On the Orlicz-Brunn-Minkowski theory

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Abstract. Recently, Gardner, Hug and Weil developed an Orlicz-Brunn-Minkowski theory. Following this, in the paper we further consider the Orlicz-Brunn-Minkowski theory. The fundamental notions of mixed quermassintegrals, mixed p-quermassintegrals and inequalities are extended to an Orlicz setting. Inequalities of Orlicz Minkowski and Brunn-Minkowski type for Orlicz mixed quermassintegrals are obtained. One of these has connections with the conjectured log-Brunn-Minkowski inequality and we prove a new log-Minkowski-type inequality. A new version of Orlicz Minkowski's inequality is proved. Finally, we show Simon's characterization of relative spheres for the Orlicz mixed quermassintegrals.

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1 Introduction

One of the most important operations in geometry is vector addition. As an operation between sets K and L, defined by

$$K + L = \{x + y \mid x \in K, y \in L\},$$

it is usually called Minkowski addition and combine volume play an important role in the Brunn-Minkowski theory. During the last few decades, the theory has been extended to L_p-Brunn-Minkowski theory. The first, a set called as L_p addition, introduced by Firey in [6] and [7]. Denoted by $+_p$, for $1 \le p \le \infty$, defined by

$$h(K +_p L, x)^p = h(K, x)^p + h(L, x)^p, \tag{1.1}$$

for all $x \in \mathbb{R}^n$ and compact convex sets K and L in $\mathbb{R}^n$ containing the origin. When $p = \infty$, (1.1) is interpreted as $h(K +_\infty L, x) = \max\{h(K, x), h(L, x)\}$, as is customary. Here the functions are the support functions. If K is a nonempty closed (not necessarily bounded) convex set in $\mathbb{R}^n$, then

$$h(K, x) = \max\{x \cdot y \mid y \in K\},$$

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for $x \in \mathbb{R}^n$, defines the support function $h(K, x)$ of K. A nonempty closed convex set is uniquely determined by its support function. L_p addition and inequalities are the fundamental and core content in the L_p Brunn-Minkowski theory. For recent important results and more information from this theory, we refer to [12], [13], [14], [15], [20], [22], [23], [24], [25], [26], [27], [30], [31], [35], [36], [37] and the references therein. In recent years, a new extension of L_p-Brunn-Minkowski theory is to Orlicz-Brunn-Minkowski theory, initiated by Lutwak, Yang, and Zhang [28] and [29]. In these papers the notions of L_p-centroid body and L_p-projection body were extended to an Orlicz setting. The Orlicz centroid inequality for star bodies was introduced in [39] which is an extension from convex to star bodies. The other articles advance the theory can be found in literatures [11], [17], [18] and [32]. Very recently, Gardner, Hug and Weil ([9]) constructed a general framework for the Orlicz-Brunn-Minkowski theory, and made clear for the first time the relation to Orlicz spaces and norms. They introduced the Orlicz addition $K +_\varphi L$ of compact convex sets K and L in $\mathbb{R}^n$ containing the origin, implicitly, by

$$\varphi\left(\frac{h(K,x)}{h(K +_\varphi L, x)}, \frac{h(L,x)}{h(K +_\varphi L, x)}\right) = 1, \tag{1.2}$$

for $x \in \mathbb{R}^n$, if $h(K,x)+h(L,x) > 0$, and by $h(K+_\varphi L, x) = 0$, if $h(K,x) = h(L,x) = 0$. Here $\varphi \in \Phi_2$, the set of convex functions $\varphi : [0,\infty)^2 \to [0,\infty)$ that are increasing in each variable and satisfy $\varphi(0,0) = 0$ and $\varphi(1,0) = \varphi(0,1) = 1$.

Unlike the L_p case, an Orlicz scalar multiplication cannot generally be considered separately. The particular instance of interest corresponds to using (1.2) with $\varphi(x_1, x_2) = \varphi_1(x_1) + \varepsilon\varphi_2(x_2)$ for $\varepsilon > 0$ and some $\varphi_1, \varphi_2 \in \Phi$, in which case we write $K +_{\varphi,\varepsilon} L$ instead of $K +_\varphi L$, where the sets of convex function $\varphi_i : [0,\infty) \to (0,\infty)$ that are increasing and satisfy $\varphi_i(1) = 1$ and $\varphi_i(0) = 0$, where $i = 1, 2$. Orlicz addition reduces to L_p addition, $1 \le p < \infty$, when $\varphi(x_1, x_2) = x_1^p + x_2^p$, or L_∞ addition, when $\varphi(x_1, x_2) = \max\{x_1, x_2\}$. Moreover, Gardner, Hug and Weil ([9]) introduced the Orlicz mixed volume, obtaining the equation

$$\frac{(\varphi_1)'_l(1)}{n} \lim_{\varepsilon\to 0^+} \frac{V(K +_{\varphi,\varepsilon} L) - V(K)}{\varepsilon} = \frac{1}{n}\int_{S^{n-1}} \varphi_2\left(\frac{h(L,u)}{h(K,u)}\right) h(K,u) dS(K,u), \tag{1.3}$$

where $S(K, u)$ is the mixed surface area measure of K and $\varphi \in \Phi_2$, $\varphi_1, \varphi_2 \in \Phi$. Here K is a convex body containing the origin in its interior and L is a compact convex set containing the origin, assumptions we shall retain for the remainder of this introduction.

Denoting by $V_\varphi(K, L)$, for any $\varphi \in \Phi$, the integral on the right side of (1.3) with φ_2 replaced by φ, we see that either side of the equation (1.3) is equal to $V_{\varphi_2}(K, L)$ and therefore this new Orlicz mixed volume plays the same role as $V_p(K, L)$ in the L_p-Brunn-Minkowski theory. In [9], Gardner, Hug and Weil obtained the Orlicz-Minkowksi inequality.

$$V_\varphi(K, L) \ge V(K) \cdot \varphi\left(\left(\frac{V(L)}{V(K)}\right)^{1/n}\right), \tag{1.4}$$

for $\varphi \in \Phi$. If φ is strictly convex, equality holds if and only if K and L are dilates or $L = \{o\}$.

In Section 3, we compute the Orlicz first variation of quermassintegrals, call as Orlicz mixed quermassintegrals, obtaining the equation
(1.5)

$$\frac{(\varphi_1)'_l(1)}{n-i}\lim_{\varepsilon\to 0^+}\frac{W_i(K+_{\varphi,\varepsilon}L)-W_i(K)}{\varepsilon}=\frac{1}{n}\int_{S^{n-1}}\varphi_2\left(\frac{h(L,u)}{h(K,u)}\right)h(K,u)dS_i(K,u).$$

for $\varphi\in\Phi_2$, $\varphi_1,\varphi_2\in\Phi$ and $1\le i\le n$, and W_i denotes the usual quermassintegrals, and $S_i(K,u)$ is the i-th mixed surface area measure of K. Denoting by $W_{\varphi,i}(K,L)$, for any $\varphi\in\Phi$, the integral on the right side of (1.5) with φ_2 replaced by φ, we see that either side of the equation (1.5) is equal to $W_{\varphi_2,i}(K,L)$ and therefore this new Orlicz mixed volume (Orlicz mixed quermassintegrals) plays the same role as $W_{p,i}(K,L)$ in the L_p-Brunn-Minkowski theory. Note that when $i=0$, (1.5) becomes (1.3). Hence we have the following definition of Orlicz mixed quermassintegrals.

$$W_{\varphi,i}(K,L)=\frac{1}{n}\int_{S^{n-1}}\varphi\left(\frac{h(L,u)}{h(K,u)}\right)h(K,u)dS_i(K,u). \tag{1.6}$$

In Section 4, we establish Orlicz-Minkowksi inequality for the Orlicz mixed quermassintegrals.

$$W_{\varphi,i}(K,L)\ge W_i(K)\cdot\varphi\left(\left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}\right), \tag{1.7}$$

for $\varphi\in\Phi$ and $0\le i<n$. If φ is strictly convex, equality holds if and only if K and L are dilates or $L=\{o\}$. Note that when $i=0$, (1.7) becomes to (1.4). In particularly, putting $\varphi(t)=t^p$, $1\le p<\infty$ in (1.7), (1.7) reduces to the following L_p-Minkowski inequality for mixed p-quermassintegrals established by Lutwak [21].

$$W_{p,i}(K,L)^{n-i}\ge W_i(K)^{n-i-p}W_i(L)^p, \tag{1.8}$$

for $p>1$ and $0\le i\le n$, with equality if and only if K and L are dilates or $L=\{o\}$. Putting $i=0$, $\varphi(t)=t^p$ and $1\le p<\infty$ in (1.7), (1.7) reduces to the well-known L_p-Minkowski inequality established by Firey [7]. For $p>1$,

$$V_p(K,L)\ge V(K)^{(n-p)/n}V(L)^{p/n}, \tag{1.9}$$

with equality if and only if K and L are dilates or $L=\{o\}$.

In Section 5, we establish the following Orlicz-Brunn-Minkowksi inequality for quermassintegrals of Orlicz addition.

$$1\ge\varphi\left(\left(\frac{W_i(K)}{W_i(K+_\varphi L)}\right)^{1/(n-i)},\left(\frac{W_i(L)}{W_i(K+_\varphi L)}\right)^{1/(n-i)}\right), \tag{1.10}$$

for $\varphi\in\Phi_2$ and $0\le i<n$. If φ is strictly convex, equality holds if and only if K and L are dilates or $L=\{o\}$. Note that when $\varphi(x_1,x_2)=x_1^p+x_2^p$, $1\le p<\infty$ in (1.11), (1.11) reduces to the following L_p-Brunn-Minkowski inequality for quermassintegrals established by Lutwak [21]. If

$$W_i(K+_pL)^{p/(n-i)}\ge W_i(K)^{p/(n-i)}+W_i(L)^{p/(n-i)}, \tag{1.11}$$

with equality if and only if K and L are dilates or $L = \{o\}$, and where $p \geq 1$ and $0 \leq i < n$. Putting $i = 0$, $\varphi(x_1, x_2) = x_1^p + x_2^p$ and $1 \leq p < \infty$ in (1.11), (1.11) reduces to the well-known L_p-Brunn-Minkowski inequality established by Firey [7].

$$V(K +_p L)^{p/n} \geq V(K)^{p/n} + V(L)^{p/n}, \tag{1.12}$$

with equality if and only if K and L are dilates or $L = \{o\}$, and where $p > 1$. A special case of (1.10) was recently established by Gardner, Hug and Weil [9].

$$1 \geq \varphi\left(\left(\frac{V(K)}{V(K +_{\varphi,\varepsilon} L)}\right)^{1/n}, \left(\frac{V(L)}{V(K +_{\varphi} L)}\right)^{1/n}\right), \tag{1.13}$$

for $\varphi \in \Phi_2$. If φ is strictly convex, equality holds if and only if K and L are dilates or $L = \{o\}$. When $i = 0$, (1.10) becomes to (1.12). Moreover, We prove also the Orlicz Minkowski inequality (1.4) and the Orlicz Brunn-Minkowski inequality (1.12) are equivalent, and (1.7) and (1.10) also are equivalent.

When we were about to submit our paper, we were informed that G. Xiong and D. Zou [38] had also obtained Orlicz Minowski and Brunn-Mingkowski inequalities for Orlicz mixed quermassintegrals. Please note that we use a completely different approach, although the two inequalities coincide with theirs.

In 2012, Böröczky, Lutwak, Yang, and Zhang [2] conjecture a log-Minkowski inequality for origin-symmetric convex bodies K and L in $\mathbb{R}^n$.

$$\int_{S^{n-1}} \log\left(\frac{h(L,u)}{h(K,u)}\right) h(K,u)dS(K,u) \geq V(K)\log\left(\frac{V(L)}{V(K)}\right). \tag{1.14}$$

In [2], (1.14) is proved by them only when $n = 2$. Very recently, Gardner, Hug and Weil [9] proved a new version of (1.14) for convex bodies, not origin-symmetric convex bodies.

$$\int_{S^{n-1}} \log\left(1 - \frac{h(L,u)}{h(K,u)}\right) h(K,u)dS(K,u) \leq V(K)\log\left(1 - \frac{V(L)^{1/n}}{V(K)^{1/n}}\right)^n, \tag{1.15}$$

with equality if and only if K and L are dilates or $L = \{o\}$, and where $L \subset \mathrm{int}K$. They also shown that combining (1.14) and (1.15) may get the classical Brunn-Minkowski inequality. In Section 6, we give a new log-Minkowski-type inequality

$$\int_{S^{n-1}} \log\left(1 - \frac{h(L,u)}{h(K,u)}\right) h(K,u)dS_i(K,u) \leq W_i(K)\log\left(1 - \frac{W_i(L)^{1/(n-i)}}{W_i(K)^{1/(n-i)}}\right)^n, \tag{1.16}$$

with equality if and only if K and L are dilates or $L = \{o\}$. When $i = 0$, (1.16) becomes (1.15). We also point out a conjecture which is an extension of the log Minkowski inequality as follows.

$$\frac{1}{n}\int_{S^{n-1}} \log\left(\frac{h(L,u)}{h(K,u)}\right) h(K,u)dS_i(K,u) \geq \log\left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}. \tag{1.17}$$

When $i = 0$, (1.17) becomes the log-Minkowski inequality (1.14). Combining (1.16) and (1.17) together split the following classical Brunn-Minkowski inequality for quermassintegrals (see Section 6).

$$W_i(K + L)^{1/(n-i)} \geq W_i(K)^{1/(n-i)} + W_i(L)^{1/(n-i)},$$

with equality if and only if K and L are dilates or $L=\{o\}$.

In 2010, the Orlicz projection body $\mathbf{\Pi}_\varphi$ of K (K is a convex body containing the origin in its interior) defined by Lutwak, Yang and Zhang [28]

$$h(\mathbf{\Pi}_\varphi,u)=\inf\left\{\lambda>0\mid \frac{1}{nV(K)}\int_{S^{n-1}}\varphi\left(\frac{|u\cdot v|}{\lambda h(K,v)}\right)h(K,v)dS(K,v)\le 1\right\}, \tag{1.18}$$

for $\varphi\in\Phi$ and $u\in S^{n-1}$. A different Orlicz version of Minkowski's inequality (1.8) is presented in Section 7. This results from replacing the left side of (1.8) by the quantity

$$\widehat{W}_{\varphi,i}(K,L)=\inf\left\{\lambda>0\mid \frac{1}{nW_i(K)}\int_{S^{n-1}}\varphi\left(\frac{h(L,u)}{\lambda h(K,u)}\right)h(K,u)dS_i(K,u)\le 1\right\}, \tag{1.19}$$

for $\varphi\in\Phi$ and $0\le i<n$. We prove the following new Orlicz Minkowski type inequality.

$$\widehat{W}_{\varphi,i}(K,L)\ge\left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}, \tag{1.20}$$

where $\varphi\in\Phi$ and $1\le i<n$. If φ is strictly convex and $W_i(L)>0$, equality holds if and only if K and L are dilates. A special version of (1.20) was recently established by Gardner, Hug and Weil [9].

$$\widehat{V}_\varphi(K,L)\ge\left(\frac{V(L)}{V(K)}\right)^{1/n},$$

If φ is strictly convex and $V(L)>0$, then equality holds if and only if K and L are dilates and where

$$\widehat{V}_\varphi(K,L)=\inf\left\{\lambda>0\mid \frac{1}{nV(K)}\int_{S^{n-1}}\varphi\left(\frac{h(L,u)}{\lambda h(K,u)}\right)h(K,u)dS(K,u)\le 1\right\},$$

for $\varphi\in\Phi$.

Finally, in Section 8, we show Simon's characterization of relative spheres for the Orlicz mixed quermassintegrals.

2 Notations and preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}^n$ be the class of nonempty compact convex subsets of $\mathbb{R}^n$, let $\mathcal{K}^n_o$ be the class of members of $\mathcal{K}^n$ containing the origin, and let $\mathcal{K}^n_{oo}$ be those sets in $\mathcal{K}^n$ containing the origin in their interiors. A set $K\in\mathcal{K}^n$ is called a convex body if its interior is nonempty. We reserve the letter $u\in S^{n-1}$ for unit vectors, and the letter B for the unit ball centered at the origin. The surface of B is S^{n-1}. For a compact set K, we write $V(K)$ for the (n-dimensional) Lebesgue measure of K and call this the volume of K. If K is a nonempty closed (not necessarily bounded) convex set, then

$$h(K,x)=\sup\{x\cdot y\mid y\in K\},$$

for $x \in \mathbb{R}^n$, defines the *support function* of K, where $x \cdot y$ denotes the usual inner product x and y in $\mathbb{R}^n$. A nonempty closed convex set is uniquely determined by its support function. Support function is homogeneous of degree 1, that is,

$$h(K, rx) = rh(K, x),$$

for all $x \in \mathbb{R}^n$ and $r \geq 0$. Let d denote the Hausdorff metric on $\mathcal{K}^n$, i.e., for $K, L \in \mathcal{K}^n$, $d(K, L) = |h(K, u) - h(L, u)|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$.

Throughout the paper, the standard orthonormal basis for $\mathbb{R}^n$ will be $\{e_1, \ldots, e_n\}$. Let Φ_n, $n \in \mathbb{N}$, denote the set of convex functions $\varphi : [0, \infty)^n \to [0, \infty)$ that are strictly increasing in each variable and satisfy $\varphi(0) = 0$ and $\varphi(e_j) = 1 > 0$, $j = 1, \ldots, n$. When $n = 1$, we shall write Φ instead of Φ_1. The left derivative and right derivative of a real-valued function f are denoted by $(f)'_l$ and $(f)'_r$, respectively.

2.1 Mixed quermassintegrals

If $K_i \in \mathcal{K}^n$ $(i = 1, 2, \ldots, r)$ and λ_i $(i = 1, 2, \ldots, r)$ are nonnegative real numbers, then of fundamental importance is the fact that the volume of $\sum_{i=1}^r \lambda_i K_i$ is a homogeneous polynomial in λ_i given by (see e.g. [3])

$$V(\lambda_1 K_1 + \cdots + \lambda_n K_n) = \sum_{i_1, \ldots, i_n} \lambda_{i_1} \ldots \lambda_{i_n} V_{i_1 \ldots i_n}, \tag{2.1}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding r. The coefficient $V_{i_1 \ldots i_n}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$ and is uniquely determined by (2.1), it is called the mixed volume of $K_i, \ldots, K_{i_n}$, and is written as $V(K_{i_1}, \ldots, K_{i_n})$. Let $K_1 = \ldots = K_{n-i} = K$ and $K_{n-i+1} = \ldots = K_n = L$, then the mixed volume $V(K_1, \ldots, K_n)$ is written as $V(K[n-i], L[i])$. If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = B$ The mixed volumes $V_i(K[n-i], B[i])$ is written as $W_i(K)$ and call as quermassintegrals (or i-th mixed quermassintegrals) of K. We write $W_i(K, L)$ for the mixed volume $V(K[n-i-1], B[i], L[1])$ and call as mixed quermassintegrals. Aleksandrov [1] and Fenchel and Jessen [5] (also see Busemann [4] and Schneider [33]) have shown that for $K \in \mathcal{K}^n_{oo}$, and $i = 0, 1, \ldots, n-1$, there exists a regular Borel measure $S_i(K, \cdot)$ on S^{n-1}, such that the mixed quermassintegrals $W_i(K, L)$ has the following representation:

$$W_i(K, L) = \frac{1}{n-i} \lim_{\varepsilon \to 0^+} \frac{W_i(K + \varepsilon L) - W_i(K)}{\varepsilon} = \frac{1}{n} \int_{S^{n-1}} h(L, u) dS_i(K, u). \tag{2.2}$$

Associated with $K_1, \ldots, K_n \in \mathcal{K}^n$ is a Borel measure $S(K_1, \ldots, K_{n-1}, \cdot)$ on S^{n-1}, called the mixed surface area measure of $K_1, \ldots, K_{n-1}$, which has the property that for each $K \in \mathcal{K}^n$ (see e.g. [8], p.353),

$$V(K_1, \ldots, K_{n-1}, K) = \frac{1}{n} \int_{S^{n-1}} h(K, u) dS(K_1, \ldots, K_{n-1}, u). \tag{2.3}$$

In fact, the measure $S(K_1, \ldots, K_{n-1}, \cdot)$ can be defined by the propter that (2.3) holds for all $K \in \mathcal{K}^n$. Let $K_1 = \ldots = K_{n-i-1} = K$ and $K_{n-i} = \ldots = K_{n-1} = L$, then the mixed surface area measure $S(K_1, \ldots, K_{n-1}, \cdot)$ is written as $S(K[n-i], L[i], \cdot)$.

When $L = B$, $S(K[n-i], L[i], \cdot)$ is written as $S_i(K, \cdot)$ and called as i-th mixed surface area measure. A fundamental inequality for mixed quermassintegrals stats that: For $K, L \in \mathcal{K}^n$ and $0 \le i < n-1$,

$$W_i(K, L)^{n-i} \ge W_i(K)^{n-i-1} W_i(L), \tag{2.4}$$

with equality if and only if K and L are homothetic and $L = \{o\}$. Good general references for this material are [4] and [19].

2.2 Mixed p-quermassintegrals

Mixed quermassintegrals are, of course, the first variation of the ordinary quermassintegrals, with respect to Minkowski addition. The mixed quermassintegrals $W_{p,0}(K, L), W_{p,1}(K, L), \ldots, W_{p,n-1}(K, L)$, as the first variation of the ordinary quermassintegrals, with respect to Firey addition: For $K, L \in \mathcal{K}^n_{oo}$, and real $p \ge 1$, defined by (see e.g. [21])

$$W_{p,i}(K, L) = \frac{p}{n-i} \lim_{\varepsilon \to 0^+} \frac{W_i(K +_p \varepsilon \cdot L) - W_i(K)}{\varepsilon}. \tag{2.5}$$

The mixed p-quermassintegrals $W_{p,i}(K, L)$, for all $K, L \in \mathcal{K}^n_{oo}$, has the following integral representation:

$$W_{p,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p dS_{p,i}(K, u), \tag{2.6}$$

where $S_{p,i}(K, \cdot)$ denotes the Boel measure on S^{n-1}. The measure $S_{p,i}(K, \cdot)$ is absolutely continuous with respect to $S_i(K, \cdot)$, and has Radon-Nikodym derivative

$$\frac{dS_{p,i}(K, \cdot)}{dS_i(K, \cdot)} = h(K, \cdot)^{1-p}, \tag{2.7}$$

where $S_i(K, \cdot)$ is a regular Boel measure on S^{n-1}. The measure $S^{n-1}(K, \cdot)$ is independent of the body K, and is just ordinary Lebesgue measure, S, on S^{n-1}. $S_i(B, \cdot)$ denotes the i-th surface area measure of the unit ball in $\mathbb{R}^n$. In fact, $S_i(B, \cdot) = S$ for all i. The surface area measure $S_0(K, \cdot)$ just is $S(K, \cdot)$. When $i = 0$, $S_{p,i}(K, \cdot)$ is written as $S_p(K, \cdot)$ (see [25], [26]). A fundamental inequality for mixed p-quermassintegrals stats that: For $K, L \in \mathcal{K}^n_{oo}, p > 1$ and $0 \le i < n-1$,

$$W_{p,i}(K, L)^{n-i} \ge W_i(K)^{n-i-p} W_i(L)^p, \tag{2.8}$$

with equality if and only if K and L are homothetic. L_p-Brunn-Minkowski inequality for quermassintegrals established by Lutwak [21]. If $K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$ and $p \ge 1$ and $0 \le i \le n$, then

$$W_i(K +_p L)^{p/(n-i)} \ge W_i(K)^{p/(n-i)} + W_i(L)^{p/(n-i)}, \tag{2.9}$$

with equality if and only if K and L are dilates or $L = \{o\}$. Obviously, putting $i = 0$ in (2.6), the mixed p-quermassintegrals $W_{p,i}(K, L)$ become the well-known L_p-mixed volume $V_p(K, L)$, defined by (see e.g. [25])

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p dS_p(K, u). \tag{2.10}$$

2.3 The Orlicz mixed volume

For $\varphi \in \Phi$, $K \in \mathcal{K}^n_{oo}$ and $L \in \mathcal{K}^n_o$, Gardner, Hug and Weil [9] defined the *Orlicz mixed volumes*, $V_\varphi(K, L)$ by

$$V_\varphi(K, L) = \frac{1}{n} \int_{S^{n-1}} \varphi\left(\frac{h(L, u)}{h(K, u)}\right) h(K, u) dS(K, u). \tag{2.11}$$

They obtained the Orlicz-Minkowksi inequality.

$$V_\varphi(K, L) \geq V(K) \cdot \varphi\left(\left(\frac{V(L)}{V(K)}\right)^{1/n}\right), \tag{2.12}$$

for all $K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$ and $\varphi \in \Phi$. If φ is strictly convex, equality holds if and only if K and L are dilates or $L = \{o\}$.

Orlicz mixed quermassintegrals is defined in Section 3, by

$$W_{\varphi,i}(K, L) =: \frac{1}{n} \int_{S^{n-1}} \varphi\left(\frac{h(L, u)}{h(K, u)}\right) h(K, u) dS_i(K, u), \tag{2.13}$$

for all $K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$, $\varphi \in \Phi$ and $0 \leq i < n$. Obviously, when $\varphi(t) = t^p$ and $p \geq 1$, *Orlicz mixed quermassintegrals* reduces to the mixed p-quermassintegrals $W_{p,i}(K, L)$ defined in (2.6). When $i = 0$, (2.13) reduces to (2.11).

2.4 Orlicz addition

Let $m \geq 2, \varphi \in \Phi_m$, $K_j \in \mathcal{K}^n_0$ and $j = 1, \ldots, m$, we define the *Orlicz addition* of $K_1, \ldots, K_m$, denoted by $+_\varphi(K_1, \ldots, K_m)$, is defined by

$$h(+_\varphi(K_1, \ldots, K_m), x) = \inf\left\{\lambda > 0 \mid \varphi\left(\frac{h(K_1, x)}{\lambda}, \ldots, \frac{h(K_m, x)}{\lambda}\right) \leq 1\right\}, \tag{2.14}$$

for $x \in \mathbb{R}^n$. Equivalently, the Orlicz addition $+_\varphi(K_1, \ldots, K_m)$ can be defined implicitly (and uniquely) by

$$\varphi\left(\frac{h(K_1, x)}{h(+_\varphi(K_1, \ldots, K_m), x)}, \ldots, \frac{h(K_m, x)}{h(+_\varphi(K_1, \ldots, K_m), x)}\right) = 1, \tag{2.15}$$

for all $x \in \mathbb{R}^n$. An important special case is obtained when

$$\varphi(x_1, \ldots, x_m) = \sum_{j=1}^m \varphi_j(x_j),$$

for some fixed $\varphi_j \in \Phi$ such that $\varphi_1(1) = \cdots = \varphi_m(1) = 1$. We then write $+_\varphi(K_1, \ldots, K_m) = K_1 +_\varphi \cdots +_\varphi K_m$. This means that $K_1 +_\varphi \cdots +_\varphi K_m$ is defined either by

$$h(K_1 +_\varphi \cdots +_\varphi K_m, u) = \sup\left\{\lambda > 0 \mid \sum_{j=1}^m \varphi_j\left(\frac{h(K_j, x)}{\lambda}\right) \leq 1\right\}, \tag{2.16}$$

for all $x \in \mathbb{R}^n$, or by the corresponding special case of (2.15).

For real $p \geq 1$, $K, L \in \mathcal{K}_{oo}^n$ and $\alpha, \beta \geq 0$ (not both zero), the Firey linear combination $\alpha \cdot K +_p \beta \cdot L \in \mathcal{K}_o^n$ can be defined by (see [6] and [7])

$$h(\alpha \cdot K +_p \beta \cdot L, \cdot)^p = \alpha h(K, \cdot)^p + \beta h(L, \cdot)^p.$$

Obviously, Firey and Minkowski scalar multiplications are related by $\alpha \cdot K = \alpha^{1/p} K$. In [9], Gardner, Hug and Weil define *the Orlicz linear combination* $+_\varphi(K, L, \alpha, \beta)$ for $K, L \in \mathcal{K}_o^n$ and $\alpha, \beta \geq 0$, defined by

$$\alpha\varphi_1\left(\frac{h(K,x)}{h(+_\varphi(K,L,\alpha,\beta),x)}\right) + \beta\varphi_2\left(\frac{h(L,x)}{h(+_\varphi(K,L,\alpha,\beta),x)}\right) = 1, \tag{2.17}$$

if $\alpha h(K,x) + \beta h(L,x) > 0$, and by $h(+_\varphi(K,L,\alpha,\beta),x) = 0$ if $\alpha h(K,x) + \beta h(L,x) = 0$, for all $x \in \mathbb{R}^n$. It is easy to verify that when $\varphi_1(t) = \varphi_2(t) = t^p, p \geq 1$, the Orlicz linear combination $+_\varphi(K, L, \alpha, \beta)$ equals the Firey combination $\alpha \cdot K +_p \beta \cdot L$. Henceforth we shall write $K +_{\varphi,\varepsilon} L$ instead of $+_\varphi(K, L, 1, \varepsilon)$, for $\varepsilon \geq 0$, and assume throughout that this is defined by (2.17), where $\alpha = 1, \beta = \varepsilon$, and $\varphi_1, \varphi_2 \in \Phi$.

3 Orlicz mixed quermassintegrals

In order to define a new concept: Orlicz mixed quermassintegrals, we need Lemmas 3.1-3.4 and Theorem 3.5.

Lemma 3.1. *([9]) If $\varphi \in \Phi_m$, then Orlicz addition $+_\varphi : (\mathcal{K}_0^n)^m \to \mathcal{K}_0^n$ is continuous, $GL(n)$ covariant, monotonic, projection covariant and has the identity property.*

Lemma 3.2. *([9]) If $K, L \in \mathcal{K}_o^n$, then*

$$K +_{\varphi,\varepsilon} L \to K, \tag{3.1}$$

in the Hausdorff metric as $\varepsilon \to 0^+$.

Lemma 3.3. *If $K, L \in \mathcal{K}_o^n$ and $0 \leq i < n$, Then*

$$\lim_{\varepsilon \to 0^+} \frac{W_i(K +_{\varphi,\varepsilon} L) - W_i(K)}{\varepsilon} = \frac{n-i}{n} \int_{S^{n-1}} \lim_{\varepsilon \to 0^+} \frac{h(K +_{\varphi,\varepsilon} L, u) - h(K,u)}{\varepsilon} dS_i(K,u), \tag{3.2}$$

where, $\lim_{\varepsilon \to 0^+} \frac{h(K +_{\varphi,\varepsilon} L, u) - h(K,u)}{\varepsilon}$ uniformly for $u \in S^{n-1}$.

Proof. For brevity, we temporarily write $K_\varepsilon = K +_{\varphi,\varepsilon} L$. Starting with the decomposition

$$\frac{W_i(K_\varepsilon) - W_i(K)}{\varepsilon} = \sum_{j=0}^{n-i-1} \frac{W_i(K_\varepsilon[j+1], K[n-i-j-1]) - W_i(K_\varepsilon[j], K[n-i-j])}{\varepsilon}.$$

Notice that

$$\frac{W_i(K_\varepsilon[j+1], K[n-i-j-1]) - W_i(K_\varepsilon[j], K[n-i-j])}{\varepsilon} \tag{3.3}$$

$$= \frac{1}{n}\int_{S^{n-1}} \frac{h(K_\varepsilon,u)-h(K,u)}{\varepsilon} dS_i(K_\varepsilon[j],K[n-i-j-1],u)$$
$$= \frac{1}{n}\int_{S^{n-1}} \left(\frac{h(K_\varepsilon,u)-h(K,u)}{\varepsilon} - \lim_{\varepsilon\to 0^+}\frac{h(K+_{\varphi,\varepsilon}L,u)-h(K,u)}{\varepsilon}\right)\times$$
$$\times dS_i(K_\varepsilon[j],K[n-i-j-1],u)$$
$$+\frac{1}{n}\int_{S^{n-1}} \lim_{\varepsilon\to 0^+}\frac{h(K+_{\varphi,\varepsilon}L,u)-h(K,u)}{\varepsilon} dS_i(K_\varepsilon[j],K[n-i-j-1],u).$$

By assumption, the integrand in (3.3) converges uniformly to zero for $u \in S^{n-1}$. Since $K_\varepsilon \to K$ as $\varepsilon \to 0^+$, by Lemma 3.2, and the i-th mixed surface area measures $S_i(K_\varepsilon[j],K[n-i-j-1])$ are uniformly bounded for $\varepsilon \in (0,1]$, the first integral in the previous sum converges to zero. Noting that $S_i(K_\varepsilon[j],K[n-i-j-1]) \to S_i(K,u)$ weakly as $\varepsilon \to 0^+$. Hence

$$\lim_{\varepsilon\to 0^+}\frac{W_i(K+_{\varphi,\varepsilon}L)-W_i(K)}{\varepsilon} = \lim_{\varepsilon\to 0^+}\sum_{j=0}^{n-i-1}\frac{1}{n}\int_{S^{n-1}}\lim_{\varepsilon\to 0^+}\frac{h(K+_{\varphi,\varepsilon}L,u)-h(K,u)}{\varepsilon}\times$$
$$\times\, dS_i(K_\varepsilon[j],K[n-i-j-1],u)$$
$$= \frac{n-i}{n}\int_{S^{n-1}}\lim_{\varepsilon\to 0^+}\frac{h(K+_{\varphi,\varepsilon}L,u)-h(K,u)}{\varepsilon} dS_i(K,u).$$

□

Lemma 3.4. *For $\varepsilon > 0$ and $u \in S^{n-1}$, let $h_\varepsilon = h(K+_{\varphi,\varepsilon}L,u)$. If $K \in \mathcal{K}^n_{oo}$ and $L \in \mathcal{K}^n_o$, then*

$$\frac{dh_\varepsilon}{d\varepsilon} = \frac{h(K,u)\frac{d\varphi_1^{-1}(y)}{dy}\varphi_2\left(\frac{h(L,u)}{h_\varepsilon}\right)}{\left(\varphi_1{}^{-1}\left(1-\varepsilon\varphi_2\left(\frac{h(L,u)}{h_\varepsilon}\right)\right)\right)^2 + \varepsilon\cdot\frac{h(L,u)h(L_n,u)}{h_\varepsilon^2}\frac{d\varphi_1^{-1}(y)}{dy}\frac{d\varphi_2(z)}{dz}}, \tag{3.4}$$

where

$$y = 1-\varepsilon\varphi_2\left(\frac{h(L,u)}{h_\varepsilon}\right),$$

and

$$z = \frac{h(L,u)}{h_\varepsilon}.$$

Proof. Suppose $\varepsilon > 0$, $L \in \mathcal{K}^n_o, K \in \mathcal{K}^n_{oo}$ and $u \in S^{n-1}$, and notice that

$$h_\varepsilon = h(K+_{\varphi,\varepsilon}L,u),$$

we have

$$\frac{h(K,u)}{h_\varepsilon} = \varphi_1^{-1}\left(1-\varepsilon\varphi_2\left(\frac{h(L,u)}{h_\varepsilon}\right)\right).$$

On the other hand

$$\frac{dh_\varepsilon}{d\varepsilon}=\frac{d}{d\varepsilon}\left(\frac{h(K,u)}{{\varphi_1}^{-1}\left(1-\varepsilon\varphi_2\left(\frac{h(L,u)}{h_\varepsilon}\right)\right)}\right)$$

$$=\frac{h(K,u)\frac{d\varphi_1^{-1}(y)}{dy}\left[\varphi_2\left(\frac{h(L,u)}{h_\varepsilon}\right)-\varepsilon\cdot\frac{d\varphi_2(z)}{dz}\frac{h(L,u)}{h_\varepsilon^2}\frac{dh_\varepsilon}{d\varepsilon}\right]}{\left({\varphi_1}^{-1}\left(1-\varepsilon\varphi_2\left(\frac{h(L,u)}{h_\varepsilon}\right)\right)\right)^2}.$$

where

$$y=1-\varepsilon\varphi_2\left(\frac{h(L,u)}{h_\varepsilon}\right),$$

and

$$z=\frac{h(L,u)}{h_\varepsilon}.$$

By simplifying the equation from above, (3.4) easily follows. □

Theorem 3.5. *Let $\varphi\in\Phi_2$, and $\varphi_1,\varphi_2\in\Phi$. If $K\in\mathcal{K}^n_{oo}$, $L\in\mathcal{K}^n_o$ and $1\le i\le n$, then*

$$\frac{(\varphi_1)'_l(1)}{n-i}\lim_{\varepsilon\to0^+}\frac{W_i(K+_{\varphi,\varepsilon}L)-W_i(K)}{\varepsilon}=\frac{1}{n}\int_{S^{n-1}}\varphi_2\left(\frac{h(L,u)}{h(K,u)}\right)h(K,u)dS_i(K,u). \tag{3.5}$$

Proof. From Lemma 3.3, we obtain

$$\lim_{\varepsilon\to0^+}\frac{W_i(K+_{\varphi,\varepsilon}L)-W_i(K)}{\varepsilon}=\frac{n-i}{n}\int_{S^{n-1}}\lim_{\varepsilon\to0^+}\frac{h(K+_{\varphi,\varepsilon}L,u)-h(K,u)}{\varepsilon}dS_i(K,u)$$

$$=\frac{n-i}{n}\lim_{\varepsilon\to0^+}\int_{S^{n-1}}\frac{dh_\varepsilon}{d\varepsilon}dS_i(K;u).$$

From Lemmas 3.1-3.2 and Lemma 3.4, and noting that $y\to1^-$ as $\varepsilon\to0^+$, we have

$$\frac{d\varphi_1^{-1}(y)}{d\varepsilon}=\lim_{y\to1^-}\frac{{\varphi_1}^{-1}(y)-{\varphi_1}^{-1}(1)}{y-1}=\frac{1}{(\varphi_1)'_l(1)},$$

the equation (3.5) easily follows. □

The theorem plays a central role in our deriving new concept of the Orlicz mixed quermassintegrals. Here, we give the another proof.

Proof. From the hypotheses, we have for $\varepsilon>0$

$$h(K+_{\varphi,\varepsilon}L,u)=\frac{h(K,u)}{\varphi_1^{-1}\left(1-\varepsilon\varphi_2\left(\frac{h(L,u)}{h(K+_{\varphi,\varepsilon}L,u)}\right)\right)}.$$

Hence

$$\lim_{\varepsilon\to 0^+}\frac{h(K+_{\varphi,\varepsilon}L,u)-h(K,u)}{\varepsilon} \tag{3.6}$$

$$=\lim_{\varepsilon\to 0^+}\frac{\dfrac{h(K,u)}{\varphi_1^{-1}\left(1-\varepsilon\varphi_2\left(\dfrac{h(L,u)}{h(K+_{\varphi,\varepsilon}L,u)}\right)\right)}-h(K,u)}{\varepsilon}$$

$$=\lim_{\varepsilon\to 0^+}\frac{h(K,u)\varphi_2\left(\dfrac{h(L,u)}{h(K+_{\varphi,\varepsilon}L,u)}\right)}{\left(\varphi_1^{-1}\left(1-\varepsilon\varphi_2\left(\dfrac{h(L,u)}{h(K+_{\varphi,\varepsilon}L,u)}\right)\right)\right)^2}\lim_{y\to 1^-}\frac{\varphi_1^{-1}(y)-\varphi_1^{-1}(1)}{y-1},$$

where

$$y=1-\varepsilon\varphi_2\left(\frac{h(L,u)}{h(K+_{\varphi,\varepsilon}L,u)}\right),$$

and note that $y\to 1^-$ as $\varepsilon\to o^+$. Notice that

$$\lim_{y\to 1^-}\frac{\varphi_1^{-1}(y)-\varphi_1^{-1}(1)}{y-1}=\frac{1}{(\varphi_1)'_l(1)},$$

and from (2.2),(3.6) and Lemmas 3.1-3.2, (3.5) easy follows. □

Denoting by $W_{\varphi,i}(K,L)$, for any $\varphi\in\Phi$ and $1\le i<n$, the integral on the right-hand side of (3.5) with φ_2 replaced by φ, we see that either side of the equation (3.5) is equal to $W_{\varphi_2,i}(K,L)$ and therefore this new Orlicz mixed volume $W_{\varphi,i}(K,L)$ (Orlicz mixed quermassintegrals) has been born.

Definition 3.1. (Orlicz mixed quermassintegrals) For $\varphi\in\Phi$, Orlicz mixed quermassintegrals, $W_{\varphi,i}(K,L)$, for $0\le i<n$, defined by

$$W_{\varphi,i}(K,L)=:\frac{1}{n}\int_{S^{n-1}}\varphi\left(\frac{h(L,u)}{h(K,u)}\right)h(K,u)dS_i(K,u), \tag{3.7}$$

for all $K\in\mathcal{K}^n_{oo}$, $L\in\mathcal{K}^n_o$.

Remark 3.2. Let $\varphi_1(t)=\varphi_2(t)=t^p$, $p\ge 1$ in (3.5), the Orlicz sum $K+_{\varphi,\varepsilon}L$ reduces to the L_p addition $K+_p\varepsilon\cdot L$, and the Orlicz mixed quermassintegrals $W_{\varphi,i}(K,L)$ become the well-known mixed p-quermassintegrals $W_{p,i}(K,L)$. Obviously, when $i=0$, $W_{\varphi,i}(K,L)$ reduces to Orlicz mixed volumes $V_\varphi(K,L)$ defined by Gardner, Hug and Weil [9].

Theorem 3.6. *If $\varphi_1,\varphi_2\in\Phi$, $\varphi\in\Phi_2$ and $K\in\mathcal{K}^n_o, L\in\mathcal{K}^n_{oo}$, and $0\le i<n$, then*

$$W_{\varphi_2,i}(K,L)=\frac{(\varphi_1)'_l(1)}{n-i}\lim_{\varepsilon\to 0^+}\frac{W_i(K+_{\varphi,\varepsilon}L)-W_i(K)}{\varepsilon}. \tag{3.8}$$

Proof. This follows immediately from Theorem 3.5 and (3.7). □

4 Orlicz-Minkowski type inequality

In the Section, we need define a Borel measure in S^{n-1}, $\bar{W}_{n,i}(K,\upsilon)$, called as i-th normalized cone measure.

Definition 4.1. If $K \in \mathcal{K}^n_{oo}$, *i-th normalized cone measure*, $\bar{W}_{n,i}(K,\upsilon)$, defined by

$$d\bar{W}_{n,i}(K,\upsilon) = \frac{h(K,\upsilon)}{nW_i(K)} dS_i(K,\upsilon). \tag{4.1}$$

When $i = 0$, $\bar{W}_{n,i}(K,\upsilon)$ becomes to the well-known normalized cone measure $\bar{V}_n(K,\upsilon)$, by

$$d\bar{V}_n(K,\upsilon) = \frac{h(K,\upsilon)}{nV(K)} dS(K,\upsilon). \tag{4.2}$$

This was defined in [2] and [9].

In the following, we start with two auxiliary results (Lemmas 4.1 and 4.2), which will be the base of our further study. The Orlicz-Minkowski inequality for Orlicz mixed quermassintegrals is established in Theorem 4.3.

Lemma 4.1. *(Jensen's inequality) Suppose that μ is a probability measure on a space X and $g : X \to I \subset \mathbb{R}$ is a μ-integrable function, where I is a possibly infinite interval. If $\varphi : I \to \mathbb{R}$ is a convex function, then*

$$\int_X \varphi(g(x))d\mu(x) \geq \varphi\left(\int_X g(x)d\mu(x)\right). \tag{4.3}$$

If φ is strictly convex, equality holds if and only if $g(x)$ is constant for μ-almost all $x \in X$ (see [16]).

Lemma 4.2. *Let $0 < a \leq \infty$ be an extended real number, and let $I = [0, a)$ be a possibly infinite interval. Suppose that $\varphi : I \to [0,\infty)$ is convex with $\varphi(0) = 0$. If $K \in \mathcal{K}^n_{oo}$ and $L \in \mathcal{K}^n_o$ are such that $L \subset \mathrm{int}(aK)$, then*

$$\frac{1}{nW_i(K)} \int_{S^{n-1}} \varphi\left(\frac{h(L,u)}{h(K,u)}\right) h(K,u)dS_i(K,u) \geq \varphi\left(\left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}\right). \tag{4.4}$$

If φ is strictly convex, equality holds if and only if K and L are dilates or $L = \{o\}$.

Proof. In view of $L \subset \mathrm{int}(aK)$, so $0 \leq \frac{h(L,u)}{h(K,u)} < a$ for all $u \in S^{n-1}$. By (4.1) and note that (2.2) with $K = L$, it follows the i-th normalized cone measure $\bar{W}_{n,i}(K,u)$ is a probability measure on S^{n-1}. Hence by using Jensen's inequality (4.3), the Minkowski's inequality (2.4), and the fact that φ is increasing, to obtain

$$\begin{aligned}\frac{1}{nW_i(K)} \int_{S^{n-1}} \varphi\left(\frac{h(L,u)}{h(K,u)}\right) h(K,u)dS_i(K,u) &= \int_{S^{n-1}} \varphi\left(\frac{h(L,u)}{h(K,u)}\right) d\bar{W}_{n,i}(K,u)\\ &\geq \varphi\left(\frac{W_i(K,L)}{W_i(K)}\right)\end{aligned}$$

$$\geq \varphi\left(\left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}\right). \tag{4.5}$$

In the following, we discuss the equal condition of (4.4). Suppose the equality holds in (4.4) and φ is strictly convex, so that $\varphi > 0$ on $(0, a)$. Moreover, notice the injectivity of φ, we have equality in Minkowski inequality (2.4), so there are $r \geq 0$ and $x \in \mathbb{R}^n$ such that $L = rK + x$ and hence

$$h(L, u) = rh(K, u) + x \cdot u$$

for all $u \in S^{n-1}$. Since equality must hold in Jensen's inequality (4.3) as well, when φ is strictly convex we can conclude from the equality condition for Jensen's inequality that

$$\frac{1}{nW_i(K)} \int_{S^{n-1}} \frac{h(L,u)}{h(K,u)} h(K,u) dS_i(K,u) = \frac{h(L,v)}{h(K,v)}, \tag{4.6}$$

for $S_i(K, \cdot)$-almost all $v \in S^{n-1}$. Hence

$$\frac{1}{nW_i(K)} \int_{S^{n-1}} \left(r + \frac{x \cdot u}{h(K,u)}\right) h(K,u) dS_i(K,u) = r + \frac{x \cdot v}{h(K,v)},$$

for $S_i(K, \cdot)$-almost all $v \in S^{n-1}$. From this and the fact that the centroid of $S_i(K, \cdot)$ is at the origin, we get

$$0 = x \cdot \left(\frac{1}{nW_i(K)} \int_{S^{n-1}} u dS_i(K,u)\right) = \frac{1}{nW_i(K)} \int_{S^{n-1}} x \cdot u dS_i(K,u) = \frac{x \cdot v}{h(K,v)},$$

that is, $x \cdot v = 0$, for $S_i(K, \cdot)$-almost all $v \in S^{n-1}$. Hence $x = o$, namely $L = rK$. □

Theorem 4.3. *Let $\varphi \in \Phi$. If $K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$ and $0 \leq i < n$, then*

$$W_{\varphi,i}(K, L) \geq W_i(K) \cdot \varphi\left(\left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}\right). \tag{4.7}$$

If φ is strictly convex, equality holds if and only if K and L are dilates or $L = \{o\}$.

Proof. This follows immediately from (3.7) and Lemma 4.2, with $a = \infty$. □

Corollary 4.4. *([21]) If $K \in \mathcal{K}^n_{oo}$ and $L \in \mathcal{K}^n_o$, and $p > 1$ and $0 \leq i \leq n$, then*

$$W_{p,i}(K, L)^{n-i} \geq W_i(K)^{n-i-p} W_i(L)^p,$$

with equality if and only if K and L are dilates or $L = \{o\}$.

Proof. This follows immediately from (4.7) with $\varphi(t) = t^p$ and $p > 1$. □

Remark 4.2. When $a = \infty$, putting $\varphi(t) = e^t - 1$ in (4.4), we obtain

$$\log \int_{S^{n-1}} \exp\left(\frac{h(L,u)}{h(K,u)}\right) d\bar{W}_{n,i}(K,u) \geq \left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}. \tag{4.8}$$

Similarly, L_p-Minkowski inequality (1.8) can be written as

$$\left(\int_{S^{n-1}}\left(\frac{h(L,u)}{h(K,u)}\right)^p d\bar{W}_{n,i}(K,u)\right)^{1/p} \geq \left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}. \tag{4.9}$$

When $p=1$, (4.9) becomes to a new form of the Minkowski inequality (2.4). The left side of (4.9) is just the pth mean of the function $h(L,u)/h(K,u)$ with respect to $\bar{W}_{n,i}(K,\cdot)$. Notice that pth means increase with $p>1$, so we find that the Minkowski inequality (2.4) implies L_p-Minkowski inequality (2.8).

5 Orlicz-Brunn-Minkowski type inequality

In this section, we establish the Orlicz Brunn-Minkowski inequality for Orlicz mixed quermassintegrals.

Theorem 5.1. *Let $\varphi\in\Phi_2$. If $K\in\mathcal{K}^n_{oo}$, $L\in\mathcal{K}^n_o$ and $1\leq i<n$, then*

$$1\geq\varphi\left(\frac{W_i(K)^{1/(n-i)}}{W_i(K+_\varphi L)^{1/(n-i)}},\frac{W_i(L)^{1/(n-i)}}{W_i(K+_\varphi L)^{1/(n-i)}}\right). \tag{5.1}$$

If φ is strictly convex, equality holds if and only if K and L are dilates or $L=\{o\}$.

Proof. From the hypotheses and Theorem 4.3, we obtain

$$\begin{aligned} &W_i(K+_\varphi L) \qquad (5.2)\\ &=\frac{1}{n}\int_{S^{n-1}}\varphi\left(\frac{h(K,u)}{h(K+_\varphi L,u)},\frac{h(L,u)}{h(K+_\varphi L,u)}\right)h(K+_\varphi L,u)dS_i(K+_\varphi L,u)\\ &=\frac{1}{n}\int_{S^{n-1}}\left(\varphi_1\left(\frac{h(K,u)}{h(K+_\varphi L,u)}\right)+\varphi_2\left(\frac{h(L,u)}{h(K+_\varphi L,u)}\right)\right)h(K+_\varphi L,u)dS_i(K+_\varphi L,u)\\ &=W_{\varphi_1,i}(K+_\varphi L,K)+W_{\varphi_2,i}(K+_\varphi L,L)\\ &\geq W_i(K+_\varphi L)\varphi\left(\frac{W_i(K)^{1/(n-i)}}{W_i(K+_\varphi L)^{1/(n-i)}},\frac{W_i(L)^{1/(n-i)}}{W_i(K+_\varphi L)^{1/(n-i)}}\right). \end{aligned}$$

This is just (5.1).

If equality holds in (5.2), then in (5.2), with K, L and φ replaced by $K+_\varphi L$, K and φ_1 (and by $K+_\varphi L$, L and φ_2), respectively. So if φ is strictly convex, then φ_1 and φ_2 are also, so both K and L are multiples of $K+_\varphi L$, and hence are dilates of each other or $L=\{o\}$. □

Corollary 5.2. *([21]) If $p>1$, $K\in\mathcal{K}^n_{oo}$, $L\in\mathcal{K}^n_o$, while $0\leq i<n$, then*

$$W_i(K+_p L)^{p/(n-i)}\geq W_i(K)^{p/(n-i)}+W_i(L)^{p/(n-i)}, \tag{5.3}$$

with equality if and only if K and L are dilates or $L=\{o\}$.

Proof. The result follows immediately from Theorem 5.1 with $\varphi(x_1,x_2)=x_1^p+x_2^p$ and $p>1$. □

Theorem 5.3. *Orlicz Brunn-Minkowski inequality for Orlicz mixed quermassintegrals implies Orlicz Minkowski inequality for Orlicz mixed quermassintegrals.*

Proof. Since φ_1 is increasing, so φ_1^{-1} is also increasing and hence from (5.1), we obtain for $\varepsilon > 0$

$$W_i(K +_{\varphi,\varepsilon} L) \geq \frac{W_i(K)}{\left(\varphi_1^{-1}\left(1 - \varepsilon\varphi_2\left(\left(\frac{W_i(L)}{W_i(K +_{\varphi,\varepsilon} L)}\right)^{1/(n-i)}\right)\right)\right)^{n-i}}.$$

From Theorem 3.6, we obtain

$$W_{\varphi_2,i}(K, L) \geq \frac{(\varphi_1)'_l(1)}{n-i}$$

$$\times \lim_{\varepsilon\to 0^+} \frac{\dfrac{W_i(K)}{\left(\varphi_1^{-1}\left(1 - \varepsilon\varphi_2\left(\left(\frac{W_i(L)}{W_i(K +_{\varphi,\varepsilon} L)}\right)^{1/(n-i)}\right)\right)\right)^{n-i}} - W_i(K)}{\varepsilon}$$

$$= (\varphi_1)'_l(1) \lim_{\varepsilon\to 0^+} \frac{W_i(K)}{\left(\varphi_1^{-1}\left(1 - \varepsilon\varphi_2\left(\left(\frac{W_i(L)}{W_i(K +_{\varphi,\varepsilon} L)}\right)^{1/(n-i)}\right)\right)\right)^{2(n-i)}}$$

$$\times \left(\varphi_1^{-1}\left(1 - \varepsilon\varphi_2\left(\left(\frac{W_i(L)}{W_i(K +_{\varphi,\varepsilon} L)}\right)^{1/(n-i)}\right)\right)\right)^{n-i-1}$$

$$\times \varphi_2\left(\left(\frac{W_i(L)}{W_i(K +_{\varphi,\varepsilon} L)}\right)^{1/(n-i)}\right) \lim_{z\to 1^-} \frac{\varphi_1^{-1}(z) - \varphi_1^{-1}(1)}{z-1},$$

where

$$z = 1 - \varepsilon\varphi_2\left(\left(\frac{W_i(L)}{W_i(K +_{\varphi,\varepsilon} L)}\right)^{1/(n-i)}\right),$$

and note that $z \to 1^-$ as $\varepsilon \to o^+$. On the other hand, in view of

$$\lim_{z\to 0^+} \frac{\varphi_1^{-1}(z) - \varphi_1^{-1}(1)}{z-1} = \frac{1}{(\varphi_1)'_l(1)},$$

and from Lemma 3.2. Hence

$$W_{\varphi_2,i}(K, L) \geq W_i(K)\varphi_2\left(\left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}\right). \tag{5.4}$$

Replace φ_2 by φ, this yields the Orlicz Minkowski inequality in (4.7). The equality condition follows immediately from the equality of Orlicz Brunn-Minkowski inequality for Orlicz mixed quermassintegrals. □

From the proof of Theorem 5.1, we may see that Orlicz Minkowski inequality for Orlicz mixed quermassintegrals implies also Orlicz Brunn-Minkowski inequality for Orlicz mixed quermassintegrals, and this combines Theorem 5.3, we found that

Theorem 5.4. *Orlicz Brunn-Minkowski inequality for Orlicz mixed quermassintegrals is equivalent to Orlicz Minkowski inequality for Orlicz mixed quermassintegrals. Namely: Let* $\varphi_2 \in \Phi$ *and* $\varphi \in \Phi_2$*. If* $K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$ *and* $1 \le i < n$, *then*

$$W_{\varphi_2,i}(K,L) \ge W_i(K)\varphi_2\left(\left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}\right) \tag{5.5}$$

$$\Leftrightarrow 1 \ge \varphi\left(\frac{W_i(K)^{1/(n-i)}}{W_i(K+_\varphi L)^{1/(n-i)}}, \frac{W_i(L)^{1/(n-i)}}{W_i(K+_\varphi L)^{1/(n-i)}}\right).$$

If φ *is strictly convex, equality holds if and only if* K *and* L *are dilates or* $L = \{o\}$.

Corollary 5.5. *Orlicz dual Brunn-Minkowski inequality is equivalent to Orlicz dual Minkowski inequality. Namely: Let* $\varphi_2 \in \Phi$ *and* $\varphi \in \Phi_2$*. If* $K \in \mathcal{K}^n_{oo}$ *and* $L \in \mathcal{K}^n_o$, *then*

$$V_{\varphi_2}(K,L) \ge V(K)\varphi_2\left(\left(\frac{V(L)}{V(K)}\right)^{1/n}\right) \Leftrightarrow 1 \ge \varphi\left(\frac{V(K)^{1/n}}{V(K+_\varphi L)^{1/n}}, \frac{V(L)^{1/n}}{V(K+_\varphi L)^{1/n}}\right). \tag{5.6}$$

If φ *is strictly convex, equality holds if and only if* K *and* L *are dilates or* $L = \{o\}$.

Proof. The result follows immediately from Theorem 5.4 with $i = 0$. □

6 The log-Minkowski type inequality

Assume that $K, L \in \mathcal{K}^n_{oo}$, then the log Minkowski combination, $(1-\lambda)\cdot K +_o \lambda \cdot L$, is defined by

$$(1-\lambda)\cdot K +_o \lambda\cdot L = \bigcap_{u \in S^{n-1}} \{x \in \mathbb{R}^n \mid x\cdot u \le h(K,u)^{1-\lambda} h(L,u)^{\lambda}\},$$

for all real $\lambda \in [0,1]$. Böröczky, Lutwak, Yang, and Zhang [2] conjecture that for origin-symmetric convex bodies K and L in $\mathbb{R}^n$ and $0 \le \lambda \le 1$,

$$V((1-\lambda)\cdot K +_o \lambda\cdot L) \ge V(K)^{1-\lambda} V(L)^{\lambda}. \tag{6.1}$$

In [2], they proved (6.1) only when $n = 2$ and K, L are origin-symmetric convex bodies, and note that while it is not true for general convex bodies. Moreover, they also shown that (6.1), for all n, is equivalent to the following log-Minkowski inequality

$$\int_{S^{n-1}} \log\left(\frac{h(L,u)}{h(K,u)}\right) d\bar{V}_n(K,\upsilon) \ge \frac{1}{n}\log\left(\frac{V(L)}{V(K)}\right), \tag{6.2}$$

where $\bar{V_n}(K,\cdot)$ is the *normalized cone measure* for K. In fact, replacing K and L by $K + L$ and K, respectively, (6.2) becomes to the following

$$\int_{S^{n-1}} \log\left(\frac{h(K,u)}{h(K+L,u)}\right) d\bar{V}_n(K+L,u) \ge \log\left(\left(\frac{V(K)}{V(K+L)}\right)\right)^{1/n}. \tag{6.3}$$

In [9], Gardner, Hug and Weil gave a new version of (6.3) for the nonempty compact convex subsets K and L, not origin-symmetric convex bodies, as follows. If $K \in \mathcal{K}^n_{oo}$ and $L \in \mathcal{K}^n_o$, then

$$\int_{S^{n-1}} \log\left(\frac{h(K,u)}{h(K+L,u)}\right) d\bar{V}_n(K+L,u) \le \log\left(\frac{V(K+L)^{1/n} - V(L)^{1/n}}{V(K+L)^{1/n}}\right), \tag{6.4}$$

with equality if and only if K and L are dilates or $L = \{o\}$. They also shown that combining (6.3) and (6.4), may get the classical Brunn-Minkowski inequality.

$$V(K+L)^{1/n} - V(L)^{1/n} \ge V(K)^{1/n},$$

whenever $K \in \mathcal{K}^n_{oo}$ and $L \in \mathcal{K}^n_o$ and (6.2) holds with K and L replaced by $K+L$ and K, respectively. In particular, if (6.2) holds (as it does, for origin-symmetric convex bodies when $n = 2$), then (6.2) and (6.4) together split the classical Brunn-Minkowski inequality. In the following, we give a new version of (6.4).

Lemma 6.1. *If $K \in \mathcal{K}^n_{oo}$ and $L \in \mathcal{K}^n_o$ are such that $L \subset \mathrm{int}K$ and $1 \le i < n$, then*

$$\log\left(\frac{W_i(K)^{1/(n-i)} - W_i(L)^{1/(n-i)}}{W_i(K)^{1/(n-i)}}\right) \ge \int_{S^{n-1}} \log\left(\frac{h(K,u) - h(L,u)}{h(K,u)}\right) d\bar{W}_{n,i}(K,u), \tag{6.5}$$

with equality if and only if K and L are dilates or $L = \{o\}$.

Proof. Since $K \in \mathcal{K}^n_{oo}$ and $L \in \mathcal{K}^n_o$ are such that $L \subset \mathrm{int}K$. Let $\varphi(t) = -\log(1-t)$, and notice that $\varphi(0) = 0$ and φ is strictly increasing and strictly convex on $[0,1)$ with $\varphi(t) \to \infty$ as $t \to 1^-$. Hence the inequality (6.5) is a direct consequence of Lemma 4.3 with this choice of φ and $a = 1$. □

Theorem 6.2. *If $K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$ and $1 \le i < n$, then*

$$\log\left(\frac{W_i(K+L)^{1/(n-i)} - W_i(L)^{1/(n-i)}}{W_i(K+L)^{1/(n-i)}}\right) \ge \int_{S^{n-1}} \log\left(\frac{h(K,u)}{h(K+L,u)}\right) d\bar{W}_{n,i}(K+L,u), \tag{6.6}$$

with equality if and only if K and L are dilates or $L = \{o\}$.

Proof. If $K \in \mathcal{K}^n_{oo}$ and $L \in \mathcal{K}^n_o$, then $K + L \in \mathcal{K}^n_{oo}$. In view of $L \subset \mathrm{int}(K+L)$ and from Lemma 6.1 with K replaced by $K+L$, (6.6) easy follows. □

Putting $i = 0$ in (6.6), (6.6) reduces to (6.4). Here, we point out a new conjecture which is an extension of the log Minkowski inequality (6.2): *Conjecture If $K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$ and $1 \le i < n$, then*

$$\int_{S^{n-1}} \log\left(\frac{h(L,u)}{h(K,u)}\right) d\bar{W}_{n,i}(K,u) \ge \frac{1}{n-i} \log\left(\frac{W_i(L)}{W_i(K)}\right). \tag{6.7}$$

Corollary 6.3. *If $K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$ and $1 \le i < n$, then*

$$\int_{S^{n-1}} \log\left(\frac{h(K,u)}{h(K+L,u)}\right) d\bar{W}_{n,i}(K+L,u) \ge \frac{1}{n-i} \log\left(\frac{W_i(K)}{W_i(K+L)}\right). \tag{6.8}$$

Proof. The result follows immediately from (6.7) with replacing K and L by $K+L$ and K, respectively. □

It is easy that combine (6.6) and (6.8) together split the following classical Brunn-Minkowski inequality for quermassintegrals. If $K \in \mathcal{K}_{oo}^n$, $L \in \mathcal{K}_o^n$ and $0 \le i \le n$, then

$$W_i(K+L)^{1/(n-i)} \ge W_i(K)^{1/(n-i)} + W_i(L)^{1/(n-i)},$$

with equality if and only if K and L are dilates or $L = \{o\}$.

7 A new version of Orlicz Minkowski's inequality

In 2010, the Orlicz projection body $\mathbf{\Pi}_\varphi$ of K defined by Lutwak, Yang and Zhang [28]

$$h(\mathbf{\Pi}_\varphi, u) = \inf\left\{\lambda > 0 \mid \int_{S^{n-1}} \varphi\left(\frac{|u\cdot v|}{\lambda h(K,v)}\right) d\bar{V}_n(K,v) \le 1\right\}, \tag{7.1}$$

for $K \in \mathcal{K}_{oo}^n, u \in S^{n-1}$, where $\bar{V}_n(K,\cdot)$ is the normalized cone measure for K. Here, we define the i-th Orlicz mixed projection body.

Definition 7.1. Let $K \in \mathcal{K}_{oo}^n, L \in \mathcal{K}_o^n$, $\varphi \in \Phi$ and $0 \le i < n$, the *i-th Orlicz mixed projection body*, $\mathbf{\Pi}_{\varphi,i}$, define by

$$h(\mathbf{\Pi}_{\varphi,i}, u) = \inf\left\{\lambda > 0 \mid \int_{S^{n-1}} \varphi\left(\frac{|u\cdot v|}{\lambda h(K,v)}\right) d\bar{W}_{n,i}(K,v) \le 1\right\}, \tag{7.2}$$

for $u \in S^{n-1}$, where $\bar{W}_{n,i}(K,\cdot)$ is the i-th normalized cone measure for K defined in (4.1).

Obviously, when $i=0$, (7.2) becomes (7.1). In the Section, definition 7.1 of the i-th Orlicz projection body suggests defining, by analogy,

$$\widehat{W}_{\varphi,i}(K,L) = \inf\left\{\lambda > 0 \mid \int_{S^{n-1}} \varphi\left(\frac{h(L,u)}{\lambda h(K,u)}\right) d\bar{W}_{n,i}(K,u) \le 1\right\}, \tag{7.3}$$

and call as $\widehat{W}_{\varphi,i}(K,L)$ Orlicz type quermassintegrals.

Theorem 7.1. *If $\varphi \in \Phi$ and $K \in \mathcal{K}_{oo}^n$, $L \in \mathcal{K}_o^n$ and $1 \le i < n$, then*

$$\widehat{W}_{\varphi,i}(K,L) \ge \left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}. \tag{7.4}$$

If φ is strictly convex and $W_i(L) > 0$, equality holds if and only if K and L are dilates.

Proof. Replacing K by λK, $\lambda > 0$ in (4.4) with $a = \infty$, we have

$$\int_{S^{n-1}} \varphi\left(\frac{h(L,u)}{\lambda h(K,u)}\right) d\bar{W}_{n,i}(K,u) \ge \varphi\left(\frac{1}{\lambda}\left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}\right). \tag{7.5}$$

Let

$$\int_{S^{n-1}} \varphi\left(\frac{h(L,u)}{\lambda h(K,u)}\right) d\bar{W}_{n,i}(K,u) \leq 1.$$

Hence

$$\varphi\left(\frac{1}{\lambda}\left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}\right) \leq 1.$$

In view of φ is strictly increasing, we obtain

$$\left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)} \leq \lambda. \tag{7.6}$$

From (7.3) and (7.6), (7.4) easy follows.

In the following, we discuss the equality condition of (7.4). Suppose that equality holds, φ is strictly convex and $W_i(L) > 0$. From (7.3), the exist $\mu = \widehat{W}_{\varphi,i}(K,L) > 0$ satisfies

$$\int_{S^{n-1}} \varphi\left(\frac{h(L,u)}{\mu h(K,u)}\right) d\bar{W}_{n,i}(K,\upsilon) = 1.$$

Hence

$$\mu = \left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)};$$

namely:

$$\varphi\left(\frac{1}{\mu}\left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}\right) = 1.$$

Therefore the equality in (7.5) holds for $\lambda = \mu$. From the equality condition of (4.4), it follows μK and L are dilates. □

When $\varphi(t) = t^p$ and $p \geq 1$ in (7.3), it easy follows that

$$\widehat{W}_{\varphi,i}(K,L) = \left(\frac{W_{p,i}(K,L)}{W_i(K)}\right)^{1/p}.$$

Putting $\varphi(t) = t^p$ and $p \geq 1$ in (7.4), (7.4) reduces to the classical L_p-Minkowski inequality (1.8) for mixed p-quermassintegrals.

There is no direct relationship between the Orlicz-Minkowski inequalities (4.7) and (7.4). Indeed, when $\varphi > 0$ on $(0,\infty)$, these can be written in the forms

$$\frac{W_{\varphi,i}(K,L)}{W_i(K)} \geq \varphi\left(\left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}\right), \tag{7.7}$$

and

$$\varphi\left(\widehat{W}_{\varphi,i}(K,L)\right) \geq \varphi\left(\left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}\right). \tag{7.7}$$

respectively, and each of the two quantities on the left-hand sides can be larger than the other. This is very interesting.

8 Simon's characterization of relative spheres

Theorem 8.1. *Suppose* $K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$, *and* $\mathcal{S} \subset \mathcal{K}^n_o$ *is a class of bodies such that* $K, L \in \mathcal{S}$. *If* $0 \le i < n-1$ *and* $\varphi \in \Phi$, *and*

$$W_{\varphi,i}(Q,K) = W_{\varphi,i}(Q,L), \quad \text{for all } Q \in \mathcal{S}, \tag{8.1}$$

then $K = L$.

Proof. To see this take $Q = K$, and from (3.10) and Theorem 4.4, we have

$$W_i(K) = W_{\varphi,i}(K,K) = W_{\varphi,i}(K,L) \ge W_i(K)\varphi\left(\left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}\right).$$

If φ is strictly convex, equality holds if and only if K and L are dilates or $L = \{o\}$. Hence

$$\varphi\left(\left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}\right) \le 1.$$

If φ is strictly convex, equality holds if and only if K and L are dilates or $L = \{o\}$. Note that φ is increasing, we obtain

$$W_i(L) \le W_i(K).$$

Take $Q = L$, we have

$$W_i(L) = W_{\varphi,i}(L,L) = W_{\varphi,i}(L,K) \ge W_i(L)\varphi\left(\left(\frac{W_i(K)}{W_i(L)}\right)^{1/(n-i)}\right).$$

If φ is strictly convex, equality holds if and only if K and L are dilates or $L = \{o\}$. Hence

$$\varphi\left(\left(\frac{W_i(K)}{W_i(L)}\right)^{1/(n-i)}\right) \le 1.$$

If φ is strictly convex, equality holds if and only if K and L are dilates or $L = \{o\}$. Hence

$$W_i(K) \le W_i(L).$$

This yields $W_i(K) = W_i(L)$. Hence $K = L$. □

Corollary 8.2. *Suppose* $K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$, *and* $\mathcal{S} \subset \mathcal{K}^n_o$ *is a class of bodies such that* $K, L \in \mathcal{S}$. *If* $\varphi \in \Phi$, *and*

$$V_\varphi(Q,K) = V_\varphi(Q,L), \quad \text{for all } Q \in \mathcal{S}, \tag{8.2}$$

then $K = L$.

Proof. The result follows immediately from Theorem 8.1 with $i = 0$. □

Putting $\varphi(t) = t^p$ and $p > 1$ in Theorem 8.1, we obtain the following result which was proved by Lutwak [21].

Corollary 8.3. *Suppose $K \in \mathcal{K}^n_{oo}$, $L \in \mathcal{K}^n_o$, and $\mathcal{S} \subset \mathcal{K}^n_o$ is a class of bodies such that $K, L \in \mathcal{S}$. If $p > 1$, $0 \le i < n-1$, and*

$$W_{p,i}(Q,K) = W_{p,i}(Q,L), \quad \text{for all } Q \in \mathcal{S}, \tag{8.3}$$

then $K = L$.

Theorem 8.4. *Suppose $0 \le i < n$ and $\varphi \in \Phi$. For $K \in \mathcal{K}^n_{oo}$, the following statements are equivalent:*

(i) The body K is centered,
(ii) The measure $\bar{W}_{n,i}(K,\cdot)$ is even.
(iii) $W_{\varphi,i}(K,Q) = W_{\varphi,i}(K,-Q)$, for all $Q \in \mathcal{K}^n_{oo}$.
(iv) $W_{\varphi,i}(K,Q) = W_{\varphi,i}(K,-Q)$, for $Q = K$.

Proof. To see that (i) implies (ii), recall that if K is centered, then $h(K,\cdot)$ is an even function, and $S_i(K)$ is an even measure. The implication is now a consequence of the fact that $d\bar{W}_{n,i}(K,\cdot) = \frac{1}{nW_i(K)} h(K,\cdot) dS_i(K,\cdot)$.

That (ii) yields (iii) is a consequence of the following integra representation

$$W_{\varphi,i}(K,Q) = W_i(K) \int_{S^{n-1}} \varphi\left(\frac{h(Q,u)}{h(K,u)}\right) d\bar{W}_{n,i}(K,u),$$

and the fact that, in general, $h(-Q,u) = h(Q,-u)$, for all $u \in S^{n-1}$. Obviously, (iv) follows directly from (iii).

To see that (iv) implies (i), notice that (iv), for $Q = K$, gives

$$W_i(K) = W_{\varphi,i}(K,-K).$$

The desired result follows from the fact that $W_i(-K) = W_i(K)$ and the equality conditions of the Orlicz-Minkoski inequality (4.7). □

Corollary 8.5. *Suppose $\varphi \in \Phi$. For $K \in \mathcal{K}^n_{oo}$, the following statements are equivalent:*

(i) The body K is centered,
(ii) The measure $\bar{V}_n(K,\cdot)$ is even.
(iii) $V_\varphi(K,Q) = V_\varphi(K,-Q)$, for all $Q \in \mathcal{K}^n_{oo}$.
(iv) $V_\varphi(K,Q) = V_{\varphi,i}(K,-Q)$, for $Q = K$.

Proof. The results follow immediately from Theorem 8.5 with $i = 0$. □

Corollary 8.6. *Suppose $0 \le i < n$ and $p > 1$. For $K \in \mathcal{K}^n_{oo}$, the following statements are equivalent:*

(i) The body K is centered,
(ii) The measure $S_{p,i}(K,\cdot)$ is even.
(iii) $W_{p,i}(K,Q) = W_{p,i}(K,-Q)$, for all $Q \in \mathcal{K}^n_{oo}$.
(iv) $W_{p,i}(K,Q) = W_{p,i}(K,-Q)$, for $Q = K$.

Proof. The results follow immediately from Theorem 8.5 with $\varphi(t) = t^p$ and $p > 1$.□

This was proved by Lutwak [21]. That (iii) implies that K is centrally symmetric, for the case $p = 1$ and $i = 0$, was shown (using other methods) by Goodey [10].

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