Locally maximal homoclinic classes for generic diffeomorphisms

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Abstract. Let M be a closed smooth $d(\geq 2)$ dimensional Riemannian manifold and let $f : M \to M$ be a diffeomorphism. For C^1 generic f, a locally maximal homogeneous homoclinic class is hyperbolic.

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1 Introduction

Let M be a closed smooth $d(\geq 2)$ dimensional Riemannian manifold and let $f : M \to M$ be a diffeomorphism. Denote by $\mathrm{Diff}(M)$ the set of all diffeomorphisms of M endowed with the C^1 topology. Let Λ be a closed f invariant set. We say that Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s_x}\| \leq C\lambda^n \text{ and } \|D_x f^{-n}|_{E^u_x}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. If $\Lambda = M$ then f is said to be *Anosov.* A point $p \in M$ is *periodic* if there is $n > 0$ such that $f^n(p) = p$. Denote by $P(f)$ the set of all periodic points of f. It is well known that if p is a hyperbolic periodic point of f with period $\pi(p)$ then the sets

$$W^s(p) = \{x \in M : f^{\pi(p)n}(x) \to p \text{ as } n \to \infty\} \quad \text{and}$$
$$W^u(p) = \{x \in M : f^{-\pi(p)n}(x) \to p \text{ as } n \to \infty\}$$

are C^1 injectively immersed submanifolds of M. A point $x \in W^s(p) \pitchfork W^u(p)$ is called a *homoclinic point* of f associated to p. The closure of the homoclinic points of f associated to p is called the *homoclinic class* of f associated to p, and it is denoted by $H_f(p)$. It is known that $H_f(p)$ is closed, transitive and f-invariant sets. Let p and q be hyperbolic periodic points. We write $p \sim q$ if $W^s(p) \pitchfork W^u(q) \neq \varnothing$ and $W^u(p) \pitchfork W^s(q) \neq \varnothing$. We say that $p, q \in P(f)$ are *homoclinically related* if $p \sim q$. It

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is clear that if $q \sim p$ then index(p) = index(q), where index$(p) = \dim W^s(p)$. Note that a hyperbolic $p \in P(f)$, there are a C^1 neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of p such that for any $g \in \mathcal{U}(f)$, $p_g = \bigcap_{n\in\mathbb{Z}} g^n(U)$, where p_g is said to be *continuation* of p.

We say that the homoclinic class $H_f(p)$ is *homogeneous* if index(p) = index(q), for any hyperbolic $q \in H_f(p) \cap P(f)$, We say that Λ is *locally maximal* if there is a neighborhood U of Λ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$. Here the neighborhood U is called *locally maximal neighborhood* of Λ.

We say that a subset $\mathcal{G} \subset \mathrm{Diff}(M)$ is *residual* if $\mathcal{G}$ contains the intersection of a countable family of open and dense subsets of Diff(M). In this case $\mathcal{G}$ is dense in Diff(M). A property "P" is said to be *(C^1) generic* if "P" holds for all diffeomorphisms which belong to some residual subset of Diff(M).

We say that Λ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E(x)}\| \cdot \|D_x f^{-n}|_{F(f^n(x))}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. An invariant closed set Λ is called a *chain transitive* if for any $\delta > 0$ and $x, y \in \Lambda$, there is δ-pseudo orbit $\{x_i\}_{i=0}^n (n \geq 1) \subset \Lambda$ such that $x_0 = x$ and $x_n = y$. Abdenur *et al* [1] proved that C^1 generically, any chain-transitive set Λ of f, then either (a) there is a dominated splitting over Λ or (b) the set Λ is contained is the Hausdorff limit of a sequence of periodic sinks/sources of f. Recently, Lee [9] proved that C^1 generically, if a chain transitive set Λ is locally maximal then it admits a dominated splitting. We say that Λ is *Lyapunov stable* for f if for every neighborhood U of Λ there is another neighborhood V of Λ such that $f^n(V) \subset U$ for any $n \geq 1$. We say that Λ is *bi-Lyapunov stable* if it is Lyapunov stalbe for f and for f^{-1}. Potrie and Sambarino [12] proved that C^1 generic diffeomorphisms with a homoclinic class with non empty interior and in particular those admitting a codimension one dominated splitting. Potrie [13] proved that for C^1 generic $f \in \mathrm{Diff}(M)$, a Lyapunov stable homolinic class $H_f(p)$ admits a dominated splitting. Wang [14] proved that for C^1 generic $f \in \mathrm{Diff}(M)$, where M is connected, if a homoclinic class $H_f(p)$ is bi-Lyapunov stable, then we have: either $H_f(p)$ is hyperbolic, and so, $H_f(p) = M$ and f is Anosov, or f can be C^1 approximated by diffeomorphisms that have a heterodimensional cycle. From the results, we prove the following.

Theorem A *For C^1 generic $f \in \mathrm{Diff}(M)$, a locally maximal homogeneous homoclinic class $H_f(p)$ is hyperbolic, for some hyperbolic $p \in P(f)$.*

From Theorem A, we directly obtained the previous results ([3, 7, 8]). Moredetail, C^1 generically, if a diffeomorphism f has the shadowing or limit shadowing property on a locally maximal $H_f(p)$ then $H_f(p)$ is homogeneous. Thus we can easily show that C^1 generically, if a diffeomorphsm f has the shadowing property ([3, 7]), or the limit shadowing property ([8]) on a locally maximal homoclinic class then it is hyperbolic.

2 Proof of Theorem A

Let M be as before, and let $f \in \mathrm{Diff}(M)$. For $\delta > 0$, a sequence of points $\{x_i\}_{i=0}^n (n \geq 1)$ in M is called a *δ-pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for $i = 0, \ldots, n$. For given $x, y \in M$, we write $x \rightsquigarrow y$ if for any $\delta > 0$, there is a δ-pseudo orbit $\{x_i\}_{i=0}^n (n \geq 1)$ of f such that $x_0 = x$ and $x_n = y$. The set $\{x \in M : x \rightsquigarrow x\}$ is called the *chain recurrent set* of f and is denoted by $\mathcal{CR}(f)$. The relation $\leftrightsquigarrow$ induces an equivalence relation on $\mathcal{CR}(f)$ whose classes are called *chain recurrence classes* of f. For any hyperbolic periodic point p, denote by $C_f(p) = \{x \in M : x \rightsquigarrow p \text{ and } p \rightsquigarrow x\}$. The chain recurrent class $C_f(p)$ is a closed and invariant set. In general, the homoclinic class $H_f(p)$ contained in the chain recurrence class $C_f(p)$.

Lemma 2.1. *There is a residual set $\mathcal{G}_1 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}_1$,*

(a) f is Kupka-Smale, that is, any element of $P(f)$ is hyperbolic, and its invariant manifolds intersect transversely (see [11]).

(b) the chain recurrence class $C_f(p)$ is the homoclinic class $H_f(p)$, for some hyperbolic periodic point p (see [4]).

(c) an isolated chain recurrence class $C_f(p)$ is robustly isolated, that is, there are a C^1 neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of $C_f(p)$ such that for every $g \in \mathcal{U}(f)$, $\mathcal{CR}(g) \cap U = C_g(p_g)$ (see [5]).

(d) if for any C^1 neighborhood $\mathcal{U}(f)$ of f there is $g \in \mathcal{U}(f)$ such that g has two periodic points p and q with $\mathrm{index}(p) \neq \mathrm{index}(q)$ then f has two periodic points p_f and q_f with $\mathrm{index}(p_f) \neq \mathrm{index}(q_f)$ (see [10]).

For any $\delta > 0$, we say that a hyperbolic $p \in P(f)$ has a δ *weak eigenvalue* if there is an eigenvalue λ of $D_p f^{\pi(p)}$ such that

$$(1-\delta)^{\pi(p)} < |\lambda| < (1+\delta)^{\pi(p)},$$

where $\pi(p)$ is the periodic of p. The following lemma was proved by Wang [14].

Lemma 2.2. *There is a residual set $\mathcal{G}_2 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}_2$, if a homoclinic class $H_f(p)$ is not hyperbolic then there is a hyperbolic periodic point $q \in H_f(p)$ with $q \sim p$ such that q has a Lyapunov exponent arbitrarily close to* 0.

By Lemma 2.2, the hyperbolic periodic point $q \in H_f(p)$ is said to be a *weak hyperbolic periodic point* if the hyperbolic periodic point $q \in H_f(p)$ has a Lyapunov exponent arbitrarily close to 0. The notion of weak hyperbolic periodic point is a δ weak eigenvalue for the hyperbolic periodic point. We say that a periodic point p is said to be *weak hyperbolic* if p has a δ weak eigenvalue. We rewrite the result of Wang as the following.

Lemma 2.3. *There is a residual set $\mathcal{G}_2 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}_2$, any hyperbolic periodic point p of f, if a homoclinic class $H_f(p)$ is not hyperbolic then there are $\delta > 0$, and a hyperbolic periodic point $q \in H_f(p)$ with $q \sim p$ such that q is a weak hyperbolic.*

The following Franks' lemma [6] will play essential roles in our proofs.

Lemma 2.4. *Let $\mathcal{U}(f)$ be any given C^1 neighborhood of f. Then there exist $\epsilon > 0$ and a C^1 neighborhood $\mathcal{V}(f) \subset \mathcal{U}(f)$ of f such that for given $g \in \mathcal{V}(f)$, a finite set $\{x_1, x_2, \cdots, x_k\}$, a neighborhood U of $\{x_1, x_2, \cdots, x_k\}$ and linear maps $L_i : T_{x_i}M \to T_{g(x_i)}M$ satisfying $\|L_i - D_{x_i}g\| \le \epsilon$ for all $1 \le i \le k$, there exists $\widetilde{g} \in \mathcal{U}(f)$ such that $\widetilde{g}(x) = g(x)$ if $x \in \{x_1, x_2, \cdots, x_k\} \cup (M \setminus U)$ and $D_{x_i}\widetilde{g} = L_i$ for all $1 \le i \le k$.*

Lemma 2.5. *Let $\mathcal{U}(f)$ be a C^1 neighborhood of f and let U be a locally maximal neighborhood of $H_f(p)$. If a weak periodic point $q \in H_f(p)$ then there are $g \in \mathcal{U}(f)$ and $q_1 \in \Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$ such that* $\text{index}(q_1) \neq \text{index}(p_g)$*, where $\Lambda_g(U)$ is the continuation of $H_f(p)$.*

Proof. Let $\mathcal{U}(f)$ be a C^1 neighborhood of f and let U be a locally maximal neighborhood of $H_f(p)$. Suppose that there is a periodic point $q \in H_f(p)$ with the period $\pi(q)$ such that q is a weak hyperbolic. For simplicity, we may assume that $f^{\pi(q)}(q) = f(q) = q$. Since $q \in H_f(p)$ is a weak hyperbolic periodic point, for any $\delta > 0$ there is an eigenvalue λ of D_qf such that

$$(1-\delta) < |\lambda| < (1+\delta) \text{ and } q \sim p.$$

By Lemma 2.4, there is g C^1 close to f such that $g(p) = f(p) = p$ and D_pg has an eigenvalue λ such that $|\lambda| = 1$. Note that by Lemma 2.4, there is g_1 C^1 close to f such that D_pg_1 has only one eigenvalue λ with $|\lambda| = 1$. Denote by E^c_p the eigenspace corresponding to λ. In the proof we consider two cases : (i) λ is real, and (ii) λ is complex.
First, we may assume that $\lambda \in \mathbb{R}$ (other case is similar). By Lemma 2.4, there are $\alpha > 0$, $B_\alpha(p) \subset U$ and h C^1 close to g ($h \in \mathcal{U}(f)$) such that

- $h(p) = g(p) = p$,
- $h(x) = \exp_p \circ D_pg \circ \exp_p^{-1}(x)$ for $x \in B_\alpha(p)$, and
- $h(x) = g(x)$ for $x \notin B_{4\alpha}(p)$.

Let $\eta = \alpha/4$. Take a nonzero vector $v \in \exp_p(E^c_p(\alpha))$ which is corresponding to λ such that $\|v\| = \eta$. Here $E^c_p(\alpha)$ is the α-ball in E^c_p with its center at $\overrightarrow{0_p}$. Then we have

$$h(\exp_p(v)) = \exp_p \circ D_pg \circ \exp_p^{-1}(\exp_p(v)) = \exp_p(v).$$

Put $\mathcal{J}_p = \exp_p(\{tv : -\eta/4 \le t \le \eta/4\})$. Then $\mathcal{J}_p$ is center at p and $h(\mathcal{J}_p) = \mathcal{J}_p$. Since $B_\alpha(p) \subset U$ we know that $\mathcal{J}_p \subset \Lambda_h(U) = \bigcap_{n\in\mathbb{Z}} h^n(U)$. Since $h(\mathcal{J}_p) = \mathcal{J}_p$, take two end points q, r of $\mathcal{J}_p$. Then we know that

$$D_qh|_{E^c_p} = D_rh|_{E^c_p} = 1.$$

By Lemma 2.4, there is ϕ C^1 close to h ($\phi \in \mathcal{U}(f)$) such that $\text{index}(q_\phi) \neq \text{index}(r_\phi)$, where q_ϕ and r_ϕ are hyperbolic points with respect to ϕ.
Finally, we consider $\lambda \in \mathbb{C}$. For simplicity, we assume that $f(p) = p$. As in the proof of the case of $\lambda \in \mathbb{R}$, by Lemma 2.4, there are $\alpha > 0, B_\alpha(p) \subset U$ and $g \in \mathcal{U}(f)$ such that

$$g(p) = f(p) = p \text{ and } g(x) = \exp_p \circ D_pg \circ \exp_p^{-1}(x)$$

for $x \in B_\alpha(p)$. Since $\lambda = 1$, there is $n > 0$ such that $D_p g^n(v) = v$ for any $v \in \exp_p^{-1}(E_p^c(\alpha))$. Let $v \in \exp_p(E_p^c(\alpha))$ such that $\|v\| = \alpha/4$. Then we have a small arc

$$\exp_p(\{tv : 0 \le t \le 1 + \alpha/4\}) = \mathcal{I}_p \subset \Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$$

such that

(i) $g^i(\mathcal{I}_p) \cap g^j(\mathcal{I}_p) = \varnothing$ if $0 \le i \ne j \le n-1$,

(ii) $g^n(\mathcal{I}_p) = \mathcal{I}_p$, and

(iii) $g^n_{|\mathcal{I}_p} : \mathcal{I}_p \to \mathcal{I}_p$ is the identity map.

Then we take two point $q, r \in \mathcal{I}_p$ such that the points are the end points of $\mathcal{I}_p$. As in the previous arguments, there is g_1 C^1 close to g such that index$(q_{g_1}) \ne$ index(r_{g_1}) where q_{g_1} and r_{g_1} are hyperbolic with respect to g_1. □

Proof of Theorem A. Let $f \in \mathcal{G} = \mathcal{G}_1 \cap \mathcal{G}_2$ and let p be a hyperbolic periodic point of f. Suppose, by contradiction, that a homogeneous homoclinic class $H_f(p)$ is not hyperbolic. Since $H_f(p)$ is homogeneous, we assume that index$(p) = j$. Let U be a locally maximal neighborhood of $H_f(p)$. Since $H_f(p)$ is not hyperbolic, by Lemma 2.3 there is a periodic point $q \in H_f(p)$ with $q \sim p$ such that q is a weak hyperbolic point. Since $H_f(p)$ is locally maximal in U, by Lemma 2.5 there is g C^1 close to f such that g has two hyperbolic periodic points $r, s \in \Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$ with index$(r) \ne$ index(s). Since $f \in \mathcal{G}_1$, $H_f(p) = C_f(p)$ and it is robust isolated, we have that

$$\bigcap_{n\in\mathbb{Z}} g^n(\mathcal{CR}(g) \cap U) = \mathcal{CR}(g) \cap \Lambda_g(U) \subset \mathcal{CR}(g) \cap U = C_g(p_g) = H_g(p_g), \tag{2.1}$$

where p_g is the continuation of p.

Since $r, s \in \Lambda_g(U)$ as hyperbolic periodic points of g, we know that $r, s \in \mathcal{CR}(g) \cap U$. Then by (1) we have

$$r, s \in \mathcal{CR}(g) \cap \Lambda_g(U) \subset \mathcal{CR}(g) \cap U = H_g(p_g) = C_g(p_g).$$

Thus we have $r, s \in H_g(p_g)$ with index$(s) \ne$ index(r). By Lemma 2.1, we have two hyperbolic periodic points $r_f, s_f \in H_f(p)$ with index$(r_f) \ne$ index(s_f). Since index$(p) = j$, we know that either index$(r_f) \ne j$ or index$(s_f) \ne j$. This is a contradiction since $H_f(p)$ is homogeneous. □

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