Hypersurfaces with parallel Laguerre form in $\mathbb{R}^n$

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Abstract. For a given $(n-1)$-dimensional hypersurface $x : M \to \mathbb{R}^n$, consider the Laguerre form $\mathbf{\Phi}$, the Laguerre tensor $\mathbf{L}$ and the Laguerre second fundamental form $\mathbf{B}$ of the immersion x. In this article, we address the case when the Laguerre form of x is parallel, i.e., $\nabla\mathbf{\Phi} \equiv 0$. We prove that $\nabla\mathbf{\Phi} \equiv 0$ is equivalent to $\mathbf{\Phi} \equiv 0$, provided that either $\mathbf{L}+\lambda\mathbf{B}+\mu g = 0$ for some smooth function λ and μ, or x has constant Laguerre eigenvalues, or x has constant para-Laguerre eigenvalues, where ∇ is the Levi-Civita connection of the Laguerre metric g.

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1 Introduction

The Laguerre geometry of surfaces in $\mathbb{R}^3$ was studied by Blaschke [1], and by other authors (see Musso and Nicolodi [8], [9]). In the Laguerre geometry of submanifolds in Euclidean space $\mathbb{R}^n$, Li and Wang [4] investigated the invariants of the hypersurfaces in $\mathbb{R}^n$ under the Laguerre transformation group. We recall that the Laguerre transformations are the Lie sphere transformations which take oriented hyperplanes in $\mathbb{R}^n$ to oriented hyperplanes and preserve the tangential distance.

Let $U\mathbb{R}^n$ be the unit tangent bundle over $\mathbb{R}^n$. An oriented sphere in $\mathbb{R}^n$ centered at p with radius r can be regarded as the oriented sphere $\{(x,\xi)|x-p=r\xi\}$ in $U\mathbb{R}^n$, where x is the position vector and ξ the unit normal vector of the sphere. An oriented hyperplane in $\mathbb{R}^n$ with a constant unit normal vector ξ and a constant real number c can be regarded as the oriented hyperplane $\{(x,\xi)|x\cdot\xi=c\}$ in $U\mathbb{R}^n$. A diffeomorphism $\phi : U\mathbb{R}^n \to U\mathbb{R}^n$ which takes oriented spheres to oriented spheres, oriented hyperplanes to oriented hyperplanes, preserving the tangential distance of any two spheres, is called a *Laguerre transformation.* All the Laguerre transformations in $U\mathbb{R}^n$ form a group of dimension $(n+1)(n+2)/2$, called *the Laguerre transformation group.*

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An oriented hypersurface $x : M \to \mathbb{R}^n$ can be identified as the submanifold $(x, \xi) : M \to U\mathbb{R}^n$, where ξ is the unit normal of x. Two hypersurfaces $x, x^* : M \to \mathbb{R}^n$ are called *Laguerre equivalent*, if there is a Laguerre transformation $\phi : U\mathbb{R}^n \to U\mathbb{R}^n$ so that $(x^*, \xi^*) = \phi \circ (x, \xi)$ (see [4]). Li and Wang [4] gave a complete Laguerre invariant system of hypersurfaces in $\mathbb{R}^n$ and proved that two umbilical free oriented hypersurfaces in $\mathbb{R}^n$ with non-zero principal curvatures are Laguerre equivalent if and only if they have the same Laguerre metric g and Laguerre second fundamental form $\mathbf{B}$.

From Li and Wang [4], we know that the *Laguerre metric* g of the immersion x can be defined by $g = \langle dY, dY\rangle$. Let $\{E_1, E_2, \ldots, E_{n-1}\}$ be an orthonormal basis for g with dual basis $\{\omega_1, \omega_2, \ldots, \omega_{n-1}\}$. The *Laguerre form* $\mathbf{\Phi}$, *Laguerre tensor* $\mathbf{L}$ and *the Laguerre second fundamental form* $\mathbf{B}$ of the immersion x are defined by

$$\mathbf{\Phi} = \sum_{i=1}^{n-1} C_i\omega_i, \quad \mathbf{L} = \sum_{i,j=1}^{n-1} L_{ij}\omega_i \otimes \omega_j, \quad \mathbf{B} = \sum_{i,j=1}^{n-1} B_{ij}\omega_i \otimes \omega_j, \tag{1.1}$$

respectively, where C_i, L_{ij} and B_{ij} are defined by

$$C_i = -\rho^{-2}\big\{\tilde{E}_i(r) - \tilde{E}_i(\log\rho)(r_i - r)\big\}, \tag{1.2}$$

$$L_{ij} = \rho^{-2}\big\{\mathrm{Hess}_{ij}(\log\rho) - \tilde{E}_i(\log\rho)\tilde{E}_j(\log\rho) + \frac{1}{2}\left(|\nabla \log\rho|^2 - 1\right)\delta_{ij}\big\}, \tag{1.3}$$

$$B_{ij} = \rho^{-1}(r_i - r)\delta_{ij}, \tag{1.4}$$

where $g = \sum_i (r_i - r)^2 III = \rho^2 III$, r_i and r are the curvature radii and mean curvature radius of x, respectively. We define a symmetric $(0, 2)$ tensor

$$\mathbf{D} = \mathbf{L} + \lambda\mathbf{B}, \tag{1.5}$$

which is called *the para-Laguerre tensor* of x, where λ is a constant. We notice that g, $\mathbf{\Phi}$, $\mathbf{L}$, $\mathbf{B}$ and $\mathbf{D}$ are Laguerre invariants (see [4]).

We call an eigenvalue of the Laguerre second fundamental form *a Laguerre principal curvature*, an eigenvalue of the Laguerre tensor *a Laguerre eigenvalue*, an eigenvalue of the para-Laguerre tensor *a para-Laguerre eigenvalue* of x. An umbilic free hypersurface $x : M \to \mathbb{R}^n$ with non-zero principal curvatures and vanishing Laguerre form $\mathbf{\Phi} \equiv 0$ is called *a Laguerre isoparametric hypersurface* if the Laguerre principal curvatures of x are constants. A hypersurface with a vanishing Laguerre form is called *a Laguerre isotropic hypersurface*, if the Laguerre eigenvalues of x are equal. We should notice that the Laguerre form $\mathbf{\Phi} \equiv 0$ plays an important role in the definitions of Laguerre isoparametric hypersurfaces and Laguerre isotropic hypersurfaces. In the study of Laguerre isoparametric hypersurfaces and Laguerre isotropic hypersurfaces, there have been many recent studies (see [3, 6, 10–12]). In [3] and [6], Li et al. obtained the complete classifications of all oriented Laguerre surfaces in $\mathbb{R}^3$ and all oriented Laguerre isoparametric hypersurfaces in $\mathbb{R}^4$. In [10]–[12], we firstly obtained the classification of Laguerre isoparametric hypersurfaces in $\mathbb{R}^n$ with three distinct Laguerre principal curvatures, one of which is simple and then we obtained the complete classifications of all oriented Laguerre isoparametric hypersurfaces in $\mathbb{R}^5$ and $\mathbb{R}^6$. In [5], Li, H. Li and Wang obtained the classification of all the Laguerre isotropic hypersurfaces.

If $\nabla\mathbf{\Phi} = \sum_{i,j} C_{i,j}\omega_i \otimes \omega_j \equiv 0$, we call x has *parallel Laguerre form*, where ∇ is the Levi-Civita connection of the Laguerre metric g. We notice that if $\mathbf{\Phi} \equiv 0$ then $\nabla\mathbf{\Phi} \equiv 0$, conversely, if $\nabla\mathbf{\Phi} \equiv 0$ then $\mathbf{\Phi} \equiv 0$ not necessarily holds. Thus, we see that the condition $\nabla\mathbf{\Phi} \equiv 0$ is weaker than $\mathbf{\Phi} \equiv 0$. Then the next question follows: in what conditions may we have $\nabla\mathbf{\Phi} \equiv 0$ if and only if $\mathbf{\Phi} \equiv 0$?

In this article, we try to give some answers to the above question. We notice that Fang [2] and Zhong et al. [13] recently proved independently that if the Laguerre principal curvatures of x are constants, then $\nabla\mathbf{\Phi} \equiv 0$ if and only if $\mathbf{\Phi} \equiv 0$. Since we know that the Laguerre eigenvalues and the para-Laguerre eigenvalues of x are also the important Laguerre invariants, we prove the following results:

Theorem 1.1. *Let $x : M \to \mathbb{R}^n$ be an umbilic free hypersurface with non-zero principal curvatures. If $\mathbf{L} + \lambda\mathbf{B} + \mu g = 0$ for some smooth function λ and μ, then $\nabla\mathbf{\Phi} \equiv 0$ if and only if $\mathbf{\Phi} \equiv 0$.*

Theorem 1.2. *Let $x : M \to \mathbb{R}^n$ be an umbilic free hypersurface with non-zero principal curvatures. If the Laguerre eigenvalues of x are constants, then $\nabla\mathbf{\Phi} \equiv 0$ if and only if $\mathbf{\Phi} \equiv 0$.*

Theorem 1.3. *Let $x : M \to \mathbb{R}^n$ be an umbilic free hypersurface with non-zero principal curvatures. If the para-Laguerre eigenvalues of x are constants, then $\nabla\mathbf{\Phi} \equiv 0$ if and only if $\mathbf{\Phi} \equiv 0$.*

Thus, from Theorem 1.2, Theorem 1.3 and Theorem 1.1 of [2] or [13], we easily see that

Theorem 1.4. *Let $x : M \to \mathbb{R}^n$ be an umbilic free hypersurface with non-zero principal curvatures. If the Laguerre principal curvatures, or the Laguerre eigenvalues, or the para-Laguerre eigenvalues of x are constants, then $\nabla\mathbf{\Phi} \equiv 0$ if and only if $\mathbf{\Phi} \equiv 0$.*

Remark 1.1. If $\lambda \equiv 0$, then $\mathbf{L} + \mu g = 0$ and x is a Laguerre isotropic hypersurface, we see that Theorem 1.1 reduce to (2) of Theorem 1.1 of Zhong et al. [13]. From Theorem 1.1, we see that if we replace $\mathbf{\Phi} \equiv 0$ by the weaker condition $\nabla\mathbf{\Phi} \equiv 0$ in the definition of Laguerre isotropic hypersurfaces, then Theorem 1.1 of Li, H. Li and Wang [5] also holds.

Remark 1.2. From Theorem 1.2 and Theorem 1.3, we see that if we replace $\mathbf{\Phi} \equiv 0$ by the weaker condition $\nabla\mathbf{\Phi} \equiv 0$ in Theorem 1.2 of [5] and Theorem 1.4 of [9], then Theorem 1.2 of [5] and Theorem 1.4 of [9] also hold.

2 Fundamental formulas of Laguerre Geometry

We recall the fundamental formulas on Laguerre geometry of hypersurfaces in $\mathbb{R}^n$. Let $x : M \to \mathbb{R}^n$ be an $(n-1)$-dimensional umbilical free hypersurface with vanishing Laguerre form in $\mathbb{R}^n$. Let $\{E_1, \ldots, E_{n-1}\}$ denote a local orthonormal frame for Laguerre metric $g = \langle dY, dY\rangle$ with dual frame $\{\omega_1, \ldots, \omega_{n-1}\}$. Putting $Y_i = E_i(Y)$, we have

$$N = \frac{1}{n-1}\Delta Y + \frac{1}{2(n-1)^2}\langle \Delta Y, \Delta Y\rangle Y, \tag{2.1}$$

$$\langle Y, Y\rangle = \langle N, N\rangle = 0, \quad \langle Y, N\rangle = -1, \tag{2.2}$$

and the following orthogonal decomposition:

$$\mathbb{R}^{n+3}_2 = \mathrm{Span}\{Y, N\} \oplus \mathrm{Span}\{Y_1, \ldots, Y_{n-1}\} \oplus \mathbb{V}, \tag{2.3}$$

where $\{Y, N, Y_1, \ldots, Y_{n-1}, \eta, \wp\}$ forms a moving frame in $\mathbb{R}^{n+3}_2$ and $\mathbb{V} = \{\eta, \wp\}$ is called *the Laguerre normal bundle* of x. We use the following range of indices throughout this paper: $1 \le i, j, k, l, m \le n-1$.

The structure equations of x with respect to the Laguerre metric g can be written as

$$dY = \sum_i \omega_i Y_i, \tag{2.4}$$

$$dN = \sum_i \psi_i Y_i + \varphi\eta, \tag{2.5}$$

$$dY_i = \psi_i Y + \omega_i N + \sum_j \omega_{ij} Y_j + \omega_{in}\eta, \tag{2.6}$$

$$d\wp = -\varphi Y + \sum_i \omega_{in} Y_i, \tag{2.7}$$

where $\{\psi_i, \omega_{ij}, \omega_{in}, \varphi\}$ are 1-forms on x with

$$\omega_{ij} + \omega_{ji} = 0, \quad d\omega_i = \sum_j \omega_j \wedge \omega_{ji}, \tag{2.8}$$

and

$$\psi_i = \sum_j L_{ij}\omega_j, \quad L_{ij} = L_{ji}, \quad \omega_{in} = \sum_j B_{ij}\omega_j, \quad B_{ij} = B_{ji}, \quad \varphi = \sum_i C_i\omega_i. \tag{2.9}$$

We define the covariant derivative of C_i, L_{ij}, B_{ij} by

$$\sum_j C_{i,j}\omega_j = dC_i + \sum_j C_j\omega_{ji}, \tag{2.10}$$

$$\sum_k L_{ij,k}\omega_k = dL_{ij} + \sum_k L_{ik}\omega_{kj} + \sum_k L_{kj}\omega_{ki}, \tag{2.11}$$

$$\sum_k B_{ij,k}\omega_k = dB_{ij} + \sum_k B_{ik}\omega_{kj} + \sum_k B_{kj}\omega_{ki}, \tag{2.12}$$

and using [4], we infer

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \quad R_{ijkl} = -R_{jikl}, \tag{2.13}$$

$$\sum_i B_{ii} = 0, \quad \sum_{i,j} B_{ij}^2 = 1, \quad \sum_i B_{ij,i} = (n-2)C_j, \quad \mathrm{tr}\mathbf{L} = -\frac{R}{2(n-2)}, \tag{2.14}$$

$$L_{ij,k} = L_{ik,j}, \tag{2.15}$$

$$C_{i,j} - C_{j,i} = \sum_k (B_{ik}L_{kj} - B_{jk}L_{ki}), \tag{2.16}$$

$$B_{ij,k} - B_{ik,j} = C_j\delta_{ik} - C_k\delta_{ij}, \tag{2.17}$$

$$R_{ijkl} = L_{jk}\delta_{il} + L_{il}\delta_{jk} - L_{ik}\delta_{jl} - L_{jl}\delta_{ik}, \tag{2.18}$$

where R_{ijkl} and R denote the Laguerre curvature tensor and the Laguerre scalar curvature with respect to the Laguerre metric g on x.

Denote by $\mathbf{D} = \sum_{i,j} D_{ij}\omega_i \otimes \omega_j$ the $(0,2)$ para-Laguerre tensor,

$$D_{ij} = L_{ij} + \lambda B_{ij}, \quad 1 \le i, j \le n, \tag{2.19}$$

where λ is a constant. The covariant derivative of D_{ij} is defined by

$$\sum_k D_{ij,k}\omega_k = dD_{ij} + \sum_k D_{ik}\omega_{kj} + \sum_k D_{kj}\omega_{ki}. \tag{2.20}$$

Defining the second covariant derivative of B_{ij} and C_i by

$$\sum_l B_{ij,kl}\omega_l = dB_{ij,k} + \sum_l B_{lj,k}\omega_{li} + \sum_l B_{il,k}\omega_{lj} + \sum_l B_{ij,l}\omega_{lk}, \tag{2.21}$$

$$\sum_k C_{ij,k}\omega_k = dC_{i,j} + \sum_k C_{i,k}\omega_{kj} + \sum_k C_{k,j}\omega_{ki}, \tag{2.22}$$

we have the Ricci identity

$$B_{ij,kl} - B_{ij,lk} = \sum_m B_{mj}R_{mikl} + \sum_m B_{im}R_{mjkl}, \tag{2.23}$$

$$C_{ij,k} - C_{ik,j} = \sum_m C_m R_{mijk}. \tag{2.24}$$

3 Proofs of Theorems

From (2.17) and (2.23), we see that

$$\begin{aligned} B_{ij,kk} =&(B_{ik,j} + C_j\delta_{ik} - C_k\delta_{ij})_{,k} = B_{ik,jk} + C_{j,k}\delta_{ik} - C_{k,k}\delta_{ij} \\ =&B_{kk,ij} + \sum_m B_{mk}R_{mijk} + \sum_m B_{im}R_{mkjk} \\ &+ C_{i,j}\delta_{kk} + C_{j,k}\delta_{ik} - C_{k,j}\delta_{ki} - C_{k,k}\delta_{ij}. \end{aligned} \tag{3.1}$$

From (2.14), (2.18) and (3.1), we have

$$\begin{aligned} \sum_{i,j,k} B_{ij}B_{ij,kk} =& \sum_{i,j,k,l} B_{ij}B_{lk}R_{lijk} + \sum_{i,j,k,l} B_{ij}B_{il}R_{lkjk} + n\sum_{i,j} B_{ij}C_{j,i} \\ =& \sum_{i,j,k,l} B_{ij}B_{lk}(L_{ij}\delta_{lk} + L_{lk}\delta_{ij} - L_{lj}\delta_{ik} - L_{ik}\delta_{lj}) \\ &+ \sum_{i,j,k,l} B_{ij}B_{il}(L_{kj}\delta_{lk} + L_{lk}\delta_{kj} - L_{lj}\delta_{kk} - L_{kk}\delta_{lj}) + n\sum_{i,j} B_{ij}C_{j,i} \\ =& -(n-1)\mathrm{tr}(LB^2) - \mathrm{tr}L + n\mathrm{tr}(B\nabla\Phi). \end{aligned} \tag{3.2}$$

Thus, from (2.14) and (3.2), we have

$$\begin{aligned} 0 =& \frac{1}{2}\Delta\Big(\sum_{i,j} B_{ij}^2\Big) = \sum_{i,j,k} B_{ij,k}^2 + \sum_{i,j,k} B_{ij}B_{ij,kk} \\ =& \sum_{i,j,k} B_{ij,k}^2 - (n-1)\mathrm{tr}(LB^2) - \mathrm{tr}L + n\mathrm{tr}(B\nabla\Phi). \end{aligned} \tag{3.3}$$

Proof of Theorem 1.1. If $\mathbf{\Phi} \equiv 0$, it is obvious that $\nabla\mathbf{\Phi} \equiv 0$. On the contrary, if $\nabla\mathbf{\Phi} \equiv 0$, that is, $C_{i,j} = 0$, for all i, j, from (2.16), we may choose the local orthonormal basis $E_1, E_2, \ldots, E_{n-1}$ to diagonalize the matrix (B_{ij}) and (L_{ij}), that is

$$B_{ij} = B_i\delta_{ij}, \quad L_{ij} = L_i\delta_{ij}. \tag{3.4}$$

Since $\mathbf{L} + \lambda\mathbf{B} + \mu g = 0$, we have $L_i = -\mu - \lambda B_i$. From (2.18) and (2.24), we see that

$$0 = \sum_l C_l R_{lijk} = C_j(L_j + L_i)\delta_{ik} - C_k(L_k + L_i)\delta_{ij}. \tag{3.5}$$

Putting $i = k \neq j$ in (3.5), we have

$$C_j(L_j + L_k) = 0, \quad k \neq j. \tag{3.6}$$

If there exists one point p on M, so that $\mathbf{\Phi} \neq 0$ at p, without loss of generality, we may assume $C_1 \neq 0$ at p, thus from (3.6), we see that $L_1 + L_k = 0$ at p, where $k \neq 1$. Since $L_i = -\mu - \lambda B_i$, we have $-\mu - \lambda B_1 - \mu - \lambda B_k = 0$ at p, where $k \neq 1$. Thus, at point p, we have

$$\lambda B_k = -(2\mu + \lambda B_1), \quad k \neq 1. \tag{3.7}$$

If $\lambda = 0$ at p, we see that $\mu = 0$ at p. Thus $L_i = 0$ at p for all i, which implies that $\mathrm{tr}L = 0$ at p and also $\mathrm{tr}(LB^2) = 0$ at p. From (3.3), we see that $B_{ij,k} = 0$ at p for all i, j, k. From (2.14), we have $C_1 = \sum_i B_{i1i} = 0$ at p, which is a contradiction.

If $\lambda \neq 0$ at p, from (3.7), we see that $B_2 = B_3 = \cdots = B_{n-1}$ at p. By (2.14), we know that

$$B_1 + (n-2)B_2 = 0, \quad B_1^2 + (n-2)B_2^2 = 1, \quad \text{at} \quad p.$$

Therefore,

$$B_1 = \mp\sqrt{\frac{n-2}{n-1}}, \quad B_2 = \pm\frac{1}{\sqrt{(n-1)(n-2)}}, \quad \text{at} \quad p. \tag{3.8}$$

From (2.12) and (3.8), we have

$$\sum_k B_{ij,k}\omega_k = (B_i - B_j)\omega_{ij}, \quad \text{at} \quad p. \tag{3.9}$$

Thus, at point p,

$$B_{ij,k} = 0, \quad \text{for} \quad 2 \leq i, j \leq n-1, \quad 1 \leq k \leq n-1. \tag{3.10}$$

Putting $i \neq j$, $i = k$ and $2 \leq i, j, k \leq n-1$ in (2.17), we have

$$C_j = 0, \quad \text{for} \quad 2 \leq j \leq n-1. \tag{3.11}$$

On the other hand, from (2.10) and (3.11), we have

$$0 = \sum_k C_{j,k}\omega_k = dC_j + \sum_k C_k\omega_{kj} = C_1\omega_{1j}, \quad \text{for} \quad 2 \leq j \leq n-1. \tag{3.12}$$

Since it is assumed $C_1 \neq 0$ at p, we have $\omega_{1j} = 0$ at p, $2 \leq j \leq n-1$. By (3.9), we see that $B_{1j,k} = 0$ at p for $2 \leq j \leq n-1$ and all k. Thus $B_{12,2} = 0$ at p and $B_{21,2} = 0$ at p. From (3.10), we have $B_{22,1} = 0$ at p. Putting $i = j = 2$ and $k = 1$ in (2.17), we have $C_1 = B_{21,2} - B_{22,1} = 0$ at p, which is a contradiction. Thus, it must have $\mathbf{\Phi} \equiv 0$. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. If $\mathbf{\Phi} \equiv 0$, it is obvious that $\nabla\mathbf{\Phi} \equiv 0$. On the contrary, if $\nabla\mathbf{\Phi} \equiv 0$, that is, $C_{i,j} = 0$, for all i, j, from (2.16), we may choose the local orthonormal basis $E_1, E_2, \ldots, E_{n-1}$ to diagonalize the matrix (B_{ij}) and (L_{ij}), that is, (3.4) holds.

(1) If the Laguerre eigenvalues of x are equal, from Theorem 1.1 (see Remark 1.1), we know that Theorem 1.2 is true.

(2) If the Laguerre eigenvalues of x are not equal, from (2.18) and (3.4), we know that

$$R_{ijkl} = 0, \quad \text{if three of } \{i, j, k, l\} \text{ are either the same or distinct.} \tag{3.13}$$

By (2.24) and (3.13), we obtain

$$C_i R_{ijij} = \sum_k C_k R_{kjij} = 0, \quad i \neq j. \tag{3.14}$$

If there exists one point p on M, so that $\mathbf{\Phi} \neq 0$ at p, without loss of generality, we may assume $C_1 \neq 0$ at p, thus from (3.14) and (2.18), we see that $0 = R_{1k1k} = -L_1 - L_k$ at p, where $k \neq 1$, that is, $L_k = -L_1, (k \neq 1)$ at p. Since the Laguerre eigenvalues of x are constants and not equal, we know that at all points of x,

$$L_k = -L_1 \neq 0, \quad k \neq 1. \tag{3.15}$$

Since $L_{ij,k} = L_{ik,j}$ and $L_k = -L_1 = \mathit{constant}$, from (2.11), we easily see that $L_{ij,k} = 0$ for all i, j, k. From (2.11) again, we have $(L_1 - L_j)\omega_{1j} = 0$ for $j \neq 1$, thus

$$\omega_{1j} = 0, \quad j \neq 1. \tag{3.16}$$

Taking exterior differential of (2.6) and by (2.4)–(2.7), we have

$$d\psi_i - \sum_j \psi_j \wedge \omega_{ji} + \omega_{in} \wedge \varphi = 0. \tag{3.17}$$

Since it is assumed $C_1 \neq 0$ at p, we must have $C_2 = C_3 = \cdots = C_{n-1} = 0$ at p. In fact, if there is $i_0 (2 \leq i_0 \leq n-1)$ such that $C_{i_0} \neq 0$ at p, from (3.14) and (2.18), we have

$$\begin{aligned} &- L_1 - L_k = 0, \quad k \neq 1, \quad k \neq i_0 \\ &- L_{i_0} - L_k = 0, \quad k \neq i_0, \quad k \neq 1, \\ &- L_1 - L_{i_0} = 0, \quad i_0 \neq 1. \end{aligned}$$

Thus, we have $L_1 = L_{i_0} = L_k = 0$, $k \neq 1, k \neq i_0$, which is a contradiction. Therefore, from (2.9), we have $\varphi = C_1\omega_1$ at p. By (3.17), (2.8), (2.9), (3.15) and (3.16), we see that for $i \neq 1$

$$-\omega_{in} \wedge \varphi = L_i d\omega_i - \sum_j L_j \omega_j \wedge \omega_{ji} \tag{3.18}$$
$$= - L_1 d\omega_i - L_1 \omega_1 \wedge \omega_{1i} - \sum_{j=2}^{n-1} L_j \omega_j \wedge \omega_{ji}$$
$$= - L_1 d\omega_i + L_1 \omega_1 \wedge \omega_{1i} + L_1 \sum_{j=2}^{n-1} \omega_j \wedge \omega_{ji}$$
$$= - L_1 (d\omega_i - \sum_j \omega_j \wedge \omega_{ji}) = 0.$$

Thus, from (2.9) and (3.18), we have

$$-B_i C_1 \omega_i \wedge \omega_1 = 0, \quad \text{at} \quad p, \quad i \neq 1. \tag{3.19}$$

Since $C_1 \neq 0$ at p and $i \neq 1$, from (3.19), we have $B_i = 0$ at p, $i \neq 1$. From (2.14), we see that $B_1 = 0$ at p, this contradicts with $\sum_i B_i^2 = 1$. Thus, it must have $\mathbf{\Phi} \equiv 0$. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. If $\mathbf{\Phi} \equiv 0$, it is obvious that $\nabla \mathbf{\Phi} \equiv 0$. On the contrary, if $\nabla \mathbf{\Phi} \equiv 0$, that is, $C_{i,j} = 0$, for all i, j, from (2.16) and (2.19), we can choose the local orthonormal basis $E_1, E_2, \ldots, E_{n-1}$ to diagonalize the matrix (B_{ij}), (L_{ij}) and (D_{ij}).

(1) If the para-Laguerre eigenvalues of x are equal, from Theorem 1.1, we know that Theorem 1.3 is true.

(2) If the para-Laguerre eigenvalues of x are not equal, since $\mathbf{D} = \mathbf{L} + \lambda \mathbf{B}$, if $\lambda = 0$, from Theorem 1.2, we know that Theorem 1.3 is true. If $\lambda \neq 0$, in order to prove $\mathbf{\Phi} \equiv 0$, we may consider the following four independent cases (i), (ii), (iii) and (iv).

(i) If $n \geq 3$ and x has two distinct para-Laguerre eigenvalues, we also consider the following two cases: $n = 3$ and $n \geq 4$.

If $n = 3$, from the Ricci identity (2.24), we have

$$\sum_k C_k R_{k212} = 0, \quad \sum_k C_k R_{k121} = 0. \tag{3.20}$$

Thus, we have from (3.13) and (3.20) that

$$C_1 R_{1212} = 0, \quad C_2 R_{1212} = 0.$$

If there exists one point p on M, so that $\mathbf{\Phi} \neq 0$ at p, without loss of generality, we may assume $C_1 \neq 0$ at p, thus $R_{1212} = 0$ at p. From (2.18), we have $\mathrm{tr} L = L_1 + L_2 = 0$ at p. Thus, from (2.14), we have $D_1 + D_2 = L_1 + L_2 + \lambda(B_1 + B_2) = 0$ at p, that is $D_1 = -D_2$ at p. Thus, $0 = D_1^2 - D_2^2 = L_1^2 - L_2^2 + 2\lambda L_1 (B_1 + B_2) + \lambda^2 (B_1^2 - B_2^2) = \lambda^2 (B_1^2 - B_2^2)$ at p. Since $\lambda \neq 0$, we have $B_1^2 = B_2^2$ at p. Thus, $\mathrm{tr}(LB^2) = L_1 B_1^2 + L_2 B_2^2 = B_1^2 \mathrm{tr} L = 0$ at p. From (3.3), we see that $B_{ij,k} = 0$ at p for all i, j, k. From (2.14), we have $C_1 = \sum_i B_{i1,i} = 0$ at p, which is a contradiction.

If $n \geq 4$, let D_1 and D_2 be the two distinct para-Laguerre eigenvalues of x with multiplicities m_1 and $n-1-m_1$, respectively. We agree on the following ranges of indices

$$1 \leq a, b \leq m_1, \quad m_1+1 \leq \alpha, \beta \leq n-1.$$

From (2.15), (2.17) and (2.19), we have

$$D_{ij,k} - D_{ik,j} = \lambda C_j \delta_{ik} - \lambda C_k \delta_{ij}. \tag{3.21}$$

Since D_1 and D_2 are constants, from (2.20), we have

$$D_{ab,i} = D_{\alpha\beta,i} = 0, \quad \sum_k D_{a\alpha,k}\omega_k = (D_1 - D_2)\omega_{a\alpha}. \tag{3.22}$$

From (2.21) and (3.22), we also have

$$D_{\alpha a,b} = D_{a\alpha,b} = \lambda C_\alpha \delta_{ab}, \quad D_{a\alpha,\beta} = D_{\alpha a,\beta} = \lambda C_a \delta_{\alpha\beta}. \tag{3.23}$$

By (2.14) and (2.19), we also have

$$\begin{aligned} \sum_i D_{ij,i} =& \sum_i (L_{ij,i} + \lambda B_{ij,i}) = \sum_i (L_{ii,j} + \lambda B_{ij,i}) \\ =& \Big(\sum_i L_{ii}\Big)_{,j} + \lambda \sum_i B_{ij,i} \\ =& \Big(\sum_i D_{ii}\Big)_{,j} + \lambda \sum_i B_{ij,i} = (n-2)\lambda C_j. \end{aligned} \tag{3.24}$$

By (3.22), (3.23) and (3.24), we get

$$(n-2)\lambda C_a = \sum_i D_{ia,i} = \sum_\alpha D_{\alpha a,\alpha} = (n-1-m_1)\lambda C_a, \tag{3.25}$$

$$(n-2)\lambda C_\alpha = \sum_i D_{i\alpha,i} = \sum_a D_{a\alpha,a} = m_1 \lambda C_\alpha. \tag{3.26}$$

Thus, we see that

$$(m_1 - 1)\lambda C_a = 0, \quad (m_1 - n + 2)\lambda C_\alpha = 0. \tag{3.27}$$

If there exists one point p on M, so that $\mathbf{\Phi} \neq 0$ at p, then there must be some i such that $C_i \neq 0$ at p, $1 \leq i \leq n-1$.

If $1 \leq i \leq m_1$, from (3.27) and $\lambda \neq 0$, we see that $m_1 = 1$, thus, $C_\alpha = 0$. We get that $\Phi = C_1\omega_1$ and $C_1 \neq 0$ at p. On the other hand, from (2.10) and $C_\alpha = 0$, $2 \leq \alpha \leq n-1$, we have

$$0 = \sum_j C_{\alpha,j}\omega_j = dC_\alpha + \sum_j C_j \omega_{j\alpha} = C_1 \omega_{1\alpha}.$$

Hence $\omega_{1\alpha} = 0$ at p. By (3.22) and (3.23), we see that, at this point p,

$$\begin{aligned} 0 =& (D_1 - D_2)\omega_{1\alpha} = \sum_k D_{1\alpha,k}\omega_k \\ =& D_{1\alpha,1}\omega_1 + \sum_\beta D_{1\alpha,\beta}\omega_\beta = \lambda C_\alpha \omega_1 + \sum_\beta \lambda C_1 \delta_{\alpha\beta}\omega_\beta = \lambda C_1 \omega_\alpha, \end{aligned}$$

thus, $\lambda C_1 = 0$, which is a contradiction. Therefore, we conclude that in case (i), it must have $\mathbf{\Phi} \equiv 0$.

(ii) If $n \geq 4$ and x has $n-1$ distinct constant para-Laguerre eigenvalues, let D_i be the $n-1$ distinct para-Laguerre eigenvalues of x, where $1 \leq i \leq n-1$. From (2.20), we have

$$D_{ii,k} = 0, \quad \sum_k D_{ij,k}\omega_k = (D_i - D_j)\omega_{ij}, \quad i \neq j. \tag{3.28}$$

Putting $k = i \neq j$ in (3.21) and from (3.28), we have

$$D_{ij,i} = \lambda C_j, \quad i \neq j. \tag{3.29}$$

By (2.24) and (3.13), we obtain

$$C_i R_{ijij} = \sum_k C_k R_{kjij} = 0, \quad i \neq j. \tag{3.30}$$

If there exists one point p on M, so that $\mathbf{\Phi} \neq 0$ at p, without loss of generality, we may assume $C_1 \neq 0$ at p, thus from (3.30), we see that $R_{1212} = R_{1313} = \cdots = R_{1n-11n-1} = 0$ at p.

If $C_2 = \cdots = C_{n-1} = 0$ at p, from (2.10), we have

$$0 = \sum_j C_{i,j}\omega_j = dC_i + C_1\omega_{1i}, \quad \text{at} \quad p. \tag{3.31}$$

Putting $i = 2, \ldots, n-1$ in (3.31), we have $C_1\omega_{12} = C_1\omega_{13} = \cdots = C_1\omega_{1n-1} = 0$ at p. Thus, we see that $\omega_{12} = \omega_{13} = \cdots = \omega_{1n-1} = 0$ at p. By (3.28), we have $D_{12,k} = D_{13,k} = \cdots = D_{1n-1,k} = 0$ at p. Thus, from (3.29) and $D_{12} = D_{21}$, we have $\lambda C_1 = D_{21,2} = D_{12,2} = 0$ at p, which is a contradiction. This contradiction implies that there exists at least one i, $2 \leq i \leq n-1$, so that $C_i \neq 0$, without loss of generality, we may assume $C_2 \neq 0$ at p. Since $C_1 \neq 0, C_2 \neq 0$ at p, from (3.30) and (2.18), we get, at point p, that

$$0 = R_{1212} = -L_1 - L_2 = 0, \tag{3.32}$$

$$0 = R_{1j1j} = -L_1 - L_j = 0, \quad 3 \leq j \leq n-1, \tag{3.33}$$

$$0 = R_{2j2j} = -L_2 - L_j = 0, \quad 3 \leq j \leq n-1. \tag{3.34}$$

From (3.32)–(3.34), we see that $L_1 = L_2 = L_j = 0$ at p, where $3 \leq j \leq n-1$. Thus $\mathrm{tr}L = 0$ at p and also $\mathrm{tr}(LB^2) = \sum_i L_i B_i^2 = 0$ at p. From (3.3), we see that $B_{ij,k} = 0$ at p for all i, j, k. From (2.14), we have $(n-2)C_1 = \sum_i B_{i1,i} = 0$, therefore $C_1 = 0$ at p, which is a contradiction. Thus, we conclude that in case (ii), it must have $\mathbf{\Phi} \equiv 0$.

(iii) If $n \geq 5$ and x has three distinct constant para-Laguerre eigenvalues, let D_1, D_2 and D_3 be the three distinct constant para-Laguerre eigenvalues of x with multiplicities m_1, m_2 and m_3, respectively. We agree on the following ranges of indices

$$1 \leq a, b \leq m_1, \quad m_1 + 1 \leq s, t \leq m_1 + m_2, \quad m_1 + m_2 + 1 \leq \alpha, \beta \leq n-1.$$

From (2.20), we have

$$D_{ii,j} = D_{ab,j} = D_{st,j} = D_{\alpha\beta,j} = 0, \tag{3.35}$$

$$\sum_k D_{ij,k}\omega_k = (D_i - D_j)\omega_{ij}, \quad i \neq j. \tag{3.36}$$

From (3.21) and (3.35), we have

$$D_{ij,j} = D_{ji,j} = \lambda C_i, \quad i \neq j, \tag{3.37}$$

$$0 = D_{aa,b} - D_{ab,a} = \lambda C_a\delta_{ab} - \lambda C_b\delta_{aa} = -\lambda C_b, \quad a \neq b, \tag{3.38}$$

$$0 = D_{ss,t} - D_{st,s} = \lambda C_s\delta_{st} - \lambda C_t\delta_{ss} = -\lambda C_t, \quad s \neq t, \tag{3.39}$$

$$0 = D_{\alpha\alpha,\beta} - D_{\alpha\beta,\alpha} = \lambda C_\alpha\delta_{\alpha\beta} - \lambda C_\beta\delta_{\alpha\alpha} = -\lambda C_\beta, \quad \alpha \neq \beta. \tag{3.40}$$

If there exists one point p on M, so that $\mathbf{\Phi} \neq 0$ at p, we shall prove that $m_1 = m_2 = 1$ and $m_3 = n-2$. In fact, if $2 \leq m_1 \leq m_2 \leq m_3$, from (3.38)–(3.40) and $\lambda \neq 0$, we see that $C_i = 0$ for all i and therefore $\mathbf{\Phi} = 0$, which is a contradiction.

If $m_1 = 1$ and $2 \leq m_2 \leq m_3$, then $C_s = C_\alpha = 0$ for all s, α and therefore $\mathbf{\Phi} = C_1\omega_1$. Since it is assumed $\mathbf{\Phi} \neq 0$ at p, we get $C_1 \neq 0$ at p. On the other hand, from (2.10) and $C_s = C_\alpha = 0$, we have

$$0 = \sum_j C_{i,j}\omega_j = dC_i + C_1\omega_{1i}. \tag{3.41}$$

Putting $i = s$ and $i = \alpha$ in (3.41), we have $C_1\omega_{1s} = C_1\omega_{1\alpha} = 0$, which implies that $\omega_{1s} = \omega_{1\alpha} = 0$ at p. From (3.36), we see that $D_{1s,s} = D_{1\alpha,\alpha} = 0$ at p. Thus, by (3.37) and $\lambda \neq 0$, we get $C_1 = 0$ at p, which also is a contradiction. Thus, we conclude that if $\mathbf{\Phi} \neq 0$ at p, then $m_1 = m_2 = 1$ and $m_3 = n-2$. Therefore, $3 \leq \alpha \leq n-1$, from (3.40), we have $C_\alpha = 0$. Since it is assumed $\mathbf{\Phi} \neq 0$ at point p, without loss of generality, we may assume $C_1 \neq 0$ at p. If $C_2 = 0$ at p, from (2.10) and $C_\alpha = 0$, we have

$$0 = \sum_j C_{i,j}\omega_j = dC_i + C_1\omega_{1i}, \quad \text{at} \quad p. \tag{3.42}$$

By a similar method as above, we see that $\omega_{12} = \omega_{1\alpha} = 0$ and $D_{12,2} = D_{1\alpha,\alpha} = 0$ at p. From (3.37) and $\lambda \neq 0$, we get $C_1 = 0$ at p, which is a contradiction. Thus, we infer that $C_2 \neq 0$ at p. From (2.24) and (3.13), we have

$$C_iR_{ijij} = \sum_k C_kR_{kjij} = 0, i \neq j. \tag{3.43}$$

From (2.18), (3.43), $C_1 \neq 0$ and $C_2 \neq 0$ at p, we see that at point p

$$0 = R_{1212} = -L_1 - L_2 = 0, \tag{3.44}$$

$$0 = R_{1\alpha1\alpha} = -L_1 - L_\alpha = 0, \quad 3 \leq \alpha \leq n-1, \tag{3.45}$$

$$0 = R_{2\beta2\beta} = -L_2 - L_\beta = 0, \quad 3 \leq \beta \leq n-1. \tag{3.46}$$

Hence, we have $L_1 = L_2 = L_\alpha = 0$ at p, where $3 \leq \alpha \leq n-1$. This implies that $\mathrm{tr}L = 0$ at p and also $\mathrm{tr}(LB^2) = \sum_i L_iB_i^2 = 0$ at p. From (3.3), we see that $B_{ij,k} = 0$

at p for all i, j, k. From (2.14), we have $C_1 = 0$ at p, which is a contradiction. Thus, we conclude that in case (iii), it must have $\mathbf{\Phi} \equiv 0$.

(iv) If $n \geq 6$ and x has γ $(4 \leq \gamma \leq n-2)$ distinct constant para-Laguerre eigenvalues, let $D_1, D_2, \ldots, D_\gamma$ be the γ distinct constant para-Laguerre eigenvalues of x with multiplicities $m_1, m_2, \ldots, m_\gamma$ and $m_1 \leq m_2 \leq \cdots \leq m_\gamma$, respectively. From (2.20), we have

$$D_{ii,j} = 0, \quad \sum_k D_{ij,k}\omega_k = (D_i - D_j)\omega_{ij}, \quad i \neq j. \tag{3.47}$$

From (3.21) and (3.47), we have

$$D_{ij,j} = D_{ji,j} = \lambda C_i, \quad i \neq j, \tag{3.48}$$

$$0 = D_{ii,k} - D_{ik,i} = \lambda C_i \delta_{ik} - \lambda C_k \delta_{ii} = -\lambda C_k, \quad i \neq k. \tag{3.49}$$

If there exists one point p on M, so that $\mathbf{\Phi} \neq 0$ at p, we shall prove that $m_1 = m_2 = 1$. In fact, if $2 \leq m_1 \leq m_2 \leq \cdots \leq m_\gamma$, from (3.49) and $\lambda \neq 0$, we see that $C_i = 0$ for all i and therefore $\mathbf{\Phi} = 0$, which is a contradiction.

If $m_1 = 1$ and $2 \leq m_2 \leq \cdots \leq m_\gamma$, then $C_i = 0$ for all $2 \leq i \leq n-1$ and therefore $\mathbf{\Phi} = C_1\omega_1$. Since it is assumed $\mathbf{\Phi} \neq 0$ at p, we get $C_1 \neq 0$ at p. By the similar method in the proof of case (iii), we see that $\omega_{1i} = 0$ and $D_{1i,i} = 0$ at p, where $2 \leq i \leq n-1$. Thus, by (3.48) and $\lambda \neq 0$, we get $C_1 = 0$ at p, also a contradiction. Therefore, we conclude $m_1 = m_2 = 1$.

Since it is assumed $\mathbf{\Phi} \neq 0$ at point p, without loss of generality, we assume $C_1 \neq 0$ at p. If $C_i = 0$ at p, where $2 \leq i \leq n-1$, from (2.10), we have

$$0 = \sum_j C_{i,j}\omega_j = dC_i + C_1\omega_{1i}, \quad \text{at} \quad p. \tag{3.50}$$

By the similar method in the proof of case (iii), we see that $\omega_{1i} = 0$ and $D_{1i,i} = 0$ at p. Thus, from (3.48), we get $C_1 = 0$ at p, which is a contradiction. This implies that at least one of C_i is not zero at p, where $2 \leq i \leq n-1$, without loss of generality, we may assume $C_2 \neq 0$ at p. From (2.24) and (3.13), we have

$$C_i R_{ijij} = \sum_k C_k R_{kjij} = 0, i \neq j. \tag{3.51}$$

From (2.18), (3.51), $C_1 \neq 0$ and $C_2 \neq 0$ at p, we see that at point p

$$0 = R_{1212} = -L_1 - L_2 = 0, \tag{3.52}$$

$$0 = R_{1i1i} = -L_1 - L_i = 0, \quad 3 \leq i \leq n-1, \tag{3.53}$$

$$0 = R_{2j2j} = -L_2 - L_j = 0, \quad 3 \leq j \leq n-1. \tag{3.54}$$

Thus, we have $L_1 = L_2 = L_i = 0$ at p, where $3 \leq i \leq n-1$. This implies that $\mathrm{tr}L = 0$ at p and also $\mathrm{tr}(LB^2) = 0$ at p. From (3.3), we see that $B_{ij,k} = 0$ at p for all i, j, k. From (2.14), we have $C_1 = 0$ at p, which is a contradiction. Thus, we conclude that in case (iv), it must have $\mathbf{\Phi} \equiv 0$. This completes the proof of Theorem 1.3. □

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