3a^2}{2} + 2h_a^2 \geq 4\sqrt{3}\Delta$, which is true, because

$$\tfrac{3a^2}{2} + 2h_a^2 \geq 2\sqrt{\tfrac{3a^2}{2} \cdot 2h_a^2} = 2\sqrt{3}ah_a = 4\sqrt{3}\Delta. \qquad \square$$

Given a triangle ABC and a point P not a vertex of triangle ABC, we define the A_1- vertex of the circumcevian triangle as the point other than A in which the line AP meets the circumcircle of triangle ABC, and similarly for B_1 and C_1. Then the triangle $A_1B_1C_1$ is called the $P-$ circumcevian triangle of ABC [7]. The circumcevian triangle associated to the incenter I is called *circumcircle mid-arc triangle.* Next, we give a similar approach as in [12] related to Ionescu-Weitzenböck' s using the circumcircle mid-arc triangle.

Theorem 2.2. *Ionescu-Weitzenböck's inequality and Finsler-Hadwiger's inequality are equivalent.*

Proof. From inequality (1.2), we obtain that $a^2+b^2+c^2 \geq 4\sqrt{3}\Delta$. Therefore, it is easy to see that Finsler - Hadwiger's inequality implies Ionescu-Weitzenböck's inequality.

Now, we show that Ionescu-Weitzenböck's inequality implies Finsler - Hadwiger's inequality. Figure 1

Denote by a_1, b_1, c_1 the opposite sides of circumcircle mid-arc triangle $A_1B_1C_1$, A_1, B_1, C_1 the angles, s_1 the semiperimeter, Δ_1 the area, r_1 the inradius and R_1 the circumradius (see Figure 1). We observe that $R_1 = R$. In triangle $A_1B_1C_1$ we have the following: $m(\widehat{BA_1C}) = \frac{B+C}{2} = \frac{\pi}{2} - \frac{A}{2}$, which implies, using the sine law, that $a_1 = 2R\cos\frac{A}{2} = 2R\sqrt{\frac{s(s-a)}{bc}} = 2R\sqrt{\frac{s}{abc}}\sqrt{a(s-a)} = \sqrt{\frac{R}{r}}\sqrt{a(s-a)}$. In analogous way, we obtain $b_1 = \sqrt{\frac{R}{r}}\sqrt{b(s-b)}$ and $c_1 = \sqrt{\frac{R}{r}}\sqrt{c(s-c)}$. We make some calculations and we obtain the following

$$\begin{aligned}\sum_{cyclic} a_1^2 &= \sum_{cyclic}\left(\sqrt{\frac{R}{r}}\sqrt{a(s-a)}\right)^2 = \frac{R}{r}\sum_{cyclic} a(s-a)\\ &= \frac{R}{2r}\left(\sum_{cyclic} a^2 - \sum_{cyclic}(a-b)^2\right)\end{aligned} \tag{2.2}$$

and

$$\Delta_1 = \frac{a_1b_1c_1}{4R_1} = \frac{8R^3\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{4R} = 2R^2\prod_{cyclic}\cos\frac{A}{2} = 2R^2\frac{s}{4R} = \frac{R}{2r}\Delta. \tag{2.3}$$

By applying Ionescu-Weitzenböck's inequality in the triangle $A_1B_1C_1$, we deduce the following relations $a_1^2+b_1^2+c_1^2 \geq 4\sqrt{3}\Delta_1$, which implies the inequality, using relations (2.2) and (2.3),

$$\sum_{cyclic} a_1^2 = \frac{R}{2r}\left(\sum_{cyclic} a^2 - \sum_{cyclic}(a-b)^2\right) \geq 4\sqrt{3}\Delta_1 = \frac{R}{2r}4\sqrt{3}\Delta,$$

which is equivalent to

$$\sum_{cyclic} a^2 - \sum_{cyclic} (a-b)^2 \geq 4\sqrt{3}\Delta.$$

This is in fact Finsler-Hadwiger's inequality. $\square$

Remark 2.1. By (2.3), using the Euler inequality ($R \geq 2r$), we obtain that $\Delta_1 \geq \Delta$.

Remark 2.2. The orthocenter of circumcircle mid-arc triangle $A_1B_1C_1$ is the incenter I of triangle ABC.

Remark 2.3. If H is the orthocenter of the triangle ABC and $A_2B_2C_2$ is $H-$circumcevian triangle of ABC, then the lines A_2A, B_2B, C_2C are the bisectors of the angles of triangle $A_2B_2C_2$. From Remark 1, we get $\Delta \geq \Delta_2$, where Δ_2 is the area of the triangle $A_2B_2C_2$.

Lemma 2.3. *In any triangle ABC there is the following equality:*

$$a^2 + b^2 + c^2 = 4\Delta\left(\cot A + \cot B + \cot C\right).$$

Proof. We observe that $a^2 + b^2 + c^2 = b^2 + c^2 - a^2 + a^2 - b^2 + c^2 + a^2 + b^2 - c^2 =$

$$= \sum_{cyclic} 2bc\cos A = 4\Delta \sum_{cyclic} \frac{\cos A}{\sin A},$$

which implies the equality of the statement. $\square$

Theorem 2.4. *Any triangle ABC satisfies the following equality*

$$a^2 + b^2 + c^2 = 4\Delta \sum_{cyclic} \tan\frac{A}{2} + \sum_{cyclic} (a-b)^2. \tag{2.4}$$

Proof. If we apply the equality of Lemma 2.3 in the triangle $A_1B_1C_1$, we obtain the relation $a_1^2 + b_1^2 + c_1^2 = 4\Delta_1\left(\cot A_1 + \cot B_1 + \cot C_1\right)$. Therefore, we have

$$\sum_{cyclic} a_1^2 = \frac{R}{2r}\left(\sum_{cyclic} a^2 - \sum_{cyclic}(a-b)^2\right) = 4\frac{R}{2r}\Delta \sum_{cyclic} \tan\frac{A}{2},$$

which proves the statement. $\square$

Remark 2.4. If use the inequality $\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} \geq \sqrt{3}$, which can be proved by Jensen's inequality, in relation (2.4), then we deduce Finsler-Hadwiger's inequality, which proved Ionescu-Weitzenböck's inequality.

Next we refined the Finsler-Hadwiger's inequality.

Theorem 2.5. *In any triangle there are the following inequalities:*

1) $a^2 + b^2 + c^2 \geq 4\sqrt{3}\Delta + \sum_{cyclic}(a-b)^2 + 4Rr\sin^2\frac{B-C}{2}$,

2) $a^2 + b^2 + c^2 \geq 4\sqrt{3}\Delta + \sum_{cyclic}(a-b)^2 + 2\sum_{cyclic}\left(\sqrt{a(s-a)} - \sqrt{b(s-b)}\right)^2$.

Proof. 1) If we apply inequality (2.1) in the triangle $A_1B_1C_1$, we obtain the relation

$$a_1^2 + b_1^2 + c_1^2 - 4\sqrt{3}\Delta_1 \geq 2\left(m_{a_1}^2 - h_{a_1}^2\right). \tag{2.5}$$

In any triangle, we have the relations

$$\begin{cases} 16\Delta^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4 \\ 4a^2m_a^2 - 4a^2h_a^2 = 2a^2b^2 + 2c^2a^2 - a^4 - 16\Delta^2 = b^4 - 2b^2c^2 + c^4 = \left(b^2 - c^2\right)^2, \end{cases}$$

from which we infer the relation $2\left(m_a^2 - h_a^2\right) = \frac{\left(b^2-c^2\right)^2}{2a^2}$. Therefore, this equality applied in the triangle $A_1B_1C_1$ becomes

$$2\left(m_{a_1}^2 - h_{a_1}^2\right) = \frac{\left(b_1^2 - c_1^2\right)^2}{2a_1^2} = \frac{4R^4\left(\cos B - \cos C\right)^2}{8R^2\cos^2\frac{A}{2}} = 2R^2\sin^2\frac{B-C}{2}. \tag{2.6}$$

But, combining relations (2.5), (2.6) and the equality

$$\sum_{cyclic} a_1^2 - 4\sqrt{3}\Delta_1 = \frac{R}{2r}\left(\sum_{cyclic} a^2 - \sum_{cyclic}(a-b)^2 - 4\sqrt{3}\Delta\right),$$

it follows the inequality of statement.

2) From equality (2.4) applied in the triangle $A_1B_1C_1$, we deduce the equality:

$$a^2+b^2+c^2 = 4\Delta\sum_{cyclic}\tan\frac{\pi-A}{4} + \sum_{cyclic}(a-b)^2 + 2\sum_{cyclic}\left(\sqrt{a(s-a)} - \sqrt{b(s-b)}\right)^2.$$

Using Jensen's inequality we have $\sum_{cyclic}\tan\frac{\pi-A}{4} \geq \sqrt{3}$, which implies the inequality of the statement. □

3 The Ionescu-Weitzenböck inequality in an Euclidean vector space

Let X be an Euclidean vector space. The inner product $< \cdot, \cdot >$ induces an associated norm, given by $||x|| = \sqrt{< x, x >}$, for all $x \in X$, which is called the *Euclidean norm*, thus X is a normed vector space.

Lemma 3.1. *In an Euclidean vector space X, we have*

$$||b + \frac{1}{2}a|| \geq ||b - \frac{\langle a, b\rangle}{||a||^2}a||, \text{ for all } a, b \in X. \tag{3.1}$$

Proof. The inequality of statement is equivalently with $||b + \frac{1}{2}a||^2 \geq ||b - \frac{\langle a,b\rangle}{||a||^2}a||^2$, which implies

$$||b||^2 + \langle a, b\rangle + \frac{1}{4}||a||^2 \geq ||b||^2 - 2\frac{\langle a, b\rangle^2}{||a||^2} + \frac{\langle a, b\rangle^2}{||a||^2},$$

and so, it follows that $\left(\frac{\langle a,b\rangle}{||a||} + \frac{1}{2}||a||\right)^2 \geq 0$, for all $a, b \in X$. □

Remark 3.1. It ie easy to see that $||b - \frac{1}{2}a|| \geq ||b - \frac{\langle a,b\rangle}{||a||^2}a||$, for all $a, b \in X$.

Theorem 3.2. *In an Euclidean vector space X, we have*

$$||a||^2 + ||b||^2 + ||a+b||^2 \geq 2\sqrt{3}\sqrt{||a||^2||b||^2 - \langle a,b\rangle^2}, \text{ for all } a, b \in X. \tag{3.2}$$

Proof. From the parallelogram law, for every $a, b \in X$, we deduce the following equality:

$$2(||a+b||^2 + ||b||^2) = ||a+2b||^2 + ||a||^2,$$

which is equivalent to $2(||a+b||^2 + ||b||^2) - ||a||^2 = 4||b + \frac{1}{2}a||^2$, and hence

$$||b + \frac{1}{2}a||^2 = \frac{||a+b||^2 + ||b||^2}{2} - \frac{||a||^2}{4}. \tag{3.3}$$

Therefore, combining the relations (3.1) and (3.3), we obtain the following

$$||a||^2 + ||b||^2 + ||a+b||^2 = \frac{1}{2}\left[2(||a+b||^2 + ||b||^2) - ||a||^2\right] + \frac{3}{2}||a||^2 = 2||b + \frac{1}{2}a||^2 + \frac{3}{2}||a||^2$$
$$\geq 2\sqrt{3}||a||||b + \tfrac{1}{2}a|| \geq 2\sqrt{3}||a||||b - \tfrac{\langle a,b\rangle}{||a||^2}a|| = 2\sqrt{3}\sqrt{||a||^2||b||^2 - \langle a,b\rangle^2},$$

which proves the inequality of the statement. □

Corollary 3.3. *In an Euclidean vector space X, we have*

$$||a||^2 + ||b||^2 + ||a-b||^2 \geq 2\sqrt{3}\sqrt{||a||^2||b||^2 - \langle a,b\rangle^2}, \text{ for all } a, b \in X. \tag{3.4}$$

Proof. If we replace the vector b by the vector $-b$ in inequality (3.2), we deduce the inequality of the statement. □

Remark 3.2. Inequality (13) represents the Ionescu-Weitzenböck inequality in an Euclidean vector space X over the field of real numbers $\mathbb{R}$.

Remark 3.3. Let E_3 be the Euclidean punctual space [11]. If we take the vectors $a = \overrightarrow{BC}, b = \overrightarrow{AC}$, $c = \overrightarrow{AB}$ in inequality (3.4), then using the Lagrange identity, $||a||^2||b||^2 - \langle a,b\rangle^2 = ||a \times b||^2$, we obtain the following inequality:

$$||\overrightarrow{BC}||^2 + ||\overrightarrow{AC}||^2 + ||\overrightarrow{BA}||^2 \geq 2\sqrt{3}||\overrightarrow{BC} \times \overrightarrow{AC}|| = 4\sqrt{3}\Delta,$$

which is in fact Ionescu-Weitzenböck inequality, from relation (1.2).

4 Applications to Differential Geometry

If $r : I \subset \mathbb{R} \to \mathbb{R}^3, r(t) = (x(t), y(t), z(t))$, is a parametrized curve in the space, then the curvature is $K(t) = \frac{||\dot{r}(t)\times\ddot{r}(t)||}{||\dot{r}(t)||^3}$ and the torsion is $\tau(t) = \frac{\left(\dot{r}(t),\ddot{r}(t),\dddot{r}(t)\right)}{||\dot{r}(t)\times\ddot{r}(t)||^2}$. We choose the curves with the velocity $||\dot{r}(t)|| = 1$. Therefore, we obtain $K(t) = ||\dot{r}(t)\times\ddot{r}(t)||$ and $\tau(t)K^2(t) = (\dot{r}(t), \ddot{r}(t), \dddot{r}(t))$. We take $a = \dot{r}(t)$ and $b = \ddot{r}(t)$ in Ionescu-Weitzenbock's inequality and we deduce the inequality for the curvature:

$$1 + ||\ddot{r}(t)||^2 + ||\dot{r}(t) - \ddot{r}(t)||^2 \geq 2\sqrt{3}K(t).$$

From Ionescu-Weitzenbock's inequality, for $a \to a \times b$ and $b \to c$, we have the inequality

$$||a \times b||^2 + ||c||^2 + ||a \times b - c||^2 \geq 2\sqrt{3}\sqrt{||a \times b||^2||c||^2 - (a, b, c)^2}.$$

We take $a = \dot{r}(t), b = \ddot{r}(t)$ and $c = \dddot{r}(t)$ in Ionescu-Weitzenbock's inequality and we deduce the inequality for the curvature:

$$K^2(t) + ||\dddot{r}(t)||^2 + ||\dot{r}(t) \times \ddot{r}(t) - \dddot{r}(t)||^2 \geq 2\sqrt{3}\,|K(t)|\,\sqrt{||\dddot{r}(t)||^2 - [K(t)\tau(t)]^2}.$$

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