On α-para Kenmotsu 3-manifolds with Ricci solitons

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Abstract. The object of the present paper is to study α-para Kenmotsu Ricci solitons of dimension three. It is shown that an α-para Kenmotsu Ricci soliton of dimension three is expanding and a manifold endowed with such a soliton is manifold of constant negative curvature. It is also established that for an α-para Kenmotsu Ricci soliton, if the potential vector field V is pointwise collinear with ξ, then V is constant multiple of ξ. It is proved that if an α-para Kenmotsu Ricci soliton of dimension three is gradient Ricci soliton corresponding to the potential function f, then either $Df = 0$ or Df is collinear with the Reeb vector field ξ.

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1 Introduction

The theory of almost contact and almost para contact manifolds is an important branch of research. Almost contact manifolds are of prime importance due to its significant applications in geometric optics, thermodynamics and string theory. Ricci and other geometric flows ([4], [5]) were introduced in Mathematics by Hamilton [9] and in Physics by Friedan [7] around almost in the same time, though with different motivations. More recently, such geometric flows have become popular, largely, because of Perelman's [13] work which lead to the proof of the well-known Poincare Conjecture. The notion of Ricci soliton was introduced by Hamilton [9]. This is considered as a natural generalization of Einstein metric and is defined on a Riemannian manifold (M, g) by

$$(\pounds_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \tag{1.1}$$

where $\pounds_V$ denotes the Lie derivative operator along the vector field V. V is known as potential vector field. It is assumed that V is complete. Here λ is a constant, called soliton constant. S is the Ricci tensor and g is the metric. X, Y are the arbitrary vector fields on M. A Ricci soliton can be considered as a fixed point of Hamilton's Ricci flow:

$$\frac{\partial}{\partial t} g_{ij} = -2S_{ij}$$

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viewed as a dynamical system on the space of Riemannian metrics modulo diffeomorphisms and scaling. The Ricci soliton is said to be shrinking, steady or expanding according as λ is negative, zero or positive respectively. If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci soliton. Ricci solitons have been studied in the papers [3], [8], [15], [14].

Para contact geometry is now an active branch of research. For some important works on para contact geometry we refer [12], [11], [16], [18], [19], [21]. Ricci solitons on para contact manifolds have been studied in [1].

In this paper we study α-para Kenmotsu manifolds of dimension three with Ricci solitons. The present paper is organized as follows:

After the introduction, we give the required preliminaries in Section 2. In Section 3, we show that an α-para Kenmotsu Ricci soliton of dimension three is expanding, and a manifold endowed with such a soliton is manifold of constant negative curvature. It is also established that for an α-para Kenmotsu Ricci soliton, if the potential vector field V is point wise collinear with ξ, then V is constant multiple of ξ. In Section 4, we prove that if an α-para Kenmotsu Ricci soliton of dimension three is gradient Ricci soliton corresponding to the potential function $-f$, then either $Df = 0$ or Df is collinear with the Reeb vector field ξ. The last section contains an example.

2 Preliminaries

Let M be a $2n+1$-dimensional differentiable manifold. Let ϕ be a 1-1 tensor field, ξ a vector field and η a 1-form on M. Then (ϕ, ξ, η) is called an almost para contact structure on M if

$$\phi^2 X = X - \eta(X)\xi. \tag{2.1}$$

The tensor field ϕ induces an almost paracomplex structure on the distribution $\mathcal{D} = \ker\eta$, that is, the eigen distributions $\mathcal{D}^+$, $\mathcal{D}^-$ corresponding to the eigen values 1, -1 of ϕ, respectively, have equal dimension n.

The manifold M is said to be almost paracontact manifold if it is endowed with an almost paracontact structure [2], [6], [16], [18]. An almost para contact manifold is called an almost paracontact metric manifold if it is additionally endowed with a pseudo-Riemannian metric g of signature $(n+1, n)$ and such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y) \tag{2.2}$$

for all $X, Y \in \chi(M)$. For almost para contact metric manifolds, we readily obtain

$$g(X, \xi) = \eta(X), \qquad \eta(\xi) = 1, \qquad \phi\xi = 0, \qquad \eta(\phi) = 0. \tag{2.3}$$

The fundamental skew-symmetric 2-form Φ is defined by $\Phi(X, Y) = g(X, \phi Y)$. Note that $\eta \wedge \Phi^n$ is, up to a constant factor, the Riemannian volume element of M. On an almost paracontact manifold, one defines the (2,1)- tensor field $N^{(1)}$ by

$$N^{(1)}(X, Y) = [\phi, \phi](X, Y) - 2d\eta(X, Y)\xi, \tag{2.4}$$

where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ given by

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]. \tag{2.5}$$

If $N^{(1)}$ vanishes identically, then the almost paracontact manifold is said to be normal. The normality condition says that the almost paracomplex structure J defined on $M \times \mathbb{R}$ by

$$J(X, \lambda\frac{d}{dt}) = (\phi X + \lambda\xi, \eta(X)\frac{d}{dt})$$

is integrable.

Our interest is on three dimension, because, sometimes the three dimensional results are strikingly different for higher dimensions. In the following we mention two important results from [19]. For a three-dimensional almost para contact metric manifold M, the following three conditions are mutually equivalent

(a) M is normal,

(b) there exist functions α, β on M such that

$$(\nabla_X\phi)Y = \beta(g(X,Y)\xi - \eta(Y)X) + \alpha(g(\phi X,Y)\xi - \eta(Y)\phi X), \tag{2.6}$$

(c) there exist functions α, β on M such that

$$\nabla_X\xi = \alpha(X - \eta(X)\xi) + \beta\phi X. \tag{2.7}$$

Here ∇ is the Levi-Civita connection of g. The functions α, β appearing in the above equations are given by

$$2\alpha = \text{Trace}\{X \to \nabla_X\xi\}, \quad 2\beta = \text{Trace}\{X \to \phi\nabla_X\xi\}. \tag{2.8}$$

A three-dimensional normal almost para contact metric manifold is said to be

- paracosymplectic if $\alpha = \beta = 0$,
- quasi-para Sasakian if and only if $\alpha = 0$ and $\beta \neq 0$,
- β-para Sasakian if and only if $\alpha = 0$ and β is constant, in particular para Sasakian if $\beta = -1$.
- α-para Kenmotsu if α is a non-zero constant and $\beta = 0$.

Recently, the Riemann curvature tensor of a three-dimensional α-para Kenmotsu manifold is deduced by K. Srivastava and S. K. Srivastava [16]. The Ricci tensor of a three-dimensional α-para Kenmotsu manifold is given by

$$S(X,Y) = (\frac{r}{2} + \alpha^2)g(X,Y) - (\frac{r}{2} + 3\alpha^2)\eta(X)\eta(Y), \tag{2.9}$$

where α is a constant and r is the scalar curvature of the manifold. The Riemann curvature tensor of a three-dimensional α-para Kenmotsu manifold is given by

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2} + 2\alpha^2)[g(Y,Z)X - g(X,Z)Y] \\ &- (\frac{r}{2} + 3\alpha^2)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi \\ &+ (\frac{r}{2} + 3\alpha^2)[\eta(X)Y - \eta(Y)X]\eta(Z). \end{aligned} \tag{2.10}$$

3 α-para Kenmotsu 3-manifolds with Ricci solitons

Theorem 3.1. *As a Ricci soliton, an α-para Kenmotsu 3-metric is expanding.*

Proof. Consider an α-para Kenmotsu 3-manifold which admits a Ricci soliton. From the commutativity of Lie derivative and covariant derivative [20], we obtain

$$\begin{aligned} & (\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[X,Y]} g)(Y, Z) \\ = \; & g\Big((\pounds_V \nabla)(X, Y), Z\Big) - g\Big((\pounds_V \nabla)(X, Z), Y\Big). \end{aligned}$$

The above equation can also be written as

$$(\nabla_X \pounds_V g)(Y, Z) = g\Big((\pounds_V \nabla)(X, Y), Z\Big) + g\Big((\pounds_V \nabla)(X, Z), Y\Big). \tag{3.1}$$

Differentiating (1.1) and using it in (3.1) it can be shown that

$$g\Big((\pounds_V \nabla)(X, Y), Z\Big) = (\nabla_Z S)(X, Y) - (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) \tag{3.2}$$

by permutation of X, Y, Z and necessary straight forward computations. Let $\{e_i\}$, $i = 1, 2, 3$ be an orthonormal basis of the tangent space at each point of the manifold. Putting $X = Y = e_i$, and taking summation over i, we get from (3.2)

$$(\pounds_V \nabla)(e_i, e_i) = 0. \tag{3.3}$$

In view of (1.1) and (2.9), it follows that

$$(\pounds_V g)(Y, Z) = -(r + 2\alpha^2 + 2\lambda) g(Y, Z) + (r + 6\alpha^2)\eta(Y)\eta(Z). \tag{3.4}$$

Differentiating both sides of (3.4) along the vector field X, we get

$$\begin{aligned} (\nabla_X \pounds_V g)(Y, Z) \; = \; & -dr(X)\Big(g(Y, Z) - \eta(Y)\eta(Z)\Big) \\ + \; & (r + 6\alpha^2)\eta(Y)\Big(\nabla_X \eta(Z) - \eta(\nabla_X Z)\Big) \\ + \; & (r + 6\alpha^2)\eta(Z)\Big(\nabla_X \eta(Y) - \eta(\nabla_X Y)\Big). \end{aligned} \tag{3.5}$$

Since ∇ is Levi-Civita connection, we have

$$(\nabla_X g)(Y, \xi) = 0.$$

The above equation implies

$$\nabla_X \eta(Y) - \eta(\nabla_X Y) = \alpha\Big(g(X, Y) - \eta(X)\eta(Y)\Big). \tag{3.6}$$

Using (3.5) and (3.6) in (3.1), we get

$$\begin{aligned} & g\Big((\pounds_V \nabla)(X, Y), Z\Big) + g\Big((\pounds_V \nabla)(X, Z), Y\Big) \\ = \; & -dr(X)\Big(g(Y, Z) - \eta(Y)\eta(Z)\Big) \\ + \; & \alpha(r + 6\alpha^2)\Big(g(X, Y)\eta(Z) + g(X, Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)\Big). \end{aligned} \tag{3.7}$$

Interchanging X, Y, Z cyclically, in the above equation we obtain

$$\begin{aligned} & g\Big((\pounds_V\nabla)(Y,Z),X\Big)+g\Big((\pounds_V\nabla)(Y,X),Z\Big) \\ = & -dr(Y)\Big(g(Z,X)-\eta(X)\eta(Z)\Big) \\ + & \alpha(r+6)\Big(g(Z,Y)\eta(X)+g(X,Y)\eta(Z)-2\eta(X)\eta(Y)\eta(Z)\Big). \end{aligned} \tag{3.8}$$

Again interchanging X, Y, Z cyclically in (3.8), we have

$$\begin{aligned} & g\Big((\pounds_V\nabla)(Z,X),Y\Big)+g\Big((\pounds_V\nabla)(Z,Y),X\Big) \\ = & -dr(Z)\Big(g(X,Y)-\eta(X)\eta(Y)\Big) \\ + & \alpha(r+6)\Big(g(X,Z)\eta(Y)+g(Y,Z)\eta(X)-2\eta(X)\eta(Y)\eta(Z)\Big). \end{aligned} \tag{3.9}$$

Subtracting (3.9) from (3.8), we have

$$\begin{aligned} & g\Big((\pounds_V\nabla)(Y,X),Z\Big)-g\Big((\pounds_V\nabla)(Z,X),Y\Big) \\ = & -dr(Y)\Big(g(Z,X)-\eta(X)\eta(Z)\Big) \\ + & dr(Z)\Big(g(X,Y)-\eta(X)\eta(Y)\Big) \\ + & \alpha(r+6\alpha^2)\Big(g(X,Y)\eta(Z)-g(X,Z)\eta(Y)\Big). \end{aligned} \tag{3.10}$$

Addition of (3.7) and (3.10) yields

$$\begin{aligned} 2g\Big((\pounds_V\nabla)(Y,X),Z\Big) = & -dr(X)\Big(g(Y,Z)-\eta(Z)\eta(Y)\Big) \\ - & dr(Y)\Big(g(X,Z)-\eta(Z)\eta(X)\Big) \\ + & dr(Z)\Big(g(X,Y)-\eta(X)\eta(Y)\Big) \\ + & 2\alpha(r+6\alpha^2)\Big(g(X,Y)\eta(Z) \\ - & \eta(X)\eta(Y)\eta(Z)\Big). \end{aligned} \tag{3.11}$$

Since $dr(Z) = g(\text{grad}r, Z)$, we have from above

$$\begin{aligned} 2g\Big((\pounds_V\nabla)(Y,X),Z\Big) = & dr(X)\Big(g(Y,Z)-\eta(Z)\eta(Y)\Big) \\ - & dr(Y)\Big(g(X,Z)-\eta(Z)\eta(X)\Big) \\ + & \Big(g(X,Y)-\eta(X)\eta(Y)\Big)g(\text{grad}r,Z) \\ + & 2\alpha(r+6\alpha^2)\Big(g(X,Y)\eta(Z) \\ - & \eta(X)\eta(Y)\eta(Z)\Big). \end{aligned} \tag{3.12}$$

Comparing both sides of the above equation, we obtain

$$\begin{aligned} 2(\pounds_V\nabla)(Y,X) &= -dr(X)\Big(Y-\eta(Y)\xi\Big) \\ &- dr(Y)\Big(X-\eta(X)\xi\Big) \\ &+ \Big(g(X,Y)-\eta(X)\eta(Y)\Big)gradr \\ &+ 2\alpha(r+6\alpha^2)\Big(g(X,Y)\xi-\eta(X)\eta(Y)\xi\Big). \end{aligned} \tag{3.13}$$

In the above equation putting $X=Y=e_i$ and taking summation over i

$$\begin{aligned} 2\Sigma_{i=1}^{3}(\pounds_V\nabla)(e_i,e_i) &= -2\Sigma_{i=1}^{2}dr(e_i)e_i+2\text{grad}r \\ &+ 4\alpha(6\alpha^2+r)\xi \\ &+ 2dr(\xi)\xi+4\alpha(6\alpha^2+r)\xi. \end{aligned} \tag{3.14}$$

In view of (3.3) and (3.14), we get

$$\xi r+2\alpha(6\alpha^2+r)=0. \tag{3.15}$$

In (3.13) putting $X=\xi$, it follows that

$$2(\pounds_V\nabla)(Y,\xi)=-dr(\xi)\phi^2Y. \tag{3.16}$$

It is well known that $(\pounds_V\nabla)$ is a symmetric tensor of type (1,2). Its covariant derivative is given by [20]

$$\begin{aligned} (\nabla_X\pounds_V\nabla)(Y,Z) &= \nabla_X(\pounds_V\nabla(Y,Z))-\pounds_V\nabla(\nabla_XY,Z) \\ &- \pounds_V\nabla(Y,\nabla_XZ). \end{aligned} \tag{3.17}$$

Putting $Z=\xi$ in (3.17) and using (3.16) and (2.7) for α-para Kenmotsu case and the fact that $(\pounds_V\nabla)(Y,fZ)=f(\pounds_V\nabla)(Y,Z)$ for a scalar valued function f, we obtain

$$\begin{aligned} 2\Big(\nabla_X\pounds_V\nabla\Big)(Y,\xi) &= (\nabla_X(\xi r))\phi^2Y \\ &+ \alpha\xi r\Big(g(X,Y)\xi+\eta(Y)X \\ &- \eta(X)Y-\eta(X)\eta(Y)\xi\Big). \end{aligned} \tag{3.18}$$

Again, it is well known that [20]

$$(\pounds_VR)(X,Y)Z=\Big(\nabla_X\pounds_V\nabla\Big)(Y,Z)-\Big(\nabla_Y\pounds_V\nabla\Big)(X,Z). \tag{3.19}$$

In (3.19) putting $Z=\xi$ and using (3.18), we obtain

$$\begin{aligned} 2(\pounds_VR)(X,Y)\xi &= -\Big(\nabla_X(\xi r)\Big)\phi^2Y+\Big(\nabla_Y(\xi r)\Big)\phi^2X \\ &+ 2\alpha(\xi r)\Big(\eta(Y)X-\eta(X)Y\Big). \end{aligned} \tag{3.20}$$

From (2.10), we get

$$R(X,Y)\xi = \alpha^2\Big(\eta(X)Y - \eta(Y)X\Big). \tag{3.21}$$

It is well known that

$$\begin{aligned}(\pounds_V R)(X,Y)Z &= \pounds_V R(X,Y)Z - R(\pounds_V X,Y)Z - R(X,\pounds_V Y)Z\\ &- R(X,Y)\pounds_V Z.\end{aligned}$$

Putting $Z = \xi$ in the above equation and using (3.21) we have

$$(\pounds_V R)(X,Y)\xi = -R(X,Y)\pounds_V\xi + \alpha^2\Big((\pounds_V\eta)(X)Y - (\pounds_V\eta)(Y)X\Big). \tag{3.22}$$

In (3.4) putting $Z = \xi$, we obtain

$$(\pounds_V\eta)(Y) = g(Y,\pounds_V\xi) + 2(2\alpha^2 - \lambda)\eta(Y). \tag{3.23}$$

By virtue of (3.22) and (3.23), it follows that

$$\begin{aligned}(\pounds_V R)(X,Y)\xi &= -R(X,Y)\pounds_V\xi + 2\alpha^2(2\alpha^2-\lambda)\Big(\eta(X)Y - \eta(Y)X\Big)\\ &+ \alpha^2\Big(g(X,\pounds_V\xi)Y - g(Y,\pounds_V\xi)X\Big).\end{aligned} \tag{3.24}$$

By virtue of (2.1), (3.20) and (3.24)

$$\begin{aligned}&-(X(\xi r))\Big(Y - \eta(Y)\xi\Big) + (Y(\xi r))\Big(X - \eta(X)\xi\Big)\\ &+ 2\alpha(\xi r)\Big(\eta(Y)X - \eta(X)Y\Big)\\ &= -2R(X,Y)\pounds_V\xi + 4\alpha^2(2\alpha^2-\lambda)\Big(\eta(X)Y - \eta(Y)X\Big)\\ &+ 2\alpha\Big(g(X,\pounds_V\xi)Y - g(Y,\pounds_V\xi)X\Big).\end{aligned}$$

Taking inner product in both sides with respect to X, we get

$$\begin{aligned}&-X(\xi r)g(Y,X) + X(\xi r)\eta(Y)\eta(X) + Y(\xi r)g(X,X)\\ &- Y(\xi r)\eta(X)\eta(X) - 2(\xi r)\Big(\eta(Y)g(X,X) - \eta(X)g(Y,X)\Big)\\ &= -2g\Big(R(X,Y)\pounds_V\xi, X\Big) + 4\alpha^2(2\alpha^2-\lambda)\Big(\eta(X)g(Y,X) - \eta(Y)g(X,X)\Big)\\ &+ 2\alpha\Big(g(X,\pounds_V\xi)g(Y,X) - g(Y,\pounds_V\xi)g(X,X)\Big).\end{aligned}$$

Taking $X = Y = e_i$, where $\{e_i\}$, $i = 1,2,3$ is an orthonormal basis with $e_3 = \xi$ of the tangent space of the manifold, we get after simplification from the above equation

$$\begin{aligned}&-Y(\xi r) - \Big(\xi(\xi r) + 4\alpha\xi r + 8\alpha^2(2\alpha^2-\lambda)\Big)\eta(Y)\\ &- 2S(Y,\pounds_V\xi) - 4\alpha^2 g(Y,\pounds_V\xi) = 0.\end{aligned} \tag{3.25}$$

Using (2.9) in (3.25), we have

$$(3.26)\quad \begin{aligned} & (r+6\alpha^2)g(Y,\pounds_V\xi)-(r+6\alpha^2)\eta(Y)\eta(\pounds_V\xi) \\ = \; & -Y(\xi r)-\xi(\xi r)\eta(Y)-4\Big(2\alpha^2(2\alpha^2-\lambda)+\alpha\xi r\Big)\eta(Y) \end{aligned}$$

The equation (3.26) is true for all Y. Putting $Y=\xi$ in the above equation, we obtain

$$(3.27)\quad -2\xi(\xi r)-4\alpha\xi r-\Big(2\alpha^2(2\alpha^2-\lambda)\Big)=0.$$

Using (3.15) in (3.27), we get

$$(3.28)\quad \lambda=2\alpha^2.$$

Hence, the theorem follows. □

Theorem 3.2. *If the metric of a three-dimensional α-para Kenmotsu manifold is a Ricci soliton, then the manifold is of constant negative curvature* $-\alpha^2$.

Proof. Let $\{e_1,\phi e_1,\xi\}$ be an orthonormal ϕ-basis of the tangent space at any point of the manifold. Putting $Y=e_i$ in (3.26), it follows that

$$(3.29)\quad r=-6\alpha^2.$$

Thus by virtue of (2.10) we obtain the theorem. □

Theorem 3.3. *If in a three-dimensional α-para Kenmotsu manifold, the metric is Ricci soliton and V is point wise collinear with ξ, then V is constant multiple of ξ and consequently ξ is complete.*

Proof. In (1.1) putting $Y=\xi$, using (2.7) for α-para Kenmotsu manifold, and using (2.9) and (3.28), we have

$$\begin{aligned} & \pounds_V\eta(X)-g(\nabla_V X,\xi)+g(\nabla_X V,\xi)-\alpha g(X,V) \\ + \; & \alpha\eta(V)\eta(X)+g(X,\nabla_\xi V)=0. \end{aligned}$$

Let V be pointwise collinear with ξ.
i.e., $V=b\xi$, for a function b on the manifold.
Then the above equation yields

$$\begin{aligned} & \pounds_{(b\xi)}\eta(X)-bg(\nabla_\xi X,\xi)+bg(\nabla_X\xi,\xi) \\ + \; & b'g(\xi,\xi)-\alpha g(X,b\xi) \\ + \; & \alpha\eta(b\xi)\eta(X)+bg(X,\nabla_\xi\xi)+b'g(X,\xi)=0. \end{aligned}$$

Putting $X=\xi$ in the above equation, we get $b'=0$, consequently b=constant and ξ is a constant multiple of V. By definition V is compact. So, we the theorem is proved.

□

4 Gradient Ricci solitons

Theorem 4.1. *If a α-para Kenmotsu Ricci soliton of dimension three is gradient Ricci soliton corresponding to the potential function $-f$, then either f is constant or Df is collinear with the Reeb vector field ξ.*

Proof. A Ricci soliton is called gradient Ricci soliton if the vector field V is the gradient of a potential function $-f$. If the Ricci soliton is gradient Ricci soliton, then (1.1) is of the form [10], [17]

$$\nabla\nabla f = S + \lambda g. \tag{4.1}$$

The above equation reduces to

$$\nabla_Y Df = QY + \lambda Y, \tag{4.2}$$

where D is the gradient operator of g and Q is Ricci operator. From (4.2) it follows that

$$R(X,Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X. \tag{4.3}$$

From (2.9), we have

$$QX = (\frac{r}{2} + \alpha^2)X - (\frac{r}{2} + 3\alpha^2)\eta(X)\xi.$$

Differentiating the above equation with respect to W, we get after simplification

$$(\nabla_W Q)X = \frac{dr}{2}(X - \eta(X)\xi) - (\frac{r}{2} + 3\alpha^2)((\nabla_W \eta)(X)\xi - \eta(X)\nabla_W \xi). \tag{4.4}$$

In (4.4) putting $W = \xi$, we obtain

$$(\nabla_\xi Q)X = \frac{dr}{2}(X - \eta(X)\xi).$$

The above equation implies

$$g((\nabla_\xi Q)X - (\nabla_X Q)\xi, \xi) = 0. \tag{4.5}$$

Using (4.5) in (4.3), we get

$$g(R(\xi, X)Df, \xi) = 0. \tag{4.6}$$

By (2.10), we have

$$g(R(\xi, X)Df, \xi) = -2\alpha^2(g(Y, Df) - \eta(Y)\eta(Df)).$$

Using (4.6) in the above equation, we obtain for $\alpha \neq 0$

$$g(Y, Df) = \eta(Y)\eta(Df).$$

The above equation gives

$$Df = (\xi f)\xi. \tag{4.7}$$

From (4.2), we get

$$S(X,Y) + \lambda g(X,Y) = g(\nabla_Y Df, X).$$

Using (4.7) in the above equation, we have

$$S(X,Y) + \lambda g(X,Y) = \nabla_Y(\xi f)\eta(X) + \xi f g(\nabla_Y \xi, X). \tag{4.8}$$

Using (2.7) for α-para Kenmotsu manifold, we obtain

$$S(X,Y) + \lambda g(X,Y) = \nabla_Y(\xi f)\eta(X) + \alpha(\xi f) g(\phi X, \phi Y). \tag{4.9}$$

In (4.9), putting $X = \xi$, we have

$$S(Y,\xi) + \lambda \eta(Y) = \nabla_Y(\xi f). \tag{4.10}$$

By (2.9), we obtain from above

$$(\lambda - 2\alpha^2)\eta(Y) = \nabla_Y(\xi f). \tag{4.11}$$

Using (3.28) in the above equation, it follows that

$$\nabla_Y(\xi f) = 0. \tag{4.12}$$

By (4.12) and (4.9)

$$QY + \lambda Y = -\alpha(\xi f)\phi^2 Y. \tag{4.13}$$

Applying (4.2) in the above equation, we have

$$\nabla_Y Df = \alpha(\xi f)(\eta(Y)\xi - Y). \tag{4.14}$$

By virtue of (4.14), it follows that

$$\begin{aligned} R(X,Y)Df &= \nabla_X\Big(\alpha(\xi f)(\eta(Y)\xi - Y)\Big) \\ &- \nabla_Y\Big(\alpha(\xi f)(\eta(X)\xi - X)\Big) \\ &- \alpha(\xi f)\Big(\eta([X,Y])\xi - [X,Y]\Big). \end{aligned}$$

Putting $X = \xi$ and using (2.7) for α-para Kenmotsu manifold, we get after simplification

$$\begin{aligned} R(\xi,Y)Df &= (\xi\alpha)(\xi f)\eta(Y)\xi + \alpha(\xi f)\nabla_\xi \eta(Y)\xi \\ &+ \alpha^2(\xi f)\eta(Y)\xi - \xi\alpha(\xi f)Y - \alpha^2(\xi f)Y \\ &- \alpha(\xi f)\eta(\nabla_\xi Y) + \alpha(\xi f)\eta(\nabla_Y \xi)\xi. \end{aligned} \tag{4.15}$$

By (2.10)

$$\begin{aligned} R(\xi,Y)Df &= (\frac{r}{2} + 2\alpha^2)\Big(g(Y,Df)\xi - \eta(Df)Y\Big) \\ &- (\frac{r}{2} + 3\alpha^2)\Big(g(Y,Df) - \eta(Z)\eta(Y)\Big)\xi \\ &+ (\frac{r}{2} + 3\alpha^2)\Big(Y - \eta(Y)\xi\Big)\eta(Df). \end{aligned} \tag{4.16}$$

By (3.29), (4.15) and (4.16), it follows that

$$\begin{aligned} & \alpha(\xi f)\nabla_\xi \eta(Y)\xi + \alpha^2(\xi f)\eta(Y)\xi - \alpha^2(\xi f)Y \\ - \; & \alpha(\xi f)\eta(\nabla_\xi Y)\xi + \alpha(\xi f)\eta(\nabla_Y \xi)\xi \\ = \; & -\alpha^2\Big(g(Y, Df)\xi - \eta(Df)Y\Big). \end{aligned} \tag{4.17}$$

Replacing Y by ϕY in the above equation, we get $-\alpha^2(\xi f)\phi Y - \alpha(\xi f)\eta(\nabla_\xi \phi Y)\xi = -\alpha^2\Big(g(\phi Y, Df)\xi - \eta(Df)\phi Y\Big)$. Taking inner product in both sides of the above equation with ξ, we get

$$\alpha^2 g(\phi Y, Df) = 0.$$

Putting $Y = \phi Df$ in the above equation, we have

$$g(Df, Df) - \eta(Df)\eta(Df) = 0.$$

If $Df \neq 0$, the above equation gives

$$1 - (\eta(Df))^2 = 0.$$

Hence, $\eta(Df) = 1$. So, Df is collinear with ξ. If $Df = 0$, then f is constant. Thus we complete the proof.

□

5 Example

Consider $M^3 = \mathbb{R}^2 \times \mathbb{R}_- \subset \mathbb{R}^3$ with the standard cartesian coordinates (x, y, z). Define the almost para contact structure (ϕ, ξ, η) on M^3 by

$$\phi(e_1) = e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0, \quad \xi = e_3, \quad \eta = dz,$$

where

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M^3. Let the metric g be defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, \qquad g(e_1, e_1) = \exp(2z),$$

$$g(e_2, e_2) = \exp(-2z), \qquad g(e_3, e_3) = 1.$$

By Koszul formula, we have

$$\nabla_{e_1} e_3 = e_1, \qquad \nabla_{e_1} e_2 = 0, \qquad \nabla_{e_1} e_1 = -\exp(2z)e_3,$$

$$\nabla_{e_2} e_3 = e_2, \qquad \nabla_{e_2} e_2 = \exp(2z)e_3, \qquad \nabla_{e_2} e_1 = 0,$$

$$\nabla_{e_3} e_3 = 0, \qquad \nabla_{e_3} e_2 = e_2, \qquad \nabla_{e_3} e_1 = e_1.$$

Using the above results, we get M^3 is an α-para Kenmotsu manifold [16] for $\alpha = 1$. The non vanishing components of the curvature tensor are

$$R(e_1, e_3)e_1 = \exp(2z)e_3, \quad R(e_1, e_2)e_2 = \exp(2z)e_1,$$

$$R(e_1,e_2)e_1 = \exp(2z)e_2, \quad R(e_2,e_3)e_2 = -\exp(2z)e_3,$$
$$R(e_1,e_3)e_3 = -e_1, \quad R(e_2,e_3)e_3 = -e_2.$$

The non vanishing component of the Ricci tensor is

$$S(e_1,e_1) = \exp(2z)(\exp(2z) - 1).$$

If we choose $V = (\exp(2z) + 1)e_2 + e_3, \quad \lambda = 2$, then

$$(\pounds_V g)(e_1,e_1) + 2S(e_1,e_1) + \lambda g(e_1,e_1) = 0.$$

Hence g is Ricci soliton. Thus we obtain an example of 1-para Kenmotsu Ricci soliton.

References

[1] A.M. Balga, *η-Ricci solitons on para-Kenmotsu manifolds*, arXiv math. DG 1402022313.

[2] B. Cappelletti-Montano, *Bi Legendrian structures and paracontact geometry*, Int. J. Geom. Math. Mod. Phys. 6 (2009), 487–504.

[3] B.Y. Chen, *Some results on concircular vector fields and their applications to Ricci solitons*, Bull. Korean Math. Soc., 52 (2015), 1535–1547.

[4] B. Chow, D. Knopf, *The Ricci flow, an introduction*, Mathematical Surveys and Monographs 110, AMS, 2004.

[5] B. Chow et al, *The Ricci flow, Techniques and applications, Part-I, Geometric Aspects*, Mathematical Surveys and Monographs 135, AMS, 2007.

[6] S. Erdem, *On almost (para) contact (hyperbolic) metric manifolds and harmonicity of* $(\phi,\phi^{'})$*-holomorphic maps between them*, Houston J. Math. 48(2002), 21–45.

[7] D. Friedan, *Non linear models in* $2+\varepsilon$ *dimensions*, Ann. Phys. 163 (1985), 318–419.

[8] A. Ghosh, *3-metric as Ricci soliton*, Chaos, Solitons and Fractals 44 (2011), 647–650.

[9] R.S. Hamilton, *Three manifolds with positive Ricci curvature*, J. Diff. Geom. 17 (1982), 255–306.

[10] R.S. Hamilton, *The Ricci flow on surfaces*, Mathematics and General Relativity (Santa Cruz, CA 1986), Contemp. Math 71, 1988.

[11] S. Kaneyuki and M. Kozai, *Structures and affine symmetric spaces*, Tokyo J. Math., 8 (1985), 81–98.

[12] S. Kaneyuki and F.L. Williams, *Almost para-contact and para-Hodge structures on manifolds*, Nagoya Math. J. 99 (1985), 173–187.

[13] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arxiv math., DG 1021111599.

[14] R. Sharma, *Certain results on* K*-contact and* (k,μ)*-contact manifolds*, J. Geom. 89 (2008), 138–147.

[15] R. Sharma and A. Ghosh, *3-manifolds as a Ricci soliton represents the Heisenberg group*, Int. J. Geom. Methods Mod. Phys. 8 (2011), 149–154.

[16] K. Srivastava and S.K. Srivastava, *On a class of α-para Kenmotsu manifolds*, Mediterr. J. Math. 13 (2016), 391–399.

[17] M. Turan, U.C. De and A. Yildiz, *Ricci solitons and gradient Ricci solitons in three-dimensional trans-Sasakian manifolds*, Filomat, 26 (2012), 363–370.

[18] J. Welyczko, *Slant curves in 3-dimensional normal almost paracontact metric manifolds*, Mediterr. J. Math. 11 (2014), 965–978.

[19] J. Welyczko, *On Legendre curves in 3-dimensional normal almost paracontact metric manifolds*, Result Math. 54 (2009), 377–387.

[20] K. Yano, *Integral formulas in Riemannian geometry*, New York, Marcel Dekker 1970.

[21] S. Zamkovoy, *Canonical connections on para-contact manifolds*, Ann. Glob. Anal. Geom. 36 (2009), 37–60.

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