Higher order frame bundles

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Abstract. For a manifold M, the k'th order tangent bundle T^kM of M consists of all equivalence classes of curves in M which agree up to their accelerations of order k. It is proved that at the presence of linear connection on M, T^kM admits a vector bundle structure over M and every Riemannian metric on M can be lifted to a Riemannian metric on T^kM [22]. In this paper, we construct the principal bundle of orthogonal frames $\mathbb{O}^kM$ of the Riemannian vector bundle T^kM over M and we prove that it is the associated bundle to T^kM with respect to the identity representation of $\mathbb{O}(\mathbb{E}^k)$. Then, we develop a generalized principal bundle structure for $\mathbb{O}^\infty M$ associated with $T^\infty M = \varprojlim T^kM$ by a radical change of the notion of the classical bundle of orthogonal frames and replacing $\mathbb{O}(\mathbb{F})$ by a generalized Fréchet Lie group. Moreover, for $1 \leq k \leq \infty$, the relation between connections on T^kM and principal connections on $\mathbb{O}^kM$ and $\mathbb{GL}^kM$ is revealed. Furthermore, using the concept of higher order differential of a diffeomorphism (or an isometry), we classify higher order frame bundles up to isomorphism. Finally we will apply our results to two infinite dimensional Hilbert manifolds.

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1 Introduction

Higher order tangent bundle T^kM of a smooth, possibly infinite dimensional, manifold M consists of all equivalence classes of curves which agree up to their acceleration of order k. The bundle T^kM is a natural extension of the usual tangent bundle.
Geometry of higher order tangent bundles and prolongations have witnessed a wide interest due to the works of Dodson and Galanis [4, 5], Morimoto [19], Miron and Bucataru [18, 3], León [15] etc.

Existence of a natural vector bundle structure for T^kM over M, even for the case $k = 2$, is not as evident as in the case of tangent bundle [5, 6, 21]. The author in his previous work [22] proved that for $2 \leq k \leq \infty$, T^kM admits a vector bundle structure

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over M if and only if M is endowed with a linear connection. Moreover, applying this vector bundle structure one can lift a Riemannian metric g from M to a Riemannian metric G_k on T^kM.

On the other hand, it is known that a large number of geometric properties of a vector bundle can be carried out to the reacher framework of its bundle of orthogonal (or linear) frames. Motivated by the preceding facts, in this paper for a smooth Riemannian manifold M modeled on the Hilbert space $(\mathbb{E}, \langle , \rangle)$, we introduce its k-th order, $k < \infty$, orthogonal frame bundle $\mathbb{O}^kM$ by

$$\mathbb{O}^kM = \bigcup_{x\in M} \mathbb{O}(\mathbb{E}^k, T^k_xM),$$

where $\mathbb{O}(\mathbb{E}^k, T^k_xM)$ stands for the space of all continuous linear unitary maps from $(\mathbb{E}^k, \langle , \rangle_k)$ to $(T^k_xM, g_k(x))$. As a first step, we prove that $\mathbb{O}^kM$ is a principal bundle over M with the orthogonal group

$$\mathbb{O}(E^k) = \{g \in \mathcal{L}(\mathbb{E}^k); \langle gv, gw\rangle_k = \langle v, w\rangle_k \text{ for all } v, w \in \mathbb{E}^k\}$$

as its structure group. Moreover, we will see that $\mathbb{O}^kM$ is a subbundle of $\mathbb{GL}^kM$, where

$$\mathbb{GL}^kM = \bigcup_{x\in M} \mathbb{GL}(\mathbb{E}^k, T^k_xM)$$

and $\mathbb{GL}(\mathbb{E}^k, T^k_xM)$ is the space of all continuous and linear maps from $\mathbb{E}^k \longrightarrow T^k_xM$ with continuous inverse.

Afterward, considering the right action of $\mathbb{O}(\mathbb{E}^k)$ on $\mathbb{O}^kM \times \mathbb{E}^k$, we show that (π_k, T^kM, M) is the associated vector bundle with $\mathbb{O}^kM$.

In the sequel , for $k \in \mathbb{N}$, we briefly outline the relation between connections (and horizontal lift) on $(p_k, \mathbb{GL}^kM, M)$, $(p_k, \mathbb{O}^kM, M)$ and (π_k, T^kM, M) mainly due to [14] chapter VIII, [12] chapter II and [26].

Then, in the light of [23], we will try to classify the principal bundle structures on $(p_k, \mathbb{GL}^kM, M)$ and $(p_k, \mathbb{O}^kM, M)$.

However, in the case of $k = \infty$, even for a finite dimensional manifold M, $(\pi_\infty, T^\infty M, M)$ becomes a generalized vector bundle with fibres isomorphic to the non-Banach Fréchet space $\mathbb{F} = \mathbb{R}^\mathbb{N} = \varprojlim \mathbb{R}^k$ [22]. Due to the pathology of $\mathbb{O}(\mathbb{F})$, for the Fréchet space $\mathbb{F}$, the definition of $\mathbb{O}^\infty M$ is under question.

Nevertheless, we will overcome this obstacle by a radical change of the notion of the classical bundle of orthogonal frames and replacing $\mathbb{O}(\mathbb{F})$ by a generalized Fréchet Lie group in the sense of Galanis [9]. More precisely, following the formalism of [25], we will propose an auxiliary projective (inverse) system of principal bundles such that $\mathbb{O}^\infty M$ becomes the projective limit of this system.
In this way $\mathbb{O}^\infty M$ becomes a Fréchet principal bundle over M with the projective limit Lie group $\mathcal{O}(\mathbb{F})$ as its structure group.

Considering a suitable representation for $\mathcal{O}(\mathbb{F})$ we will prove that $T^\infty M$ is the associated vector bundle with the Fréchet principal bundle $\mathbb{O}^\infty M$.

Then, we present some of the results about connections and horizontal lift on $\mathbb{O}^\infty M$ and $\mathbb{GL}^\infty M$ mainly due to [10] and [25] and section 3.1. Moreover, we introduce the induced principal bundle morphisms on infinite order frame bundles in order to declare the isomorphism classes of these bundles.

Finally we propose two examples to support our theory. In fact we will apply our results to the Hilbert manifold of H^1 loops on a finite dimensional Riemannian manifold and an infinite dimensional symplectic group.

Throughout this paper all the manifolds and morphisms are assumed to be smooth but before the last section, a lesser degree of differentiability can be assumed.

2 Preliminaries

Let M be a manifold, possibly infinite dimensional, modeled on the Banach space $\mathbb{E}$. For any $x_0 \in M$ define

$$C_{x_0} := \{\gamma : (-\epsilon, \epsilon) \longrightarrow M \ ; \ \gamma(0) = x_0 \text{ and } \gamma \text{ is smooth } \}.$$

As a natural extension of the tangent bundle TM define the following equivalence relation. For $\gamma \in C_{x_0}$, set $\gamma^{(1)}(t) = \gamma'(t)$ and $\gamma^{(k)}(t) = {\gamma^{(k-1)}}'(t)$, where $k \in \mathbb{N}$ and $k \geq 2$. Two curves $\gamma_1, \gamma_2 \in C_{x_0}$ are said to be k-equivalent, denoted by $\gamma_1 \approx^k_{x_0} \gamma_2$, if and only if $\gamma_1^{(j)}(0) = \gamma_2^{(j)}(0)$ for all $1 \leq j \leq k$. Define $T^k_{x_0}M := C_{x_0}/ \approx^k_{x_0}$ and the **tangent bundle of order** k **or** k**-osculating bundle** of M to be $T^kM := \bigcup_{x\in M} T^k_x M$. Denote by $[\gamma, x_0]_k$ the representative of the equivalent class containing γ and define the canonical projections $\pi_k : T^kM \longrightarrow M$ which projects $[\gamma, x_0]_k$ onto x_0.

Let $\mathcal{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha\in I}$ be a C^∞ atlas for M. For any $\alpha \in I$ define

$$\begin{aligned} \phi^k_\alpha : {\pi_k}^{-1}(U_\alpha) \quad &\longrightarrow \quad \psi_\alpha(U_\alpha) \times \mathbb{E}^k \\ [\gamma, x_0]_k \quad &\longmapsto \quad \big((\phi_\alpha \circ \gamma)(0), (\phi_\alpha \circ \gamma)'(0), ..., \frac{1}{k!}(\phi_\alpha \circ \gamma)^{(k)}(0)\big). \end{aligned}$$

Proposition 2.1. *The family $\mathcal{B} = \{({\pi_k}^{-1}(U_\alpha), \phi^k_\alpha)\}_{\alpha\in I}$ declares a smooth fibre bundle (not generally a vector bundle) structure for T^kM over M [22].*

A connection map on T^kM is a vector bundle morphism

$$K = (\overset{1}{K}, \overset{2}{K} ..., \overset{k}{K}) : TT^kM \longrightarrow \Big(\oplus^k_{i=1} \pi_1, \oplus^k_{i=1} TM, \oplus^k_{i=1} M \Big)$$

for which locally on a chart $(U_\alpha, \phi^k_\alpha)$ there are maps $\overset{i}{M}_\alpha : U_\alpha \times \mathbb{E}^k \longrightarrow L(\mathbb{E}, \mathbb{E})$, $1 \leq i \leq k$, with $K|_{U_\alpha}(u; y, \eta_1, ..., \eta_k) = \bigoplus^k_{i=1} \Big(x, \eta_i + \overset{1}{M}_\alpha (u)\eta_{i-1} + \overset{2}{M}_\alpha (u)\eta_{i-2} + ... + \overset{i}{M}_\alpha (u)y\Big)$,for any $(u; y, \eta_1, ..., \eta_k) \in T_uT^kM$ (See also [2, 22]).

Keeping the formalism of [22] we state the following theorem.

Proposition 2.2. *Let ∇ be a linear connection on M. Then there exists an induced connection map on T^kM.*

Proof. See [22], Remark 2.8 or [23] Proposition 2.10 with $g = id_M$. □

For $k \geq 2$, the bundle structure defined in proposition 2.1 is quite far from being a vector bundle due to the complicated nonlinear transition functions [6, 22, 23]. However, according to [22] we have the following main theorem.

Theorem 2.3. *Let ∇ be a linear connection on M and K the induced connection map introduced in Proposition 2.2. The following trivializations define a vector bundle structure on $\pi_k : T^k M \longrightarrow M$ with the structure group $\mathbb{GL}(\mathbb{E}^k)$.*

$$\begin{aligned} \Phi^k_\alpha : \pi_k^{-1}(U_\alpha) &\longrightarrow \psi_\alpha(U_\alpha) \times \mathbb{E}^k \\ [\gamma, x]_k &\longmapsto (\gamma_\alpha(0), \gamma'_\alpha(0), z^2_\alpha([\gamma,x]_k), \dots, z^k_\alpha([\gamma,x]_k)), \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} z^2_\alpha([\gamma,x]_k) &= \frac{1}{2}\Big\{\frac{1}{1!}\gamma''_\alpha(0) + \overset{1}{M}_\alpha [\gamma_\alpha(0), \gamma'_\alpha(0)]\gamma'_\alpha(0)\Big\}, \\ &\vdots \\ z^k_\alpha([\gamma,x]_k) &= \frac{1}{k}\Big\{\frac{1}{(k-1)!}\gamma^{(k)}_\alpha(0) + \frac{1}{(k-2)!}\overset{1}{M}_\alpha [\gamma_\alpha(0), \gamma'_\alpha(0)]\gamma^{(k-1)}_\alpha(0) + \dots \\ &\quad + \overset{k-1}{M}_\alpha [\gamma_\alpha(0), \gamma'_\alpha(0), \dots, \frac{1}{(k-1)!}\gamma^{(k-1)}_\alpha(0)]\gamma'_\alpha(0)\Big\}. \end{aligned}$$

Moreover setting $\Phi^k_{\alpha\beta} = \Phi^k_\alpha \circ {\Phi^k_\beta}^{-1}$, $\phi_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ and $U_{\alpha\beta} = U_\alpha \cap U_\beta$, then the transition map $\Phi^k_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \mathbb{GL}(\mathbb{E}^k)$ is given by $\Phi^k_{\alpha\beta}(x)(\xi_1, \xi_2, \dots, \xi_k) = \Big(\phi_{\alpha\beta}(x), d\phi_{\alpha\beta}(x)\xi_1, ..., d\phi_{\alpha\beta}(x)\xi_k\Big)$.

The converse of the above theorem is also true i.e. if $\pi_k : T^k M \longrightarrow M$, $k \geq 2$, admits a vector bundle structure isomorphic to $\oplus_{i=1}^k TM$, then a linear connection on M can be defined [22].

At the presence of a linear connection on M, for any $i < k$,

$$\pi_{k,i} : T^k M \longrightarrow T^i M \quad ; \quad [\gamma, x]_k \longmapsto [\gamma, x]_i$$

also admits a vector bundle structure [22].

Definition 2.1. Let $(\mathbb{E}, \langle,\rangle)$ and $(\mathbb{E}', \langle,\rangle')$ be two Hilbert spaces and $h \in \mathcal{L}(\mathbb{E}, \mathbb{E}')$ (i.e. h is a linear and continuous map). The operator h is called an orthogonal transformation (or a Hilbert morphism [16]) if

$$\langle hv, hw\rangle' = \langle v, w\rangle \quad ; \quad v, w \in \mathbb{E}.$$

The space of all orthogonal transformations is denoted by $\mathbb{O}(\mathbb{E}, \mathbb{E}')$. If $\mathbb{E} = \mathbb{E}'$ then $\mathbb{O}(\mathbb{E}) := \mathbb{O}(\mathbb{E}, \mathbb{E})$ is a closed subgroup of

$$\mathbb{GL}(\mathbb{E}) = \{h \in \mathcal{L}(\mathbb{E}); h \text{ is invertible}\}.$$

Prolongation of the Riemannian structures from a (finite dimensional) manifold M to its higher order tangent bundles was introduced and studied by Miron and Atanasiu in [17]. However, applying the vector bundle structure of $\pi_k : T^k M \longrightarrow M$ for lifting a Riemannian metric from M to its higher order tangent bundles is proposed in [6] for $k = 2$ and in [22] for $k \in \mathbb{N}$.

Remark 2.2. Suppose that g is a Riemannian metric on M which on the chart (U_α, ϕ_α) is represented by g_α. Then, theorem 2.3 proposes the Riemannian metric G^k on T^kM, where

$$G^k_\alpha : \pi_k^{-1}(U_\alpha) \times \pi_k^{-1}(U_\alpha) \longrightarrow \mathbb{R}$$

maps $\Big([\gamma_1, x]_k, [\gamma_2, x]_k\Big)$ to $\sum_{i=1}^k g_\alpha(x)\Big(proj_i \circ \Phi^k_\alpha([\gamma_1, x]_k), proj_i \circ \Phi^k_\alpha([\gamma_2, x]_k)\Big)$ and $proj_i$, $1 \leq i \leq k$, stands for the projection to the $(i+1)$'th factor [22].

Moreover as a result of theorem 2.3 and theorem 3.1, chapter VII [16] we can assume without loss of generality that a system of local trivializations $\{(\Phi^k_\alpha, \pi_k^{-1}(U_\alpha))\}_{\alpha\in I}$ consists only orthogonal trivializations and the transition maps take values in the orthogonal (or Hilbert) group $\mathbb{O}(\mathbb{E}^k)$.

3 Higher order frame bundles

Let (M, g) be a Riemannian manifold and G^k be the induced Riemannian metric on T^kM. With the vector bundle (π_k, T^kM, M) in mind, set

$$\mathbb{O}^kM := \bigcup_{x\in M} \mathbb{O}(\mathbb{E}^k, T^k_xM),$$

where $\mathbb{O}(\mathbb{E}^k, T^k_xM)$ stands for the space of all continuous and linear orthogonal transformations from $(\mathbb{E}^k, \langle,\rangle)$ to $(T^k_xM, G^k(x))$.

Consider the triple $(p_k, \mathbb{O}^kM, M)$, where $p_k : \mathbb{O}^k(M) \longrightarrow M$ maps the **orthogonal k-frame** (or k-frame for abbreviation) $h : \mathbb{E}^k \longrightarrow T^k_xM$ onto x. In this section, we will study some geometric properties of this fibration.

With the orthogonal bundle atlas of remark 2.2 and following the formalism of [5] we have:

Theorem 3.1. *$(p_k, \mathbb{O}^kM, M)$ is a smooth principal $\mathbb{O}(\mathbb{E}^k)$-bundle over M.*

Proof. All we need is a $\mathbb{O}(\mathbb{E}^k)$-bundle structure on $(p_k, \mathbb{O}^kM, M)$ with a right action which acts on fibres by the right translation [14]. The right action of $\mathbb{O}(\mathbb{E}^k)$ on $\mathbb{O}^kM$, for any pair $(h, g) \in \mathbb{O}^k(M) \times \mathbb{O}(\mathbb{E}^k)$, is given by $h.g := h \circ g$. Moreover for any $\alpha \in I$ define

$$\Psi^k_\alpha : p_k^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{O}(\mathbb{E}^k) \ \ ; \ \ h \longmapsto (x, \Phi^k_{\alpha,x} \circ h), \tag{3.1}$$

where (U_α, ϕ_α) is a local chart for M and $\Phi^k_{\alpha,x}$ is the restriction of Φ^k_α (defined by (2.1)) to the fibre T^k_xM. Clearly Ψ^k_α is smooth and for any $g \in \mathbb{O}(\mathbb{E}^k)$, $\Phi^k_{\alpha,x}(h.g) = \Phi^k_{\alpha.x} \circ h \circ g = \Phi^k_{\alpha,x}(h) \circ g$ which means that the last component of (3.1) is $\mathbb{O}(\mathbb{E}^k)$-equivariant. For $(x, g) \in U_\alpha \times \mathbb{O}(\mathbb{E}^k)$ set $h = {\Phi^k_{\alpha,x}}^{-1} \circ g$. Then $\Psi^k_\alpha(h) = (x, g)$ i.e. Ψ^k_α is surjective. The injectivity of Ψ^k_α is a direct consequence of the invertibility of $\Phi^k_{\alpha,x}$ for any $x \in M$.

Finally if $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \varnothing$, then

$$\begin{aligned} \Psi^k_\alpha \circ {\Psi^k_\beta}^{-1} := \Psi^k_{\alpha\beta} : U_{\alpha\beta} \times \mathbb{O}(\mathbb{E}^k) \quad &\longrightarrow \quad U_{\alpha\beta} \times \mathbb{O}(\mathbb{E}^k) \\ (x, g) \quad &\longmapsto \quad (x, \Phi^k_{\alpha\beta}(x) \circ g) \end{aligned}$$

which clearly satisfies the cocycle condition. As a consequence $\{(\Psi_k^\alpha, p_k^{-1}(U_\alpha))\}_{\alpha\in I}$ is a bundle atlas which makes the triple $(p_k, \mathbb{O}^k M, M)$ into a principal $\mathbb{O}(\mathbb{E}^k)$-bundle. □

Remark 3.1. One can replace $\mathbb{O}(\mathbb{E}^k, T_x^k M)$ with $\mathbb{GL}(\mathbb{E}^k, T_x^k M)$ and prove that $\mathbb{O}^k M$ is a subbundle of the principal $\mathbb{GL}(\mathbb{E}^k)$-bundle

$$\mathbb{GL}^k M = \bigcup_{x\in M} \mathbb{GL}(\mathbb{E}^k, T_x^k M).$$

Remark 3.2. For $k > i$ and the vector bundle $(\pi_{k,i}, T^k M, T^i M)$, by repeating this procedure with appropriate modifications, one can show that

$$\mathbb{O}^{k,i} M := \bigcup_{x\in T^i M} \mathbb{O}(\mathbb{E}^{k-i}, T_x^{k,i} M)$$

is an $\mathbb{O}(\mathbb{E}^{k-i})$-principal bundle over $T^i M$, where $T_x^{k,i} M = \pi_{k,i}^{-1}(T_x^i M)$. This bundle will be denoted by $(p_{k,i}, \mathbb{O}^{k,i} M, T^i M)$.

Remark 3.3. Since the vector bundle structures proposed by theorem 2.3 or remark 2.2 depend crucially on the connection or the original Riemannian metric, then the bundle structures on $\mathbb{GL}^k M$ and $\mathbb{O}^k M$ change if we change the connection or metric. In section 3.2 we ask about the extent of this dependence.

Consider the right action of $\mathbb{O}(\mathbb{E}^k)$ on the product $\mathbb{O}^k M \times \mathbb{E}^k$ defined by

$$\Big(\mathbb{O}^k M \times \mathbb{E}^k\Big) \times \mathbb{O}(\mathbb{E}^k) \longrightarrow \mathbb{O}^k M \times \mathbb{E}^k \quad ; \quad ((h, v), g) \longmapsto (h \circ g, g^{-1} v).$$

Denote the quotient $\mathbb{O}^k M \times \mathbb{E}^k / \mathbb{O}(\mathbb{E}^k)$ by $\mathbb{O}^k M \times_{\mathbb{O}(\mathbb{E}^k)} \mathbb{E}^k$ equipped with the quotient topology. Note that the equivalence classes $[h, v]$ and $[\bar{h}, \bar{v}]$ are equal if and only if there exists a $g \in \mathbb{O}(\mathbb{E}^k)$ such that $h \circ g = \bar{h}$ and $g^{-1} v = \bar{v}$. The proof of the following theorem is a modified version of theorem 3.3 of [5].

Theorem 3.2. $\mathbb{O}^k M \times_{\mathbb{O}(\mathbb{E}^k)} \mathbb{E}^k$ *admits a vector bundle structure on* M *isomorphic to* $(\pi_k, T^k M, M)$.

Proof. Define $\tilde{\pi}_k : \mathbb{O}^k M \times_{\mathbb{O}(\mathbb{E}^k)} \mathbb{E}^k \longrightarrow M$ by $\tilde{\pi}_k([h, v]) = p_k(h) = x$, where h is a k-frame at x. Clearly $\tilde{\pi}_k$ is well defined. Our main task is to construct local (vector bundle) trivializations for this bundle. To this end, let (ϕ_a, U_α) be a local chart of M and $\Psi_\alpha^k : p_k^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{O}(\mathbb{E}^k)$ be the corresponding trivialization (3.1) for the principal bundle $(p_k, \mathbb{O}^k M, M)$. Define

$$\Theta_\alpha^k : \tilde{\pi}_k^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{E}^k \quad ; \quad [h, v] \longmapsto (x, \Phi_{\alpha,x}^k \circ h(v)).$$

If $[h \circ g, g^{-1} v]$ is another representative for the class $[h, v]$ then, $\Theta_\alpha^k([h \circ g, g^{-1} v]) = (x, \Phi_{\alpha,x}^k \circ h \circ g(g^{-1} v)) = (x, \Phi_{\alpha,x}^k \circ h(v))$ which means that Θ_α^k is well defined.

Θ_α^k is also surjective since for any $(x, v) \in U_\alpha \times \mathbb{E}^k$, we have $\Theta_\alpha^k([{\Phi_{\alpha,x}^k}^{-1}, v]) = (x, v)$. Moreover if $\Theta_\alpha^k([h, v]) = \Theta_\alpha^k([\bar{h}, \bar{v}])$, then h and $\bar{h}$ are k-frames at $p_k(h) = x$ and

$$\Phi_{\alpha,x}^k \circ h(v) = \Phi_{\alpha,x}^k \circ h(g \circ g^{-1})(v) = \Phi_{\alpha,x}^k \circ \bar{h}(\bar{v})$$

that is $h(v) = \bar{h}(\bar{v})$. By considering $g = h^{-1} \circ \bar{h}$ we see that $[h, v] = [h \circ g, g^{-1}v] = [\bar{h}, \bar{h}^{-1} \circ h(v)] = [\bar{h}, \bar{v}]$ i.e. Θ^k_α is also injective. If $U_{\alpha\beta} \neq \varnothing$, then the overlap map $\Theta^k_{\alpha\beta} := \Theta^k_\alpha \circ {\Theta^k_\beta}^{-1}$ is given by

$$\begin{aligned} \Theta^k_{\alpha\beta} : U_{\alpha\beta} \times \mathbb{E}^k &\longrightarrow U_{\alpha\beta} \times \mathbb{E}^k \\ (x, v) &\longmapsto (x, \Phi^k_{\alpha,x} \circ {\Phi^k_{\beta,x}}^{-1} v) = (x, \Phi^k_{\alpha\beta}(x)v). \end{aligned}$$

Consequently, $(\tilde{\pi}_k, \mathbb{O}^k M \times_{\mathbb{O}(\mathbb{E}^k)} \mathbb{E}^k, M)$ admits a vector bundle structure with the fibres isomorphic to $\mathbb{E}^k$ and

$$\{\Phi^k_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \mathbb{O}(\mathbb{E}^k);\ \alpha, \beta \in I \ \text{ and } \ U_{\alpha\beta} \neq \varnothing\}$$

as its transition functions.

The next step is to establish an isomorphism between $\mathbb{O}^k(M) \times_{\mathbb{O}(\mathbb{E}^k)} \mathbb{E}^k$ and T^kM. We claim that

$$\Upsilon : \mathbb{O}^k(M) \times_{\mathbb{O}(\mathbb{E}^k)} \mathbb{E}^k \longrightarrow T^kM \ \ ; \ \ [h, v] \longmapsto (x, h(v))$$

is a vector bundle isomorphism. Υ is well defined because for another representative $[h \circ g, g^{-1}v]$, $g \in \mathbb{O}(\mathbb{E}^k)$, of the class $[h, v]$ we have $\Upsilon([h \circ g, g^{-1}v]) = \big(x, h \circ g(g^{-1}(v))\big) = \Upsilon([h, v])$. Moreover Υ is injective since $\Upsilon([h, v]) = \Upsilon([\bar{h}, \bar{v}])$ implies that h and $\bar{h}$ are frames at the same point x and $h(v) = \bar{h}(\bar{v})$. Taking $g = h^{-1} \circ \bar{h} \in \mathbb{O}(\mathbb{E}^k)$ we get $[h, v] = [h \circ g, g^{-1}v] = [\bar{h}, \bar{v}]$ as desired. Υ is also surjective since, locally, $(x, v) \in T^kM$ is the image of $[{\Phi^k_{\alpha,x}}^{-1}, v]$ via Υ. Finally let $[h, v]$ be an arbitrary element of $\mathbb{O}^k(M) \times_{\mathbb{O}(\mathbb{E}^k)} \mathbb{E}^k$, $\tilde{\pi}_k([h, v]) = x$ and (ϕ_α, U_α) be a chart of M around x. Considering the trivializations $\big(\Theta^k_\alpha, {\tilde{\pi}_k}^{-1}(U_\alpha)\big)$ and $\big(\Phi^k_\alpha, {\pi_k}^{-1}(U_\alpha)\big)$ we observe that

$$\Phi^k_\alpha \circ \Upsilon \circ {\Theta^k_\alpha}^{-1} : U_\alpha \times \mathbb{E}^k \longrightarrow U_\alpha \times \mathbb{E}^k \ \ ; \ \ (x, v) \longmapsto (x, v)$$

which means that Υ and its inverse are differentiable vector bundle morphisms. □

Remark 3.4. In a similar way one can show that $\mathbb{GL}^k M \times_{\mathbb{GL}(\mathbb{E}^k)} \mathbb{E}^k \simeq T^kM$. Moreover for $k > i$, a similar result holds true for $\mathbb{O}^{k,i}M \times_{\mathbb{O}(\mathbb{E}^{k-i})} \mathbb{E}^{k-i}$ and $(\pi_{k,i}, T^kM, T^iM)$.

3.1 Connections on higher order frame bundles

In this section, for $k \in \mathbb{N}$, we briefly outline the relation between connections on $(p_k, \mathbb{GL}^k M, M)$, $(p_k, \mathbb{O}^k M, M)$ and (π_k, T^kM, M) mainly due to [14] chapter VIII, [12] chapter II and [26].

Let G be a Lie group and $p : P \longrightarrow M$ be a principal G-bundle . A principal connection Γ on $p : P \longrightarrow M$ is a smooth assignment of a subspace H_xP of T_xP, for each $x \in P$, such that
1. $T_xP = H_xP \oplus V_xP$, where $V_xP = ker(d_xp) : T_xP \longrightarrow T_{p(x)}M$,
2. $R_{g_*}H_x = H_{gx}$ for any $g \in G$.

Moreover, given a principal connection one can define the (Lie Algebra) $\mathfrak{g}$-valued 1-form ω on P which annihilates the horizontal distribution [12, 26].

Let V be a vector space and $\rho : G \longrightarrow \mathbb{GL}(V)$ be a representation of G on V. Then any principal connection on P gives rise to a connection on the associated vector bundle $P \times_G V$ with the horizontal distribution $H_{[x,v]} = \kappa_{v*}(H_x P)$ and $\kappa_v : P \longrightarrow P \times_G V;\ x \longmapsto [x, v]$.

In our case, since $(p_k, \mathbb{O}^k M, M)$ is a subbundle of $(p_k, \mathbb{GL}^k M, M)$, under certain conditions a principal connection Γ on $(p_k, \mathbb{GL}^k M, M)$ can be reduced to a principal connection Γ' on the bundle of orthogonal frames. More precisely Γ on $\mathbb{GL}^k M$ can be reduced to Γ' on $\mathbb{O}^k M$ if and only if the parallel translation with respect to Γ preserves the inner product on the fibres. In this case we say that Γ is a metric connection [12].

On the other hand, a connection Γ' on $\mathbb{O}^k M$ can be extended to a metric connection Γ on $\mathbb{GL}^k M$ by the right action. Indeed for any $x \in \mathbb{GL}^k M$ there exists $g \in \mathbb{GL}(\mathbb{E}^k)$ such that $y := xg^{-1} \in \mathbb{O}^k M$. Then we define $H_x \mathbb{GL}^k M := R_{g*}(H_y \mathbb{O}^k M)$.

We note that, for a metric connection Γ on $\mathbb{GL}^k M$, given a curve γ in M its horizontal lift started form an orthogonal k-frame remains in $\mathbb{O}^k M$ and it is the same as the horizontal lift with respect to the (reduced) connection Γ'.

More precisely let $x \in \mathbb{O}^k M$ (the initial condition) and $\gamma : (-\epsilon, \epsilon) \longrightarrow M$ be a differentiable curve. Then there exists a unique curve $\tilde{\gamma}$ in $\mathbb{GL}^k M$ such that $p_k \circ \tilde{\gamma} = \gamma$ and the tangent to $\tilde{\gamma}$ is always horizontal (For the local conditions see [12] and [14]). Since $\tilde{\gamma}(0)$ is an orthogonal frame and Γ is metric then $\tilde{\gamma}(t)$ remain in $\mathbb{O}^k M$ for all $t \in (-\epsilon, \epsilon)$.

Finally, a linear connection on $(\pi_k, T^k M, M)$ induces a principal connection on $\mathbb{GL}^k M$ as follows. Suppose that ∇ is a linear connection on π_k with the family of Christoffel symbols $\Gamma_\alpha : U_\alpha \longrightarrow \mathcal{L}(\mathbb{E}^k, \mathcal{L}(\mathbb{E}, \mathbb{E}^k))$, $\alpha \in I$. Then one can associate a unique connection Γ with the form $\omega \in \Lambda^1\big(L^k M, \mathcal{L}(\mathbb{E}^k, \mathbb{E}^k)\big)$ and the local connection forms

$$\Gamma_\alpha(x)(y, v) = [\phi_\alpha^{-1*} \omega_\alpha(x) y] v \tag{3.2}$$

for any $x \in \phi_a(U_\alpha)$, $y \in \mathbb{E}$ and $v \in \mathbb{E}^k$ (for more details see [26]; Corollary 2.3). If parallel translation with respect to Γ preserves the inner product, then the connection form ω takes its values in the Lie algebra $\mathfrak{o}(\mathbb{E}^k)$ of $\mathbb{O}(\mathbb{E}^k)$.

3.2 Higher order differentials and the induced principal bundle morphisms

In this section, in the light of [23], we will try to classify the principal bundle structures on $(p_k, \mathbb{GL}^k M, M)$ and $(p_k, \mathbb{O}^k M, M)$ up to isomorphism.

For a differentiable map $f : M \longrightarrow N$, the k'th order differential of f is defined by

$$T^k f : T^k M \longrightarrow T^k N \quad ; \quad [\gamma, x]_k \longmapsto [f \circ \gamma, f(x)]_k.$$

We have the following main theorem from [23].

Theorem 3.3. **i**. *If ∇_M and ∇_N are f-related, then $T^k f : T^k M \longrightarrow T^k N$ becomes a vector bundle morphism.*
ii. *Let M and N be two (Riemannian) manifolds. If f is a diffeomorphism (isometry), then $T^k f$ is an isomorphism between (isometry between Riemannian) vector bundles.*

We remind that a principal bundle morphism between (p, P, G, M) and (p', P', G', N) is a triple (f'', f', f), where $f'' : P \longrightarrow P'$ is a fibre preserving map above $f : M \longrightarrow N$, that is, $p' \circ f'' = f \circ p$. Furthermore, for any $g \in G$ and $x \in P$, $f''(xg) = f''(x)f'(g)$ [12, 14].

Now, suppose that $f : M \longrightarrow N$ is a diffeomorphism and ∇_M and ∇_N be two f-related connections on M and N respectively. Since M and N are diffeomorphic, then we assume that the model space for M and N is $\mathbb{E}$. With the above facts in mind, we have the following proposition.

Proposition 3.4. **i.** *The vector bundle morphism $T^k f$ induces the principal bundle isomorphism $(\mathbb{T}^k f, id_{\mathbb{GL}(\mathbb{E}^k)}, f)$ where*

$$\mathbb{T}^k f : \mathbb{GL}^k M \longrightarrow \mathbb{GL}^k N \quad ; \quad b \longmapsto T^k_x f \circ b$$

and $x = p_k(b)$.
ii. *If f is an isometry, then $(\mathbb{T}^k f, id_{\mathbb{O}(\mathbb{E}^k)}, f)$ given by $\mathbb{T}^k f(b) = T^k_x f \circ b, b \in \mathbb{O}^k M$ is a principal bundle isomorphism.*

Now, suppose that $M = N$. As a consequence of proposition 3.4, if we replace ∇_M with a g-related connection where g is a diffeomorphism (isometry), then the bundle structures on $\mathbb{GL}^k M$ (and $\mathbb{O}^k M$) remain isomorphic.

4 Infinite order frame bundles

In order to introduce $\mathbb{O}^\infty M$ we will consider it as an appropriate limit (projective or inverse limit) of the finite factors $\mathbb{O}^k M$. To make our exposition as self-contained as possible, we state some preliminaries about projective limit of sets, topological vector spaces, manifolds and bundles from [24], [8], [9], [10], [20] and [1].

Consider the family $\mathcal{M} = \{M_i, \varphi_{ji}\}_{i,j \in \mathbb{N}}$, where M_i, $i \in \mathbb{N}$, is a **manifold** modeled on the Banach space $\mathbb{E}_i$ and $\varphi_{ji} : M_j \longrightarrow M_i$, $j \geq i$, is a differentiable map. Moreover we need to
i) the model spaces $\{\mathbb{E}_i, \rho_{ji}\}_{i,j \in \mathbb{N}}$ form a projective system of vector spaces,
ii) for any $x = (x_i) \in M = \varprojlim M_i$ there exists a projective family of charts $\{(\phi_i, U_i)\}$ such that $x_i \in U_i \subseteq M_i$ and for $j \geq i$, $\rho_{ji} \circ \phi_j = \phi_i \circ \varphi_{ji}$.
In this case $M = \varprojlim M_i$ may be considered as a generalized Fréchet manifold modeled on the Fréchet space $\mathbb{E} = \varprojlim \mathbb{E}_i$. If in addition the manifolds $\{M_i\}_{i \in \mathbb{N}}$ have **Lie group** structures then $M := \varprojlim M_i$ is called a Projective Limit Banach (PLB) Lie group [9].

Note that in the case of non-Banach Fréchet spaces $\mathbb{E}$ and $\mathbb{F}$, $\mathcal{L}(\mathbb{E}, \mathbb{F})$ does not remain in the category of Fréchet spaces and $\mathbb{GL}(\mathbb{E})$ and hence $\mathbb{O}(\mathbb{E}^k)$ does not even admit a reasonable topological group structure (see also [23], Remark 4.1).

In what follows, the indices i, j are natural numbers with $j \geq i$ unless otherwise stated.

Let $\{\mathbb{E}_i, \rho_{ji}\}$ and $\{\mathbb{F}_i, \nu_{ji}\}$ be two projective systems of Hilbert spaces with the limits $\mathbf{E}$ and $\mathbf{F}$ respectively. For any $i \in \mathbb{N}$, set

$$\mathcal{H}_i(\mathbf{E}, \mathbf{F}) := \{(f_1, \ldots, f_i) \in \prod_{1 \leq k \leq i} \mathcal{L}(\mathbb{E}_k, \mathbb{F}_k) \ ; \nu_{kl} \circ f_k = f_l \circ \rho_{kl} \text{ for } i \geq k \geq l\},$$

and

$$\mathcal{O}_i(\mathbf{E},\mathbf{F}) := \{(f_1,\ldots,f_i) \in \prod_{1\le k\le i} \mathbb{O}(\mathbb{E}_k,\mathbb{F}_k) \; ; \nu_{kl}\circ f_k = f_l\circ\rho_{kl} \text{ for } i\ge k\ge l\}.$$

Notice that $\mathcal{H}_i(\mathbf{E},\mathbf{F})$ is a Banach space since it is a closed subspace of $\prod_{1\le k\le i}\mathcal{L}(\mathbb{E}_k,\mathbb{F}_k)$. Moreover $\mathcal{O}_i(\mathbf{E},\mathbf{F})$ is a Lie group because $\mathcal{O}_i(\mathbf{E},\mathbf{F}) = \mathcal{GL}_i(\mathbf{E},\mathbf{F})\cap\prod_{1\le k\le i}\mathbb{O}(\mathbb{E}_k,\mathbb{F}_k)$ and $\mathcal{O}_i(\mathbf{E},\mathbf{F})$ is closed in

$$\mathcal{GL}_i(\mathbf{E},\mathbf{F}) := \{(f_1,\ldots,f_i) \in \prod_{1\le k\le i} \mathbb{GL}(\mathbb{E}_k,\mathbb{F}_k) \text{ s.t } \nu_{kl}\circ f_k = f_l\circ\rho_{kl} \text{ for } i\ge k\ge l\}.$$

Furthermore it is easy to check that $\{\mathcal{H}_i(\mathbf{E},\mathbf{F}),h_{ji}\}$, $\{\mathcal{GL}_i(\mathbf{E},\mathbf{F}),h_{ji}\}$ and $\{\mathcal{O}_i(\mathbf{E},\mathbf{F}),h_{ji}\}$ are projective systems ([8]) with the connecting morphisms $h_{ji}(f_1,\ldots,f_j) = (f_1,\ldots,f_i)$.

For our purposes we will consider the Fréchet space $\mathcal{H}(\mathbf{E},\mathbf{F}) = \varprojlim \mathcal{H}_i(\mathbf{E},\mathbf{F})$ and the generalized Lie groups $\mathcal{GL}(\mathbf{E},\mathbf{F}) = \varprojlim \mathcal{GL}_i(\mathbf{E},\mathbf{F})$ and $\mathcal{O}(\mathbf{E},\mathbf{F}) = \varprojlim \mathcal{O}_i(\mathbf{E},\mathbf{F})$. In the case of $\mathbb{E}_i = \mathbb{F}_i$, $i\in\mathbb{N}$, we write $\mathcal{H}(\mathbf{E})$, $\mathcal{GL}(\mathbf{E})$ and $\mathcal{O}(\mathbf{E})$ rather than $\mathcal{H}(\mathbf{E},\mathbf{E})$, $\mathcal{GL}(\mathbf{E},\mathbf{E})$ and $\mathcal{O}(\mathbf{E},\mathbf{E})$, respectively.

Finally suppose we are given a sequence of Banach **vector bundles (principal bundles)** $\{\pi_i : E_i \longrightarrow M\}_{i\in\mathbb{N}}$. This system is called a projective system of vector (resp. principal) bundles if (**i**) the total spaces form a projective systems of Hilbert (resp. Banach) manifolds, (**ii**) the fibres form a projective system of vector spaces (Lie groups) and (**iii**) for any $x\in M$ the exists a projective system of trivializations $\{(\tau_i,U_i)\}_{i\in\mathbb{N}}$ of $\{(\pi_i,E_i,M)\}_{i\in\mathbb{N}}$ respectively ([8], [10] and [25]).

The following theorem is a modified version of theorem 3.4 from [22].

Theorem 4.1. *At the presence of a linear connection on the Hilbert manifold M, $T^\infty M = \varprojlim T^iM$, admits a generalized vector bundle structure over M with fibres isomorphic to $\mathbb{E}^\infty = \varprojlim \mathbb{E}^i$ and the structure group $\mathcal{O}(\mathbb{E}^\infty)$.*

4.1 The bundle $\mathbb{O}^\infty M$

In this section we focus on constructing a principal bundle, say $\mathbb{O}^\infty M$, over the base M (or $T^{i_0}M$) associated with $T^\infty M$. But if we argue as the ordinary case $\mathbb{O}^kM$, we encounter a principal bundle with $\mathbb{O}(\mathbb{E}^\infty)$ as its structure group which is not a Fréchet Lie group. We construct a generalized principal bundle $\mathbb{O}^\infty M$ associated with $T^\infty M$ by a radical change of the notion of the classical bundle of orthogonal frames and replacing $\mathbb{O}(\mathbb{E}^\infty)$ by the generalized Fréchet Lie group $\mathcal{O}(\mathbb{E}^\infty)$ (see also [9, 10, 25]).

In order to define the infinite order orthogonal frame bundle $\mathbb{O}^\infty M$ we consider an auxiliary projective system of Banach principal bundles $\big(\bar{p}_k,\bar{\mathbb{O}}^kM,M\big)$, $k\in\mathbb{N}$, where

$$\bar{\mathbb{O}}^kM = \bigcup_{x\in M}\mathcal{O}_k(\mathbb{E}^\infty,T^\infty_xM)$$

and $\bar{p}_k : \bar{\mathbb{O}}^kM \longrightarrow M$ maps $(h_1,\ldots,h_k)\in\mathcal{O}_k(\mathbb{E}^\infty,T^\infty_xM)$ onto x.

Lemma 4.2. *$(\bar{p}_k,\bar{\mathbb{O}}^kM,M)$ is a principal fibre bundle with the structure group $\mathcal{O}_k(\mathbb{E}^\infty)$.*

Proof. Let $x \in M$, (U_α, ϕ_α) be a chart around x in M and Φ_α^k, $k \in \mathbb{N}$, be the trivialization given by remark 2.2. The corresponding trivialization for $\bar{\mathbb{O}}^k M$ is defined by

$$\begin{array}{rcl} \bar{\Psi}_\alpha^k : \bar{p}_k^{-1}(U_\alpha) & \longrightarrow & U_\alpha \times \mathcal{O}_k(\mathbb{E}^\infty) \\ (h_1, \ldots, h_k) & \longmapsto & (x, \Phi_{\alpha,x}^1 \circ h_1, \ldots, \Phi_{\alpha,x}^k \circ h_k). \end{array} \tag{4.1}$$

Similar to the proof of theorem 3.1 (and also [25]), one can deduce that $\bar{\Psi}_\alpha^k$ is a bijection and $\mathcal{O}_k(\mathbb{E}^\infty)$ has a right action on $\bar{\mathbb{O}}^k M$. Moreover the transition map, for suitable $\alpha, \beta \in I$, is given by

$$\begin{array}{rcl} \bar{\Psi}_\alpha^k \circ \bar{\Psi}_\beta^{k^{-1}} := \bar{\Psi}_{\alpha\beta}^k : U_{\alpha\beta} \times \mathcal{O}_k(\mathbb{E}^\infty) & \longrightarrow & U_{\alpha\beta} \times \mathcal{O}_k(\mathbb{E}^\infty) \\ (x, g_1, \ldots, g_k) & \longmapsto & (x, \Phi_{\alpha\beta}^1(x) \circ g_1, \ldots, \Phi_{\alpha\beta}^k(x) \circ g_k) \end{array}$$

which clearly is a smooth map. □

We proceed by defining the connecting morphisms

$$r_{ji} : \bar{\mathbb{O}}^j(M) \longrightarrow \bar{\mathbb{O}}^i(M) \quad ; \quad (h_1, \ldots, h_j) \longmapsto (h_1, \ldots, h_i)$$

in order to introduce $\mathbb{O}^\infty M$.

Theorem 4.3. *$\{(\bar{p}_i, \bar{\mathbb{O}}^i M, M), h_{ji}, r_{ji}\}_{i,j \in \mathbb{N}}$ is a projective system of Banach principal bundles and the limit $(p_\infty, \mathbb{O}^\infty M, M)$ is a generalized principal bundle on M with the structural group $\mathcal{O}(\mathbb{E}^\infty)$.*

Proof. It is clear that for any $i \in \mathbb{N}$, $r_{ii} = id$ and for natural numbers $i \leq j \leq k$ we have $r_{ji} \circ r_{kj} = r_{ki}$ which means that $\mathbb{O}^\infty M := \varprojlim \bar{\mathbb{O}}^k M$ (as a set) exists. To tell whether $\mathbb{O}^\infty M$ is a generalized Fréchet principal bundle, it suffices to show that for any $x \in M$ and the chart (ϕ_α, U_α) around it in M, the trivializations $\{(\bar{p}_i^{-1}(U_\alpha), \bar{\Psi}_\alpha^i)\}_{i \in \mathbb{N}}$ form a projective system [25]. However for $j \geq i$

$$\begin{array}{rcl} (id_{U_\alpha} \times h_{ji}) \circ \bar{\Psi}_\alpha^j(h_1, \ldots, h_j) & = & (id_{U_\alpha} \times h_{ji})(x, \Phi_{\alpha,x}^1 \circ h_1, \ldots, \Phi_{\alpha,x}^j \circ h_j) \\ & = & (x, \Phi_{\alpha,x}^1 \circ h_1, \ldots, \Phi_{\alpha,x}^i \circ h_i) \\ & = & \bar{\Psi}_\alpha^i \circ r_{ji}(h_1, \ldots, h_j). \end{array}$$

No we define a generalized principal bundle structure by setting $\{\big(\bar{p}_\infty^{-1}(U_\alpha), \bar{\Psi}_\alpha^\infty\big)\}_{\alpha \in I}$ as a bundle atlas for $\mathbb{O}^\infty M$, where $\bar{\Psi}_\alpha^\infty := \varprojlim \bar{\Psi}_\alpha^i$ and $p_\infty := \varprojlim \bar{p}_i$. □

In what follows, we shall try to show that $\mathbb{O}^\infty M$ is a principal bundle associated with the vector bundle $\pi_\infty : T^\infty M \longrightarrow M$. To this end, we first consider the right action of $\mathcal{O}(\mathbb{E}^\infty)$ on $\mathbb{O}^\infty M \times \mathbb{E}^\infty$ given by

$$\Big(\big((h_i), (e_i)\big), (g_i)\Big)_{i \in \mathbb{N}} \longmapsto \big((h_i \circ g_i), (g_i^{-1} e_i)\big)_{i \in \mathbb{N}}.$$

This action induces the quotient $(\mathbb{O}^\infty M \times \mathbb{E}^\infty)/\mathcal{O}(\mathbb{E}^\infty) := \mathbb{O}^\infty M \times_{\mathcal{O}(\mathbb{E}^\infty)} \mathbb{E}^\infty$.

Theorem 4.4. *$\mathbb{O}^\infty M \times_{\mathcal{O}(\mathbb{E}^\infty)} \mathbb{E}^\infty$ admits a Fréchet vector bundle structure on M isomorphic to $(\pi_\infty, T^\infty M, M)$.*

Proof. Keeping in mind the index $i \in \mathbb{N}$ for all sequences, consider the projection $\bar{\pi} : \mathbb{O}^\infty M \times_{\mathbb{O}(\mathbb{E}^\infty)} \mathbb{E}^\infty \longrightarrow M$ mapping $[(h_i),(e_i)]$ onto $p_\infty\big((h_i)\big)$. We will propose a family of trivializations to endow the above-mentioned fibration with a vector bundle structure. For any $x \in M$ contained in the chart (ϕ_a, U_α) of M define

$$\begin{array}{rcl} \bar{\Theta}^\infty_\alpha : \bar{\pi}^{-1}(U_\alpha) & \longrightarrow & U_\alpha \times \mathbb{E}^\infty \\ [(h_i),(e_i)] & \longmapsto & \big(\bar{p}_\infty((h_i)), \Phi^\infty_{\alpha,x}((h_i e_i))\big), \end{array}$$

where $\Phi^\infty_{\alpha,x} = \varprojlim \Phi^i_{\alpha,x}$. It is easily checked that $\bar{\Theta}^\infty_\alpha$ is well defined and bijective (e.g. [25, 26]).

Let α and β be a pair of indices such that $U_\alpha \cap U_\beta \neq \varnothing$ and $\bar{\Theta}^\infty_\alpha$ and $\bar{\Theta}^\infty_\beta$ be the corresponding trivializations. Then,

$$\begin{array}{rcl} \bar{\Theta}^\infty_{\alpha\beta} := \bar{\Theta}^\infty_\alpha \circ {\bar{\Theta}^\infty_\beta}^{-1} : U_{\alpha\beta} \times \mathbb{E}^\infty & \longrightarrow & U_{\alpha\beta} \times \mathbb{E}^\infty \\ \big(x,(e_i)\big) & \longmapsto & \big(x, (\Phi^i_{\alpha\beta}(x)(e_i))\big) = \big(x, \Phi^\infty_{\alpha\beta}(x)((e_i))\big), \end{array}$$

where $\Phi^i_{\alpha\beta}$ are those which are defined in remark 2.2.

Finally, the map

$$\bar{\Upsilon} : \mathbb{O}^\infty(M) \times_{\mathcal{O}(\mathbb{E}^\infty)} \mathbb{E}^\infty \longrightarrow T^\infty M \quad ; \quad [(h_i)_{i\in\mathbb{N}}, (e_i)_{i\in\mathbb{N}}] \longmapsto (h_i e_i)_{i\in\mathbb{N}}$$

establishes the desired vector bundle isomorphism. □

Remark 4.1. The methods of theorems 4.3 and 4.4 can be adopted to prove that $\mathbb{O}^\infty M$ is a (generalized) principal bundle associated with the vector bundle $T^\infty M$ on $T^{i_0}M$, $i_0 \in \mathbb{N}$.

Remark 4.2. One can replace $\mathcal{O}(\mathbb{E}^\infty)$ with $\mathcal{GL}(\mathbb{E}^\infty)$ and prove the same results for $\mathbb{GL}^\infty M = \varprojlim \overline{\mathbb{GL}}^k M$, where

$$\overline{\mathbb{GL}}^k M = \bigcup_{x\in M} \mathcal{GL}_k(\mathbb{E}^\infty, T^\infty_x M).$$

In fact $(p_\infty, \mathbb{GL}^\infty M, M)$ is a (generalized) principal bundle with the structure group $\mathcal{GL}(\mathbb{E}^\infty)$.

4.2 Connections on $\mathbb{O}^\infty M$

The purpose of this section is to present some of the results about connections on $\mathbb{O}^\infty M$ and $\mathbb{GL}^\infty M$ mainly due to [10] and [25] and section 3.1.

Since the structure groups of $\mathbb{O}^\infty M$ and $\mathbb{GL}^\infty M$, that is $\mathcal{O}(\mathbb{E}^\infty)$ and $\mathcal{GL}(\mathbb{E}^\infty)$ respectively, are obtained as projective limits of Banach Lie groups, then we have:

Theorem 4.5. i. *Any connection on $(p_\infty, \mathbb{O}^\infty M, M)$ or $(p_\infty, \mathbb{GL}^\infty M, M)$ can be considered as the limit of a projective system of connections (Theorem 3.7, [10]).*
ii. *There is a bijective correspondence between linear connections $\nabla = \varprojlim \nabla_i$ on $(\pi_\infty, T^\infty M, M)$ and connection forms ω on $(p_\infty, \mathbb{GL}^\infty M,\ M)$ (Corollary 3.4, [25]).*

Moreover, since $\mathcal{O}(\mathbb{E}^\infty) = \varprojlim \mathcal{O}_i(\mathbb{E}^\infty)$ and $\mathcal{GL}(\mathbb{E}^\infty) = \varprojlim \mathcal{GL}_i(\mathbb{E}^\infty)$ are projective limit Lie groups, then according to [9], for the corresponding Lie algebras we have $\mathfrak{o}(\mathbb{E}^\infty) = \varprojlim \mathfrak{o}_i(\mathbb{E}^\infty)$ and $\mathfrak{gl}(\mathbb{E}^\infty) = \varprojlim \mathfrak{gl}_i(\mathbb{E}^\infty)$. Now, suppose that for any $i \in \mathbb{N}$, ω_i is the connection form a metric connection on $(\bar{p}_i, \overline{\mathbb{GL}}_i M, M)$ with the limit $\omega = \varprojlim \omega_i$. Then, ω_i can be reduced to a connection on $(\bar{p}_i, \overline{\mathbb{O}}_i M, M)$ and the limit $\omega = \varprojlim \omega_i$ is a connection form on $\mathcal{O}^\infty M$ with values in $\mathfrak{o}(\mathbb{E}^\infty)$.

As a consequence of theorem 3.8 of [10], for any connection on $(p_\infty, \mathbb{O}^\infty M, M)$ or $(p_\infty, \mathbb{GL}^\infty M, M)$, and a given initial condition, the horizontal lift of $\gamma : (-\epsilon, \epsilon) \longrightarrow M$ exists and it is unique. This curve is of the form $\gamma = \varprojlim \gamma_i$, where for any $i \in \mathbb{N}$, γ_i is the horizontal lift of γ with respect to ω_i. As a consequence, the horizontal lift of γ with respect to any connection on $\mathcal{O}(\mathbb{E}^\infty)$ (or any metric connection on $\mathcal{GL}(\mathbb{E}^\infty)$) exists and for a given initial condition it is unique. Finally, any connection on $\mathcal{O}(\mathbb{E}^\infty)$ can be extended to a connection on $\mathcal{GL}(\mathbb{E}^\infty)$ in the obvious way.

4.3 Induced morphisms on infinite order frame bundles

Let $g : M \longrightarrow N$ be a diffeomorphism (isometry). As we have seen in section 3.2, at the presence of g-related connections on M and N for any $k \in \mathbb{N}$, $(\mathbb{T}^k g, id_{\mathbb{GL}(\mathbb{E}^k)}, g)$ (respectively $(\mathbb{T}^k g, id_{\mathbb{O}(\mathbb{E}^k)}, g)$) becomes a principal bundle morphism.

According to [23], theorem 4.2 and example 5.1,

Proposition 4.6. *The pair $(T^\infty g, g) : (T^\infty M, \pi_M^\infty, M) \longrightarrow (T^\infty N, \pi_N^\infty, N)$ is a generalized vector bundle isomorphism. Moreover, ∇_M and ∇_N are Riemannian connections on M and N respectively and g is an isometry, then $(T^\infty g, g)$ is a projective limit of vector bundle isometries.*

Now, we take one step further and we observe that $(\mathbb{T}^\infty g, id_{\mathbb{GL}(\mathbb{E}^k)}, g)$ and $(\mathbb{T}^\infty g, id_{\mathbb{GL}(\mathbb{E}^k)}, g)$ are generalized principal bundle morphisms. More precisely

$$\mathbb{T}^\infty g : \mathbb{GL}^\infty M \longrightarrow \mathbb{GL}^\infty N \quad ; \quad (h_i)_{i\in\mathbb{N}} \longmapsto (h_i \circ T_x^i g)_{i\in\mathbb{N}}$$

$x = p_i(h_i)$, is a limit map and $\mathbb{T}^\infty g = \varprojlim \mathbb{T}^i g$. In the same way we define $\mathbb{T}^\infty g : \mathbb{O}^\infty M \longrightarrow \mathbb{O}^\infty N$.

Now, suppose that $M = N$. As a consequence of the last proposition, if we replace ∇_M with a g-related connection, where g is a diffeomorphism (an isometry), then the bundle structures on $\mathbb{GL}^\infty M$ (and $\mathbb{O}^\infty M$) remain invariant.

5 Examples

In this section we propose two examples to support our theory. First we apply our results to the Hilbert manifold of H^1 loops on a finite dimensional Riemannian manifold. Then we describe another infinite dimensional example by employing the infinite dimensional symplectic group $Sp_2(\mathcal{H})$.

Example 5.1. An H^1 loop $c : I = [0,1] \longrightarrow \mathbb{R}^n$ is an absolutely continuous curve for which $\dot{c}(t)$ exists almost everywhere and

$$\int_{[0,1]} \langle c(t), c(t)\rangle dt + \int_{[0,1]} \langle \dot{c}(t), \dot{c}(t)\rangle dt < \infty$$

and $c(0) = c(1)$. We recall that $c : I \longrightarrow \mathbb{R}^n$ is absolutely continuous if for any $\epsilon > 0$ there exists $\delta > 0$ such that $0 \leq t_0 < t_1 < \cdots < t_{2k+1} \leq 1$ and $\sum_{i=0}^{k} |t_{2i+1} - t_{2i}| < \delta$, imply $\sum_{i=0}^{k} |c(t_{2i+1}) - c(t_{2i})| < \epsilon$ ([13], p. 159).

Now suppose that (M, g) is an n-dimensional Riemannian manifold. A loop $c : I \longrightarrow M$ is called of class H^1 if, for any chart (U, ϕ) of M the mapping $\phi \circ c : I' \subseteq c^{-1}(U) \longrightarrow \mathbb{R}^n$ is H^1. Then $H^1(M)$, formed by the loops $c : I \longrightarrow M$ of class H^1, admits a Hilbert manifold structure modeled on the Hilbert space $\mathbb{H} := H^1(c^*TM)$ [7, 13].

The space of vector fields along c is isomorphic with the space of sections $\Gamma(c^*TM)$ of the pullback c^*TM and it is identified with the tangent space $T_cH^1(M)$. The scalar product on $T_cH^1(M)$ is given by

$$H^1(g)(u, v)_c := \int_I g_{c(t)}(u(t), v(t))dt + \int_I g_{c(t)}\big(\frac{D}{dt}u(t), \frac{D}{dt}v(t)\big)dt$$

for any $u, v \in T_cH^1(M)$. (For a detailed study about the geometry of $H^1(M)$ we refer to [13] and [7].) In this case $(H^1(M), H^1(g))$ becomes an infinite dimensional Riemannian manifold. Let $\nabla_{H^1(M)}$ be the Levi-Civita connection of $H^1(M)$. Then, according to theorem 2.3, $\pi_k : T^kH^1(M) \longrightarrow H^1(M)$ admits a Riemannian vector bundle structure and the induced Riemannian metric is given by remark 2.2.

As a consequence $(p_k, \mathbb{O}^kH^1(M), M)$ is an $\mathbb{O}(\mathbb{H}^k)$ principal fiber bundle associated with the vector bundle $(\pi_k, T^kH^1(M), M)$.

Moreover, according to section 3.1, the Levi-Civita connection on the vector bundle $\pi_k : T^kH^1(M) \longrightarrow H^1(M)$ induces a principal connection on $p_k : \mathbb{O}^kH^1(M) \longrightarrow H^1(M)$ which can be extended to a principal connection on $p_k : \mathbb{GL}^kH^1(M) \longrightarrow H^1(M)$.

The existence of a generalized principal bundle structure for $p_\infty : \mathbb{O}^\infty H^1(M) \longrightarrow H^1M$ and $p_\infty : \mathbb{GL}^\infty H^1(M) \longrightarrow H^1M$ is a result of theorem 4.3.

Now, suppose that (M, g) and (N, h) are two finite dimensional Riemannian manifolds and $f : M \longrightarrow N$ is an isometry. Then $T^kf : T^kM \longrightarrow T^kN$ is a vector bundle morphism over $f : M \longrightarrow N$ ([23], example 5.1). Moreover

$$H^1(f) : \big(H^1(M), H^1(g)\big) \longrightarrow \big(H^1(N), H^1(h)\big) \quad ; \quad c \longmapsto f \circ c$$

is also an isometry ([7], p. 97). Again example 5.1 of [23] implies that

$$(T^kH^1(f), H^1f) : \Big(\pi_k^M, T^kH^1(M), H^1(M)\Big) \longrightarrow \Big(\pi_k^N, T^kH^1(N), H^1(N)\Big)$$

is also a vector bundle isometry. As a consequence of section 3.2, the induced map $\Big(\mathbb{T}^kH^1(f), id_{\mathbb{O}(\mathbb{H}^k)}, H^1f\Big)$ is a principal bundle isomorphism from $\Big(\mathbb{O}^kH^1(M), \mathbb{O}(\mathbb{H}^k), H^1M\Big)$ to $\Big(\mathbb{O}^kH^1(N), \mathbb{O}(\mathbb{H}^k),\ H^1N\Big)$.

Note that we can rephrase the last result for the k'th order frame bundles $\mathbb{GL}^kH^1(M)$ and $\mathbb{GL}^kH^1(N)$.

Finally, as a result of section 4.3, the isometry $f : M \longrightarrow N$ induces the principal bundle isomorphism $\Big(\mathbb{T}^\infty H^1(f), id_{\mathcal{O}(\mathbb{H}^\infty)}, H^1f\Big)$ from $\Big(\mathbb{O}^\infty H^1(M), \mathcal{O}(\mathbb{H}^\infty), H^1M\Big)$ to $\Big(\mathbb{O}^\infty H^1(N), \mathcal{O}(\mathbb{H}^\infty),\ H^1N\Big)$.

Example 5.2. We begin this example with a detailed analysis of the restricted symplectic group $Sp_2(\mathcal{H})$ based on [11]. Let $(\mathcal{H}, \langle , \rangle)$ be an infinite dimensional real Hilbert space. Moreover suppose that J is a complex structure on $\mathcal{H}$ that is J is a linear isometry on $\mathcal{H}$ with $J^2 = -1$ and $J^* = -J$. The symplectic group $Sp(\mathcal{H})$ is defined by

$$Sp(\mathcal{H}) = \{g \in GL(\mathcal{H}) : g^* J g = J\}.$$

and its Lie algebra $\mathfrak{sp}(\mathcal{H})$ is the set of all bounded linear operators for which $xJ = -Jx^*$. Denote by $\mathcal{B}_2(\mathcal{H})$ the Hilbert-Schmidt class $B_2(\mathcal{H}) = \{g \in \mathcal{B}(\mathcal{H}) : Tr(g^* g) < \infty\}$, where Tr is the usual trace and $\mathcal{B}(\mathcal{H})$ is the set of all bounded linear operators on $\mathcal{H}$. Define the restricted symplectic group to be

$$Sp_2(\mathcal{H}) = \{g \in Sp(\mathcal{H}) : g - 1 \in B_2(\mathcal{H})\}$$

Then the Lie algebra of $Sp_2(\mathcal{H})$ is $\mathfrak{sp}_2(\mathcal{H}) = \{x \in B_2(\mathcal{H}) : xJ = -Jx^*\}$ which is a closed subspaces of $B_2(\mathcal{H})$ and hence a Hilbert space [11]. Moreover, for any $g \in Sp_2(\mathcal{H})$,

$$(TSp_2(\mathcal{H}))_g = g\mathfrak{sp}_2(\mathcal{H}) \subset B_2(\mathcal{H})$$

is an inner product space endowed with the left invariant Riemannian metric

$$\langle v, w \rangle_g = \langle g^{-1}v, g^{-1}w \rangle = Tr((gg^*)^{-1} v w^*) \ ; \quad v, w \in T_g Sp_2(\mathcal{H}) \tag{5.1}$$

However, the Riemannian connection on $Sp_2(\mathcal{H})$ is given by the local form (Christoffel symbol)

$$2g^{-1}\Gamma_g(gx, gy) = xy + yx + x^*y + y^*x - xy^* - yx^*$$

for any $g \in Sp_2(\mathcal{H})$ and $x, y \in \mathfrak{sp}_2(\mathcal{H})$.

Now, theorems 3.1, 4.3, 3.2 and 4.4 guarantee that for any $k \in \mathbb{N} \cup \{\infty\}$, $p_k : \mathbb{O}^k Sp_2(\mathcal{H}) \longrightarrow Sp_2(\mathcal{H})$ is a principal bundle associated with the vector bundle $\pi_k : T^k Sp_2(\mathcal{H}) \longrightarrow Sp_2(\mathcal{H})$.

Finally, since the metric 5.1 is left invariant it follows that for any $f \in Sp_2(\mathcal{H})$ the mapping $\mathtt{f} : Sp_2(\mathcal{H}) \longrightarrow Sp_2(\mathcal{H})$; $g :\longmapsto f \circ g$ is an isometry of Hilbert Lie groups. Therefore, for any $k \in \mathbb{N} \cup \{\infty\}$, $T^k\mathtt{f} : T^k Sp_2(\mathcal{H}) \longrightarrow T^k Sp_2(\mathcal{H})$ is a vector bundle isomorphism and $\mathbb{T}^k\mathtt{f} : \mathbb{O}^k Sp_2(\mathcal{H}) \longrightarrow \mathbb{O}^k Sp_2(\mathcal{H})$ becomes a principal bundle isomorphism.

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