Real hypersurfaces in non-flat complex planes, in view of a contact metric condition

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Abstract. The aim of the present paper is to study real hypersurfaces in non-flat complex planes, for which the curvature satisfies $R(X,Y)Z = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) + \nu(\eta(Y)h\phi X - \eta(X)h\phi Y)$. Such manifolds are called (κ, μ, ν)-manifolds, and the relation is called (κ, μ, ν) condition. This condition has been studied for contact metric manifolds. In this work, we study it for real hypersurfaces M of the complex plane $M_2(c)$, since M always admits an almost contact metric structure - weaker than the contact metric one. One of the obtained results is that real hypersurfaces satisfying the (κ, μ, ν) condition do not admit a contact structure, even though they admit an almost contact structure. Classification results are given too, depending on the number of principal curvatures.

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1 Introduction

Contact metric manifolds have been studied by many points of view. D.E. Blair studied contact metric manifolds satisfying $R(X,Y)\xi = 0$ ([2]), where R denotes the Riemannian curvature tensor. Another type of (almost) contact manifolds, is the Sasakian one, which satisfies the condition $R(X,Y)\xi = \eta(Y)X - \eta(X)Y$. A generalization of both the $R(X,Y)\xi = 0$ and the Sasakian case, was introduced by Blair, Koufogiorgos and Papantoniou ([4]), with the condition $R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)$, where κ and μ are constants and $h = \frac{1}{2}L_\xi\phi$. These manifolds were called *(κ, μ)-manifolds*.

In 2000, Koufogiorgos and Tsichlias ([7]), considered the spaces called *generalized (κ, μ)-manifolds*; the same condition as in *(κ, μ)-manifolds* holds, but κ, μ are now functions. They showed that in dimension ≥ 5, κ and μ must be constants, while in dimension 3, they gave an example for which κ and μ are not constant. It should be mentioned that this idea is closely related to the idea of the characteristic vector

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field as a map into the tangent sphere bundle being a harmonic map. For further information on these manifolds and its applications, we refer to [3].

Following up on the above ideas, Koufogiorgos, Markellos and Papantoniou introduced the notion of a (κ,μ,ν)-manifold in [6], as a contact metric manifold whose curvature tensor satisfies

$$R(X,Y)Z = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) + \nu(\eta(Y)h\phi X - \eta(X)h\phi Y) \tag{1.1}$$

for some functions κ, μ and ν, and showed that for dimension > 3, such a manifold is a (κ,μ)-manifold. However, in dimension 3 they proved that a (κ,μ,ν)-manifold is an H-contact manifold ([3]) and conversely, a 3-dimensional H-contact manifold is a (κ,μ,ν)-manifold on an everywhere open dense set.

An n-dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called complex space form and is denoted by $M_n(c)$. A complete and simply connected complex space form is complex analytically isometric to a projective space $\mathbb{C}P^n$ if $c > 0$, a hyperbolic space $\mathbb{C}H^n$ if $c < 0$, or a Euclidean space $\mathbb{C}^n$ if $c = 0$. The induced almost contact metric structure of a real hypersurface M of $M_n(c)$ is denoted by (ϕ,ξ,η,g). The vector field ξ is defined by $\xi = -JN$ where J is the complex structure of $M_n(c)$ and N is a unit normal vector field.

Real hypersurface have been studied by many authors and from many points of view. An important class of hypersurfaces is *Hopf* hypersurfaces. Hopf hypersurfaces with constant principal curvatures have been classified in $\mathbb{C}P^n$. Any such hypersurface is an open subset of one of the following ([12]):

(A_1) Geodesic spheres.
(A_2) Tubes over totally geodesic complex projective spaces $\mathbb{C}P^k$, where $1 \le k \le n-2$.
(B) Tubes over complex quadrics and $\mathbb{R}P^n$.
(C) Tubes over the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^m$ where $2m+1 = n$ and $n \ge 5$.
(D) Tubes over the Plucker embedding of the complex Grassmann manifold $G_{2,5}$. (occur only for n = 9).
(E) Tubes over the canonical embedding of the Hermitian symmetric space SO(10)=U(5)(Occur only for n = 15).

The above list is often referred as "Takagi's list". In $\mathbb{C}H^n$, a Hopf hypersurface, all of whose principal curvatures are constant, is locally congruent to one of the following ([8]):
(A_0) The horosphere in $\mathbb{C}H^n$.
$(A_{1,0})$ A geodesic sphere of radius r $(0 < r < \infty)$.
$(A_{1,1})$ A tube of radius r around totally geodesic $\mathbb{C}H^{n-1}(c)$, where $0 < r < \infty$.
(A_2) A tube of radius r around totally geodesic $\mathbb{C}H^n(l)$, where $0 \le l \le n-2$.
(B) A tube of radius r around totally real totally geodesic$\mathbb{R}H^n(\frac{c}{4})$, where $0 < r < \infty$.

The above list can be found in [9]. The classification of these hypersurfaces was begun by S. Montiel in [10] (who also described the examples in detail) and completed by J. Berndt in [1].

In this paper, real hypersurfaces satisfying condition (1.1) are studied. In section 1 we introduce the notions and relations which will be our tools throughout the paper.

In section 2 auxiliary relations and lemmas are given. In Section 3, classification results and properties of these hypersurfaces are established. In addition, it is proved that such hypersurfaces, do not admit a contact structure, even though they admit an almost contact structure.

2 Preliminaries

Let M_n be a Kaehlerian manifold of real dimension $2n$, equipped with an almost complex structure J and a Hermitian metric tensor G. Then for any vector fields X and Y on $M_n(c)$, the following relations hold: $J^2X = -X$, $\quad G(JX, JY) = G(X, Y)$, $\widetilde{\nabla} J = 0$, where $\widetilde{\nabla}$ denotes the Riemannian connection of G of M_n.

Let M_{2n-1} be a real (2n-1)-dimensional hypersurface of $M_n(c)$, and denote by N a unit normal vector field on a neighborhood of a point in M_{2n-1} (from now on we shall write M instead of M_{2n-1}). For any vector field X tangent to M we have $JX = \phi X + \eta(X)N$, where ϕX is the tangent component of JX, $\eta(X)N$ is the normal component, and $\xi = -JN$, $\quad \eta(X) = g(X, \xi)$, $\quad g = G|_M$.

From the properties of the almost complex structure J and from the definitions of η and g, the following relations hold ([2]):

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta \circ \phi = 0, \qquad \phi\xi = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad g(X, \phi Y) = -g(\phi X, Y). \tag{2.2}$$

The above relations define an *almost contact metric structure* on M which is denoted by (ϕ, ξ, g, η). When an almost contact metric structure is defined on M, we can locally define a specific orthonormal basis $\{e_1, e_2, \dots e_{n-1}, \phi e_1, \phi e_2, \dots \phi e_{n-1}, \xi\}$, called a $\phi - basis$. We mention that the contact metric structure is similar to an almost contact one, with the additional condition $\eta \wedge (d\eta)^n \neq 0$. However we will not use this condition in our calculations, rather than make use of metric relations that only hold in a contact metric structure.

Furthermore, let A be the shape operator in the direction of N, and denote by ∇ the Riemannian connection of g on M. Then A is symmetric, and the following relations are satisfied:

$$i)\ \nabla_X \xi = \phi AX, \qquad ii)\ (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{2.3}$$

Since the ambient space $M_n(c)$ is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are respectively given by:

$$R(X, Y)Z = \frac{c}{4}[g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y$$
$$-2g(\phi X, Y)\phi Z] + g(AY, Z)AX - g(AX, Z)AY, \tag{2.4}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi]. \tag{2.5}$$

The tangent space T_pM, for every point $p \in M$, is decomposed as following: $T_pM =$

$\mathbb{D}^{\perp} \oplus \mathbb{D}$, where $\mathbb{D} = ker(\eta) = \{X \in T_pM : \eta(X) = 0\}$.

Based on the above decomposition, by virtue of (2.3), we decompose the vector field $A\xi$ in the following way:

$$A\xi = \alpha\xi + \beta U, \tag{2.6}$$

where $\beta = |\phi\nabla_\xi\xi|$, α is a smooth function on M and $U = -\frac{1}{\beta}\phi\nabla_\xi\xi \in ker(\eta)$, provided that $\beta \neq 0$. If the vector field $A\xi$ is expressed as $A\xi = \alpha\xi$, then ξ is called principal vector field.

The almost contact metric structure of a real hypersurface M is a contact one, if and only if

$$A\phi + \phi A = 2\phi \tag{2.7}$$

holds ([5]). In a 3-dimensional contact metric manifold we have

$$(\nabla_X\phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX). \tag{2.8}$$

Finally, for every vector field X, the tensor h is defined as

$$hX = \frac{1}{2}(L_\xi\phi) = \frac{1}{2}\big([\xi, \phi X] - \phi[\xi, X]\big). \tag{2.9}$$

The differentiation of X along a function f will be denoted by (Xf). All manifolds, vector fields, e.t.c., of this paper are assumed to be connected and of class C^∞.

3 Auxiliary Relations

Let $\mathcal{N} = \{p \in M : \beta \neq 0$ in a neighborhood around $p\}$. We define the open subsets $\mathcal{N}_1$ and $\mathcal{N}_2$ of $\mathcal{N}$ such that:
$\mathcal{N}_1 = \{p \in \mathcal{N} : \alpha \neq 0$ in a neighborhood around $p\}$,
$\mathcal{N}_2 = \{p \in \mathcal{N} : \alpha = 0$ in a neighborhood around $p\}$.
Then $\mathcal{N}_1 \cup \mathcal{N}_2$ is open and dense in the closure of $\mathcal{N}$.

Lemma 3.1. *Let M be a real hypersurface of a complex plane $M_2(c)$. Then the following relations hold on $\mathcal{N}_1$.*

$$AU = \Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\Big)U + \frac{\delta}{\alpha}\phi U + \beta\xi, \qquad A\phi U = \frac{\delta}{\alpha}U + (\frac{\epsilon}{\alpha} - \frac{c}{4\alpha})\phi U \tag{3.1}$$

$$\nabla_\xi\xi = \beta\phi U,\ \nabla_U\xi = -\frac{\delta}{\alpha}U + \Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\Big)\phi U, \tag{3.2}$$

$$\nabla_{\phi U}\xi = -(\frac{\epsilon}{\alpha} - \frac{c}{4\alpha})U + \frac{\delta}{\alpha}\phi U$$

(3.3) $$\nabla_\xi U=\kappa_1\phi U,\quad \nabla_U U=\kappa_2\phi U+\frac{\delta}{\alpha}\xi,\quad \nabla_{\phi U}U=\kappa_3\phi U+(\frac{\epsilon}{\alpha}-\frac{c}{4\alpha})\xi$$

(3.4) $$\nabla_\xi\phi U=-\kappa_1 U-\beta\xi,\quad \nabla_U\phi U=-\kappa_2 U-\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}+\frac{\beta^2}{\alpha}\Big)\xi,$$
$$\nabla_{\phi U}\phi U=-\kappa_3 U-\frac{\delta}{\alpha}\xi$$

where κ_1*,* κ_2*,* κ_3 *are smooth functions on* $\mathcal{N}_1$*.*

Proof.
From(1.4) we obtain

(3.5) $$lU=\frac{c}{4}U+\alpha AU-\beta A\xi,\qquad l\phi U=\frac{c}{4}\phi U+\alpha A\phi U.$$

The inner products of lU with U and ϕU yield respectively

(3.6) $$g(AU,U)=\frac{\gamma}{\alpha}-\frac{c}{4\alpha}+\frac{\beta^2}{\alpha},\qquad g(AU,\phi U)=\frac{\delta}{\alpha}$$

where $\gamma=g(lU,U)$ and $\delta=g(lU,\phi U)$.

So, (3.6) and $g(AU,\xi)=g(A\xi,U)=\beta$, yield the first of (3.1). Since l is symmetric with respect to metric g, the scalar products of the second of (3.5) with U and ϕU yield respectively

(3.7) $$g(A\phi U,U)=\frac{\delta}{\alpha},\qquad g(A\phi U,\phi U)=\frac{\epsilon}{\alpha}-\frac{c}{4\alpha},$$

where $\epsilon=g(l\phi U,\phi U)$. So, (3.7) and $g(A\phi U,\xi)=g(A\xi,\phi U)=0$, yield the second of (3.1). Combining (3.1) and (3.5), we obtain

(3.8) $$lU=\gamma U+\delta\phi U,\qquad l\phi U=\delta U+\epsilon\phi U.$$

By virtue of (2.6) and (3.1), (2.3.i) for $X=\xi$, $X=U$ and $X=\phi U$ yields (3.2).

It is well known that:

(3.9) $$Xg(Y,Z)=g(\nabla_X Y,Z)+g(Y,\nabla_X Z).$$

The relation (3.9) for $X=\xi$, $Y=Z=U$ and $X=Z=\xi$, $Y=U$, because of (3.2), implies respectively $g(\nabla_\xi U,U)=0=g(\nabla_\xi U,\xi)$. So if we put $g(\nabla_\xi U,\phi U)=\kappa_1$, we have the first of (3.3). Similarly (3.9) for $X=Y=Z=U$ and $X=Y=U$, $Z=\xi$, because of (3.2), yields respectively $g(\nabla_U U,U)=0$, $g(\nabla_U U,\xi)=\frac{\delta}{a}$. Therefore, putting $g(\nabla_U U,\phi U)=\kappa_2$, we have the second of (3.3). By the use of (3.2) and (3.9), we have that $g(\nabla_{\phi U}U,U)=0$ and $g(\nabla_{\phi U}U,\xi)=\frac{\epsilon}{\alpha}-\frac{c}{4\alpha}$. Then, if we set $g(\nabla_{\phi U}U,\phi U)=\kappa_3$, we get the third of (3.3). In a similar way using (3.9) we obtain (3.4). □ By virtue of Lemma 3.1 and (2.9) we obtain

(3.10) $$h\xi=0,\qquad hU=\frac{1}{2}(\frac{\epsilon}{\alpha}-\frac{\gamma}{\alpha}-\frac{\beta^2}{\alpha})U-\frac{\beta}{2}\xi\qquad h\phi U=(\frac{\gamma}{\alpha}-\frac{\epsilon}{\alpha}+\frac{\beta^2}{\alpha})\phi U.$$

Using (3.10) and the condition (1.1) we calculate $lU = R(U,\xi)\xi = [\kappa + \frac{\mu}{2}(\frac{\epsilon}{\alpha} - \frac{\gamma}{\alpha} - \frac{\beta^2}{\alpha})]U + \frac{\nu}{2}(\frac{\epsilon}{\alpha} - \frac{\gamma}{\alpha} - \frac{\beta^2}{\alpha})\phi U - \frac{\mu\beta}{2}\xi$. By comparing the last relation with the first of (3.5) and by virtue of (3.1), we gain

$$\kappa = \gamma, \qquad \nu(\frac{\epsilon}{\alpha} - \frac{\gamma}{\alpha} - \frac{\beta^2}{\alpha}) = \delta. \tag{3.11}$$

Similarly, from (1.1) and (3.10) we obtain $l\phi U = R(\phi U,\xi)\xi = [\kappa + \frac{\nu}{2}(\frac{\epsilon}{\alpha} - \frac{\gamma}{\alpha} - \frac{\beta^2}{\alpha})]\phi U$, which - with the aid of (3.1), (3.11) - is compared to (3.5), giving

$$\kappa = \epsilon, \qquad \delta = 0. \tag{3.12}$$

Finally, the calculation of $R(U,\phi U)\xi$ from (1.1) yields $R(U,\phi U)\xi = 0$. However, from Lemma 3.1 and equations (2.4), (3.11), (3.12), it results that $R(U,\phi U)\xi = \beta(\frac{\gamma}{\alpha} - \frac{c}{4\alpha})\phi U$. The two expressions of $R(U,\phi U)\xi$ with (2.11) and (2.12) lead to the following lemma:

Lemma 3.2. *Let M be a real hypersurface of a complex plane $M_2(c)$. Then the following relations hold on $\mathcal{N}_1$:*

$$\kappa = \gamma = \epsilon = \frac{c}{4}, \qquad \nu = \delta = 0.$$

We will now prove the following Lemma.

Lemma 3.3. *Let M be a real hypersurface of a complex plane $M_2(c)$, satisfying (0.1). The set $\mathcal{N}_1$ is the empty set: $\mathcal{N}_1 = \varnothing$.*

Proof.
Equation (2.5), for $X = U$, $Y = \xi$ yields $(\nabla_U A)\xi - (\nabla_\xi A)U = -\frac{c}{4}\phi U$, which is further developed with the aid of Lemmas 3.1, 3.2, giving

$$[(U\alpha) - (\xi\beta)]\xi + [(U\beta) - (\xi\frac{\beta^2}{\alpha})]U + (\kappa_2\beta - \frac{\kappa_1\beta^2}{\alpha} + \frac{c}{4})\phi U = 0.$$

The above relation, due to the linear independence of the vector fields U, ϕU and ξ, gives

$$(U\alpha) = (\xi\beta), \qquad (U\beta) = (\xi\frac{\beta^2}{\alpha}), \qquad \kappa_2\beta - \frac{\kappa_1\beta^2}{\alpha} + \frac{c}{4} = 0. \tag{3.13}$$

Similarly, equation (2.5) for $X = \phi U$, $Y = \xi$ yields $(\nabla_{\phi U}A)\xi - (\nabla_\xi A)\phi U = \frac{c}{4}U$, which is analyzed with the aid of Lemmas 3.1 and 3.2, resulting to

$$i)(\phi U\alpha) = \kappa_1\beta + \alpha\beta, \qquad ii)(\phi U\beta) = \kappa_1\frac{\beta^2}{\alpha} + \beta^2 - \frac{c}{4}, \qquad iii)\kappa_3 = 0. \tag{3.14}$$

In a similar way, (2.5) yields $(\nabla_U A)\phi U - (\nabla_{\phi U}A)U = -\frac{c}{2}\xi$, which is analyzed, by virtue of Lemmas 3.1, 3.2 and (3.14.*iii*), giving

$$i)\kappa_2\frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha} - \phi U(\frac{\beta^2}{\alpha}) = 0, \qquad ii)\kappa_2\beta + \beta^2 - (\phi U\beta) = -\frac{c}{2}. \tag{3.15}$$

Relation (3.15.i) is further analyzed giving $\kappa_2\beta + \beta^2 - 2(\phi U\beta) + \frac{\beta}{\alpha}(\phi U\alpha) = 0$. In the last equation, the term $\kappa_2\beta+\beta^2$ is replaced by (3.15.ii), and we take $\frac{\beta}{\alpha}(\phi U\alpha)-(\phi U\beta)-\frac{c}{2} = 0$. In the last relation, the terms $(\phi U\alpha)$, and $(\phi U\beta)$ are replaced respectively by (3.14.i) and (3.14.ii), giving $c = 0$ which is a contradiction on $\mathcal{N}_1$. Therefore $\mathcal{N}_1 = \varnothing$. □

4 Main results

Theorem 4.1. *A real hypersurface M of a complex plane $M_2(c)$, satisfying (1.1) is a Hopf hypersurface.*

Proof. From Lemma 3.3, we conclude that the set $\mathcal{N}$ coincides with $\mathcal{N}_2$, which means $\alpha = 0$ in $\mathcal{N}$. Equation (2.4) yields

$$i) lU = (\frac{c}{4} - \beta^2)U, \qquad ii) l\phi U = \frac{c}{4}\phi U. \tag{4.1}$$

Since the vector fields U, ϕU and ξ are linearly independent, from (2.6) and the symmetry of A, the following decompositions hold.

$$A\xi = \beta U, \quad AU = \alpha_1 U + \alpha_2\phi U, \quad A\phi U = \alpha_2 U + \alpha_3\phi U, \tag{4.2}$$

where α_1, α_1 and α_3 are functions. By virtue of (2.3) and (4.2), we obtain

$$\nabla_\xi\xi = \beta\phi U, \quad \nabla_U\xi = -\alpha_2 U + \alpha_1\phi U, \quad \nabla_{\phi U}\xi = -\alpha_3 U + \alpha_2\phi U. \tag{4.3}$$

Using (3.9) for $X = Z = \xi$, $Y = U$, and by making use of (4.3), we prove that $\nabla_\xi U \perp \xi$. Similarly, (4.3) and (3.9), for $X = \xi$, $Y = Z = U$, yield $\nabla_\xi U \perp U$. Therefore the vector field $\nabla_\xi U$ is decomposed as $\nabla_\xi U = \beta_1\phi U$, where β_1 is a function. By virtue of the last equation and (2.3.ii), (4.2), we obtain $\nabla_\xi\phi U = -\beta_1 U - \beta\xi$. Summing up, we have the following decompositions.

$$\nabla_\xi U = \beta_1\phi U, \qquad \nabla_\xi\phi U = -\beta_1 U - \beta\xi. \tag{4.4}$$

From (2.9), (4.3) and (4.4) we also have

$$hU = \frac{1}{2}\big((\alpha_3 - \alpha_1)U - 2\alpha_2\phi U - \beta\xi\big), \quad h\phi U = \frac{1}{2}\big((\alpha_1 - \alpha_3)\phi U - 2\alpha_2 U\big). \tag{4.5}$$

Condition (1.1), combined with (4.3) yields

$$lU = R(U,\xi)\xi = [\kappa + \frac{\mu}{2}(\alpha_3 - \alpha_1) + \nu\alpha_2]U + [-\mu\alpha_2 + \frac{\nu}{2}(\alpha_3 - \alpha_1)]\phi U - \frac{\mu\beta}{2}\xi.$$

Comparing the above relation with (4.1.i) we take

$$\mu = 0, \quad \nu(\alpha_3 - \alpha_1) = 0, \quad \frac{c}{4} - \beta^2 = \kappa + \nu\alpha_2. \tag{4.6}$$

The calculation of $l\phi U = R(\phi U,\xi)\xi$, from (1.1), (4.5) and (4.6), yields

$$l\phi U = \nu\alpha_2 U + \kappa\phi U.$$

The above equation with (4.1.ii) and (4.6), lead to $\beta = 0$, which is a contradiction on $\mathcal{N}$. Therefore we have $\mathcal{N} = \varnothing$ and $\beta = 0$ everywhere on M, that is M is a Hopf hypersurface. □

Since we have $A\xi = \alpha\xi$ and M is a 3-dimensional real hypersurface, we can define a ϕ-basis $\{e, \phi e, \xi\}$, which satisfies

$$Ae = \lambda_1 e, \quad A\phi e = \lambda_2 \phi e, \quad A\xi = \alpha\xi. \tag{4.7}$$

where $\lambda_1 = g(Ae, e)$ and $\lambda_2 = g(A\phi e, \phi e)$ are C^∞ functions and α is a constant ([11]). By virtue of (2.3.i) we calculate the following:

$$i)\nabla_\xi \xi = 0, \quad ii)\nabla_e \xi = \lambda_1 \phi e, \quad iii)\nabla_{\phi e}\xi = -\lambda_2 e. \tag{4.8}$$

Next, we make use of (3.9) for $X = \xi$, $Y = Z = e$ and prove $\nabla_\xi e \perp e$. Similarly, (3.9) for $X = Z = \xi$, $Y = e$, with the aid of (4.7.iii) and $\phi\xi = 0$ (due to (2.1)), yields $\nabla_\xi e \perp \xi$. Therefore, it must be $\nabla_\xi e = n_1 \phi e$, where n_1 is a function. In a similar way, from (3.9) we have $\nabla_e e \perp \{e, \xi\}$, which leads to $\nabla_e e = n_2 \phi e$, where n_2 is a function. Again from (3.9) we prove $\nabla_{\phi e} e = n_3 \phi e + \lambda_2 \xi$, where n_3 is a function. Summing up the equations of this paragraph, we have shown that

$$i)\nabla_\xi e = n_1 \phi e, \quad ii)\nabla_e e = n_2 \phi e, \quad iii)\nabla_{\phi e} e = n_3 \phi e + \lambda_2 \xi, n_3. \tag{4.9}$$

By virtue of (4.9) and (2.3.ii), (4.7) we take

$$i)\nabla_\xi \phi e = -n_1 e, \quad ii)\nabla_e \phi e = -n_2 - \lambda_1 \xi \phi e, \quad iii)\nabla_{\phi e}\phi e = -n_3 e. \tag{4.10}$$

From (2.8), (4.8.ii), (4.8.iii), (4.9.i), (4.9.ii) we acquire

$$he = \frac{1}{2}(\lambda_2 - \lambda_1)e, \quad h\phi e = -\frac{1}{2}(\lambda_2 - \lambda_1)\phi e. \tag{4.11}$$

By virtue of (2.4) and (4.7) we calculate

$$le = R(e, \xi)\xi = \frac{c}{4}e + \alpha\lambda_1 e, \quad l\phi e = R(e, \xi)\xi = \frac{c}{4}\phi e + \alpha\lambda_2 e. \tag{4.12}$$

However, from (1.1) and (4.11) we get

$$le = \big(\kappa + \frac{\mu}{2}(\lambda_2 - \lambda_1)\big)e + \frac{\nu}{2}(\lambda_2 - \lambda_1)\phi e, \tag{4.13}$$

$$l\phi e = \big(\kappa + \frac{\mu}{2}(\lambda_1 - \lambda_2)\big)\phi e - \frac{\nu}{2}(\lambda_1 - \lambda_2)e.$$

By comparing (4.12) with (4.13), we obtain

$$i)\kappa + \frac{\mu}{2}(\lambda_2 - \lambda_1) = \frac{c}{4} + \alpha\lambda_1, \ \ ii)\kappa + \frac{\mu}{2}(\lambda_1 - \lambda_2) = \frac{c}{4} + \alpha\lambda_2, \tag{4.14}$$

$$\nu(\lambda_1 - \lambda_2) = 0.$$

Equation (2.5) for $X = e$, $Y = \xi$ yields $(\nabla_e A)\xi - (\nabla_\xi A)e = -\frac{c}{4}\phi e$, which is developed with the help of (4.7), (4.8.ii) and (4.9.i), leading to

$$(\xi\lambda_1) = 0, \quad \alpha\lambda_1 - n_1(\lambda_1 - \lambda_2) - \lambda_1\lambda_2 = -\frac{c}{4}. \tag{4.15}$$

Similarly, (2.5) yields $(\nabla_{\phi e}A)\xi - (\nabla_{\xi}A)\phi e = \frac{c}{4}e$, which is developed with the help of (4.7), (4.8.*iii*) and (4.10.*i*), giving

$$(\xi\lambda_2) = 0, \quad \alpha\lambda_2 + n_1(\lambda_1 - \lambda_2) - \lambda_1\lambda_2 = -\frac{c}{4}. \tag{4.16}$$

Finally, we use (4.9.*iii*), (4.10.*ii*) to develop $(\nabla_{\phi e}A)\phi e - (\nabla_e A)\phi e = -\frac{c}{2}\xi$ (that holds due to (2.5)) and get

$$i)(e\lambda_2) = n_3(\lambda_1 - \lambda_2), \quad ii)(\phi e\lambda_1) = n_2(\lambda_1 - \lambda_2), \tag{4.17}$$
$$iii)\lambda_1\lambda_2 = \frac{\lambda_1 + \lambda_2}{2}\alpha + \frac{c}{4}.$$

Before proceeding with the proves of main results, we mention that the principal curvatures can not satisfy $\alpha = \lambda_1 = \lambda_2$ since, in this case, (4.17.*iii*) yields $c = 0$, which is a contradiction.

Proposition 4.2. *Let M be a real hypersurface of a complex plane $M_2(c)$, satisfying (1.1), with $\alpha \neq 0$. Then M has a principal curvature $\lambda = \lambda_1 = \lambda_2$, of multiplicity 2, if and only if M is one of the following: type A_1 in $\mathbb{C}P^2$, or types A_0, $A_{1,0}$, $A_{1,1}$, in a complex hyperbolic space.*

Let us assume there exists a point $p_1 \in M$ such that $\lambda_1 = \lambda_2 \neq \alpha \neq 0$ in a neighborhood around p_1. Then from (4.16) and (4.17) we have $(e\lambda) = (\phi e\lambda) = (\xi\lambda) = 0$, that is λ is a constant. Based on [12] and [9], the only spaces with a constant principal curvature $\lambda(\neq \alpha)$ are of type A_1 in $\mathbb{C}P^2$, or of types A_0, $A_{1,0}$, $A_{1,1}$ in a complex hyperbolic space. □

Proposition 4.3. *A (κ, μ, ν)-real hypersurface M of a complex plane $M_2(c)$ admits no contact structure.*

Proof. Let us assume that M admits a contact structure in a neighborhood around a point p. Then (2.7) yields $A\phi e + \phi Ae = 2\phi e$ which is combined with (4.7), giving

$$\lambda_1 + \lambda_2 = 2. \tag{4.18}$$

However (2.8) for $X = Y = e$, combined with (4.9.*ii*) and (4.11), gives

$$(\nabla_e\phi)e = \big(1 + \frac{1}{2}(\lambda_2 - \lambda_1)\big)\xi. \tag{4.19}$$

Moreover, (2.3.*ii*) for $X = Y = e$, with the aid of (4.7), yields $(\nabla_e\phi)e = -\lambda_1\xi$. The last equation and (4.19), lead to

$$\lambda_1 + \lambda_2 = -2. \tag{4.20}$$

From (4.18) and (4.20), we have a contradiction and so M does not admit a contact structure.

Lemma 4.4. *Let M be (κ, μ, ν)-real hypersurface M a complex plane $M_2(c)$ with $\alpha \neq 0$. If the principal curvatures satisfy locally $\lambda_1 \neq \lambda_2$, then $\nu = 0$, $\mu = -\alpha$ and $\lambda_1\lambda_2 = \kappa$.*

Proof. If we have locally $\lambda_1 \neq \lambda_2$, then (4.14.*iii*) gives $\nu = 0$. Moreover, subtracting (4.14.*i*) form (4.14.*ii*) we obtain $\mu = -\alpha$. Finally, adding (4.14.*i*) with (4.14.*ii*) we infer

$$\kappa = \frac{\lambda_1 + \lambda_2}{2}\alpha + \frac{c}{4}. \tag{4.21}$$

By comparing the last equation with (4.17.*ii*) we take $\lambda_1\lambda_2 = \kappa$.

Proposition 4.5. *Let M be (κ, μ, ν)-real hypersurface M a complex plane $M_2(c)$ with $\alpha \neq 0$. Then the following hold:*

- *If the principal curvatures satisfy $\alpha \neq \lambda_1 \neq \lambda_2 \neq \alpha$ then the sectional curvature c is negative.*
- *If the principal curvatures satisfy $\alpha = \lambda_1 \neq \lambda_2$, then M is of type B in $\mathbb{C}H^2$.*

Proof. Let as assume that the principal curvatures satisfy $\alpha \neq \lambda_1 \neq \lambda_2 \neq \alpha$. Then Lemma 4.4 and (4.17.*iii*) give $\lambda_1\lambda_2 = \kappa$ and $\lambda_1 + \lambda_2 = \frac{2}{\alpha}(\kappa - \frac{c}{4})$, which means that λ_1, λ_2 are roots of the quadric equation $X^2 + \frac{2}{\alpha}(\frac{c}{4} - \kappa)X + \kappa = 0$. Since it has discrete roots $\lambda_1 \neq \lambda_2$, the discriminant must be strictly positive. So we have $D = \frac{4}{\alpha^2}(\frac{c}{4} - \kappa)^2 - 4\kappa > 0$. The last inequality is rewritten as $D = \frac{4}{\alpha^2}\kappa^2 - (\frac{2c}{\alpha^2} + 4)\kappa + \frac{c^2}{4\alpha^2} > 0$. Therefore, the discriminant is a quadric equation $f(\kappa)$, which is always positive. So, the discriminant D_κ of $f(\kappa)$ must be negative: $D_\kappa = (\frac{2c}{\alpha^2} + 4)^2 - \frac{4c^2}{\alpha^4} < 0$. The last inequality is rewritten as $c < -\alpha^2$ and so $c < 0$.

Now, let as assume that the principal curvatures satisfy $\alpha = \lambda_1 \neq \lambda_2$. Then from (3.17.*iii*) we obtain $\lambda_2 = \frac{c}{2\alpha} + \alpha =$ constant. So λ_1 and λ_2 are constants.

In the case $M_n(c) = \mathbb{C}P^n$, according to [12], M can only be of type A_2 or B. If M is of type A_2, then $\alpha = \lambda_1 = 2cot2r$, $\lambda_2 = \frac{c}{2\alpha} + \alpha = cotr$. Combining the last two relations we obtain $r = 0$ which is a contradiction. If M is of type B, then from $\lambda_2 = \frac{c}{2\alpha} + \alpha = cot(r - \frac{\pi}{4})$, $\alpha = \lambda_1 = -tan(r - \frac{\pi}{4})$, we take $c = -2(1 + \alpha^2) < 0$, which is a contradiction in $\mathbb{C}P^n$.

In case $M_n(c) = \mathbb{C}H^2$, based on [9] M can only be of type B. □

Remark. We mention that a hypersurface of type B in $\mathbb{C}H^2$ with $\alpha = \lambda_1 \neq \lambda_2$ satisfying the following specific characteristics: $r = \frac{1}{\sqrt{|c|}}ln(2 + \sqrt{3})$, $\lambda_1 = \alpha = \frac{\sqrt{3|c|}}{2}$, $\lambda_2 = \frac{\sqrt{|c|}}{2\sqrt{3}}$.

Proposition 4.6. *Let M be a (κ, μ, ν)-real hypersurface of a complex plane $M_2(c)$ with $\alpha \neq 0$. If the principal curvatures satisfy $\alpha \neq \lambda_1 \neq \lambda_2 \neq \alpha$, then we have the following equivalence: the function κ is constant if and only if λ_1, λ_2 are constants and M is of type B in $\mathbb{C}H^2$.*

Proof. Let us assume that κ is a constant. Then we differentiate $\lambda_1\lambda_2 = \kappa$ (Lemma 3.4) along the vector fields e and ϕe, to obtain, respectively

$$(e\lambda_1)\lambda_2 + (e\lambda_2)\lambda_1 = 0, \quad (\phi e\lambda_1)\lambda_2 + (\phi e\lambda_2)\lambda_1 = 0. \tag{4.22}$$

We also differentiate (4.21) along the vector fields e and ϕe (κ is a constant) to obtain, respectively

$$(e\lambda_1) + (e\lambda_2) = 0, \quad (\phi e\lambda_1) + (\phi e\lambda_2) = 0. \tag{4.23}$$

Combining (4.22), with (4.23) and since $\lambda_1 \neq \lambda_2$, we get $(e\lambda_1) = (\phi e\lambda_1) = (e\lambda_2) = (\phi e\lambda_2) = 0$. We also have $(\xi\lambda_1) = (\xi\lambda_2) = 0$, from (4.16) and (4.17). So the principal curvatures λ_1 and λ_2 are constants. Moreover, from Proposition 4.5 we infer $M_2(c) = \mathbb{C}H^2$.

From [9] the only spaces with three distinct constant principal curvatures in $\mathbb{C}H^2$, are type A_2 and B. However, a real hypersurface M is of type A if and only if M satisfies $\phi A = A\phi$ on M ([11]). So in type A_2, we must have $\phi Ae = A\phi e \Rightarrow \lambda_1\phi e = \lambda_2\phi e \Rightarrow \lambda_1 = \lambda_2$, which is a contradiction. So M can only be of type B in $\mathbb{C}H^2$. □

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