Orbit space of cohomogeneity two flat Riemannian manifolds

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Abstract. We give a topological classification of the orbit space of cohomogeneity two isometric actions on flat Riemannian manifolds.

M.S.C. 2010: 53C30, 57S25.
Key words: Orbit space; isometry; Riemannian manifold.

1 Introduction

Let $G \times M \to M$ be a differentiable action of a Lie group G on a differentiable manifold M and consider the orbit space $\frac{M}{G}$ with the quotient topology. The dimension of $\frac{M}{G}$ which we will denote by $\mathrm{Coh}(M,G)$, is called the cohomogeneity of the action of G on M. The study of orbit spaces has many important applications in invariant function theory and G-invariant variational problems associated to M. Many G-invariant objects associated to M can be related to similar objects associated to the orbit space.

Therefore, we can effectively reduce many problems about G-invariant objects of M to generally easier problems on $\frac{M}{G}$. Because of this motivation, many mathematicians studied topological properties of the orbit spaces of Lie group actions on manifolds. A pioneer theorem in this area is the following theorem proved by P. Mostert in 1957 ([11]): *If M is a differentiable manifold and G is a compact Lie group acting on M such that $Coh(M,G) = 1$, then the orbit space $\frac{M}{G}$ is homeomorphic to one of the spaces $[0,1], (0,1], S^1$ or $\mathbb{R}$.*

This theorem has been generalized to noncompact Lie groups with proper actions on manifolds. Moreover, If M is endowed with a Riemannian metric, and G is a closed and connected subgroup of the isometries of M, which acts by cohomogeneity one on M, there are more interesting results about the orbit space and orbits (see [10], [11], [13]). It is proved in [13] that if M is a Riemannian manifold of negative curvature and G is a connected and closed subgroup of isometries of M, acting on M with $\mathrm{Coh}(M,G) = 1$, then the orbit space is not homeomorphic to $[0,1]$, so by (generalized) Mostert's theorem, it would be homeomorphic to $(0,1)$ or S^1 or $\mathbb{R}$, and if in addition M is simply connected then the orbit space is homeomorphic to $(0,1)$

Balkan Journal of Geometry and Its Applications, Vol.23, No.2, 2018, pp. 25-33.
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or $\mathbb{R}$. This result, generalized to flat Riemannian manifolds in [10], and recently it is proved for Riemannian manifolds of non-positive curvature. To extend Mostert's theorem, it is natural to ask, what may be the orbit space $\frac{M}{G}$, when $\mathrm{Coh}(M, G) = 2$. There is no classification for orbit spaces of cohomogeneity two G-manifolds in general. Cohomogeneity two actions of compact Lie groups on $\mathbb{R}^n$, $n > 1$, are polar (in the sense of Dadok) and all such actions and their orbits are classified (see [12]). It is clear in this case that the orbit space is homeomorphic to plane or half-plane. Also, It is proved in [8] that if G is a connected (compact or non-compact) group of the isometries of $\mathbb{R}^n$ such that $\mathrm{Coh}(\mathbb{R}^n, G) = 2$, then the orbit space $\frac{\mathbb{R}^n}{G}$ is homeomorphic to plane or half-plane. Classification of orbit spaces of cohomogeneity two actions on the standard sphere S^n has been described in [1].

This article follows a series of papers [6]-[9], where we are trying to study orbits and orbit spaces of cohomogeneity two Riemannian manifolds under conditions on curvature. In [7] the following theorem is proved which gives a topological description of cohomogeneity two flat riemannian manifolds and their orbits.

Theorem A. *Let M^n, $n \geq 3$, be a complete connected nonsimply connected and flat Riemannian manifold, which is of cohomogeneity two under the action of a closed and connected Lie group G of isometries. Then, one of the following is true:*

(a) *$\pi_1(M) = Z$ and each principal orbit is isometric to $S^{n-2}(c)$, for some $c > 0$ (c depends on orbits).*

(b) *There is a positive integer l, such that $\pi_1(M) = Z^l$ and one of the following is true:*
(b1) *There is a positive integer m, $2 < m < n$, such that each principal orbit is covered by $N^{m-2}(c) \times \mathbb{R}^{n-m}$, where $N^{m-2}(c)$ is a homogeneous hypersurface of $S^{m-1}(c)$ ($c > 0$ depends on orbits). There is a unique orbit diffeomorphic to $T^l \times \mathbb{R}^{n-m-l}$.*
(b2) *Each principal orbit is covered by $S^r \times \mathbb{R}^{n-r-2}$, for some positive integer r.*
(b3) *Each principal orbit is covered by $H \times \mathbb{R}^{n-m}$, such that H is a helix in $\mathbb{R}^m$. There is an orbit diffeomorphic to $T^l \times \mathbb{R}^t$, for some non-negative integer t.*

(c) *Each orbit is diffeomorphic to $\mathbb{R}^t \times T^l$, for some non-negative integer t.*

To complete the study of flat cohomogeneity two Riemannian manifolds, it remains to characterize the orbit space, which is the aim of the present paper. For any flat surface S there exists a cohomogeneity two flat Riemannian G-manifold M such that all orbits are flat and $\frac{M}{G}$ is homeomorphic to S (put $M = S \times \mathbb{R}^n$, $G = \{I\} \times H$ such that I is the identity map on S and H is a closed and connected subgroups of $\mathrm{Iso}(\mathbb{R}^n)$ which acts transitively on $\mathbb{R}^n$).

Thus, study of the orbit space of cohomogeneity two flat Riemannian manifolds is interesting when there are some non-flat orbits. We will prove the following theorem.

Theorem B. *Let M be a flat Riemannian manifold and G be a closed and connected subgroup of the isometries of M such that $\mathrm{Coh}(M, G) = 2$. If there are some* *non-flat orbits then* $\frac{M}{G}$ *is homeomorphic to one of the following spaces:*

$$[0,+\infty)\times\mathbb{R}, S^1\times\mathbb{R}, S^1\times[0,\infty), \mathbb{R}^2$$

2 Preliminaries

In the following, M^n is a Riemannian manifold of dimension n, G is a closed and connected subgroup of $\mathrm{Iso}(M)$, and $\pi: M \to \frac{M}{G}$ denotes the projection on to the orbit space. We know that the fixed point set of the action of G on M, given by

$$M^G = \{x \in M : \quad G(x) = x\}$$

is a totally geodesic submanifold of M.
We will write $A = B$ if A and B are homeomorphic topological spaces, isomorphic groups or diffeomorphic manifolds.

Fact 2.1. If $\mathrm{Coh}(G, M) = m \geq 1$ then there are two types of points in M called principal and singular points (for definition and details about singular and principal points, we refer to [1] and [13]. If x is a principal(singular) point then $\pi(x)$ is an interior(boundary) point of $\frac{M}{G}$. Also, if x is a principal point, the orbit $G(x)$ is called a principal (singular) orbit and we have $\dim G(x) = n - m$ ($\dim G(x) \leq n - m$). The union of all principal orbits is an open and dense subset of M.

Remark 2.2. If $\mathrm{Coh}(G, \mathbb{R}^n) = 1$ then one of the following is true:
(1) All orbits are isometric to $\mathbb{R}^{n-1}$. So, by suitable choice of coordinates, each orbit will be equal to $\{b\} \times \mathbb{R}^{n-1}$ for some $b \in \mathbb{R}$ related to the orbit, and $\frac{\mathbb{R}^n}{G} = \mathbb{R}$.
(2) Each principal orbit is diffeomorphic to $S^{n-m-1} \times \mathbb{R}^m$ for some $m \geq 0$, there is a unique singular orbit isometric to $\mathbb{R}^m$ and $\frac{\mathbb{R}^n}{G} = [0, +\infty)$.
(3) If G is compact then each principal orbit is diffeomorphic to S^{n-1}, the unique singular orbit is a one point set, and $\frac{\mathbb{R}^n}{G} = [0, \infty)$.

Proof. See [10], proof of the theorems 3.1 and 3.5. □

Definition 2.3. If $G, H \subset \mathrm{Iso}(M)$ then we say that G and H are orbit equivalent and we denote it by $G \simeq H$, if for each $x \in M$, $G(x) = H(x)$.

We recall that the connected component of $\mathrm{Iso}(\mathbb{R}^n)$ is equal to $SO(n) \times \mathbb{R}^n$, such that the standard action of $SO(n) \times \mathbb{R}^n$ on $\mathbb{R}^n$ is in the following way:

$$(A, b)x = Ax + b,\ (A, b) \in SO(n) \times \mathbb{R}^n,\ x \in \mathbb{R}^n.$$

Also, $SO(d) \times \mathbb{R}^e$ acts on $\mathbb{R}^d \times \mathbb{R}^e$ in the following way, which is called direct product action:

$$(A, b)(x, y) = Ax + (y + b), (A, b) \in SO(d) \times \mathbb{R}^e,\ x \in \mathbb{R}^d,\ y \in \mathbb{R}^e$$

Definition 2.4.

(a) Let G be a connected subgroup of Iso$(\mathbb{R}^n)$ and d,e be positive integers such that $d+e=n$. If G is not compact and it is a subgroup of $SO(d)\times\mathbb{R}^e$ (direct product), then we say that G is *d-helicoidical* on $\mathbb{R}^n$.
(b) Following (a), let

$$K=\{A\in SO(d):\quad (A,b)\in G,\text{ for some } b\in\mathbb{R}^e\}$$
$$T=\{b\in\mathbb{R}^e:\quad (A,b)\in G,\text{ for some } A\in SO(d)\}$$

If $x=(x_1,x_2)\in(\mathbb{R}^d-\{o\})\times\mathbb{R}^e$, $T(x_2)=\mathbb{R}^e$ and $K(x_1)=S^{d-1}(|x_1|)$, then $G(x)$ is called a *d-helix* around $S^{d-1}(|x_1|)\times\mathbb{R}^e$.

Definition 2.5. Let G be a closed and connected subgroup of Iso$(\mathbb{R}^n)$, $n\geq 3$. We say that G has compact (or helicoidical) factor, if there is an integer $0<m<n$ and there are Lie groups $G_1\subset$ Iso$(\mathbb{R}^{n-m})$, $G_2\subset$ Iso$(\mathbb{R}^m)$, such that
(1) G_2 is compact (or helicoidical on $\mathbb{R}^m$).
(2) $G\simeq G_2\times G_1$.
(3) For some(so each) $x\in\mathbb{R}^{n-m}$, $G_1(x)=\mathbb{R}^{n-m}$.

Corollary 2.6 ([7]). *If G is a connected and closed subgroup of* Iso$(\mathbb{R}^n)$,$n\geq 3$, *and* Coh$(G,\mathbb{R}^n)=2$. *Then one of the following is true:*
(I) *G is compact.* (II) *G has compact factor on $\mathbb{R}^n$.* (III) *G is helicoidical on $\mathbb{R}^n$.*
(IV) *G has helicoidical factor on $\mathbb{R}^n$.* (V) *All G-orbits are Euclidean.*

3 Orbit spaces

By Lemma 3.6 in [7] and its proof, we get the following fact.

Fact 3.1. If the action of G on $\mathbb{R}^n$ is helicoidical then one of the following assertions is true:
(1) G action on $\mathbb{R}^n$ is orbit equivalent to the action of a product $H\times T\subset SO(d)\times\mathbb{R}^e$ on $\mathbb{R}^d\times\mathbb{R}^e$, $d+e=n$, such that each principal H-orbit in $\mathbb{R}^d$ is isometric to $S^{d-1}(r)$, $r>0$, and T acts by cohomogeneity one on $\mathbb{R}^e$ such that all T-orbits on $\mathbb{R}^e$ are isometric to $\mathbb{R}^{e-1}$.
(2) Each principal G-orbit is isometric to a d-helix around $S^{d-1}(r)\times\mathbb{R}^e$, $e>1$, $r>0$, and G acts transitively on $\{o\}\times\mathbb{R}^e=\mathbb{R}^e$.

Fact 3.2. Let M be a Riemannian manifold and $\widetilde{M}$ be the Riemannian universal covering of M, by the covering map $k:\widetilde{M}\to M$, and let G be a closed and connected subgroup of Iso(M). Then there is a connected covering $\widetilde{G}$ for G such that $\widetilde{G}$ acts isometrically on $\widetilde{M}$ and the following assertions are true:
(1) Coh$(G,M)=$Coh$(\widetilde{G},\widetilde{M})$.
(2) If $D=\widetilde{G}(x)$ is a $\widetilde{G}$-orbit in $\widetilde{M}$ then $k(D)$ is a G-orbit in M, and each G-orbit in M is equal to $k(D)$ for some $\widetilde{G}$-orbit D in $\widetilde{M}$.
(3) If Δ is the deck transformation group of the covering $k:\widetilde{M}\to M$ then for each $\delta\in\Delta$ and each $g\in\widetilde{G}$, $\delta o g=g o\delta$. Thus δ maps $\widetilde{G}$-orbits in $\widetilde{M}$ on to $\widetilde{G}$-orbits.
(4) $\widetilde{M}^{\widetilde{G}}=\kappa^{-1}(M^G)$.

Proof. See [1], pages 63-64. □

Fact 3.3. Let Δ be a discrete subgroup of the isometries of $\mathbb{R}^m$, $m > 1$, and suppose that for each $a \in \mathbb{R}$, there is $a_1 \in \mathbb{R}$ such that $\Delta(\{a\} \times \mathbb{R}^{m-1}) = \{a_1\} \times \mathbb{R}^{m-1}$. Put

$$\Gamma = \{\delta \in \Delta : \delta(\{a\} \times \mathbb{R}^{m-1}) = \{a\} \times \mathbb{R}^{m-1} \ \ for \ \ all \ \ a \in \mathbb{R}\}.$$

Then, Γ is a normal subgroup of Δ and we have $\frac{\Delta}{\Gamma} = Z$.

Proof. It is clear from the definition of Γ that Γ is normal in Δ. Consider the function $p : \mathbb{R}^m (= \mathbb{R} \times \mathbb{R}^{m-1}) \to \mathbb{R}$ defined by $p(a, x) = a$, and put

$$\theta : \Delta \times \mathbb{R} \to \mathbb{R}, \ \ \theta(\delta, a) = p\delta(a, o), o = (0, ..., 0) \in \mathbb{R}^{m-1}.$$

Since for all $a \in \mathbb{R}$, $\Delta(\{a\} \times \mathbb{R}^{m-1}) = \{a_1\} \times \mathbb{R}^{m-1}$ for some a_1 related to a, then for each $x = (a, b) \in \mathbb{R} \times \mathbb{R}^{m-1}$ and $\delta \in \Delta$, we have $p\delta(a, b) = p\delta(a, o)$, so

$$p\delta(x) = p\delta(px, o) \quad (*)$$

Therefore, if $\delta_1, \delta_2 \in \Delta$ then

$$\theta(\delta_1, \theta(\delta_2, a)) = \theta(\delta_1, p\delta_2(a, o)) = p\delta_1(p\delta_2(a, o), o).$$

We get from (*) that

$$p\delta_1(p\delta_2(a, o), o) = p\delta_1\delta_2(a, o).$$

Thus, $\theta(\delta_1, \theta(\delta_2, a)) = \theta(\delta_1\delta_2, a)$. This means that θ is an action of Δ on $\mathbb{R}$. The action of Δ induces an effective action of $\frac{\Delta}{\Gamma}$ on $\mathbb{R}$, which is clearly an isometric action and no element of $\frac{\Delta}{\Gamma}$ has a fixed point in $\mathbb{R}$. So, $\frac{\Delta}{\Gamma}$ can be considered as a discrete subgroup of $(\mathbb{R}, +)$ and must be isomorphic to $(Z, +)$. □

Lemma 3.4 ([9]). *If M is a connected and complete cohomogeneity k Riemannian G-manifold then $k > dim M^G$.*

Theorem 3.5 ([8]). *If G is a closed and connected subgroup of* $\mathrm{Iso}\mathbb{R}^n$, $n \geq 2$, *and* $\mathrm{Coh}(G, \mathbb{R}^n) = 2$, *then* $\frac{\mathbb{R}^n}{G} = [0, \infty) \times \mathbb{R}$ *or* $\mathbb{R}^2$.

Lemma 3.6. *Let M be a flat Riemannian manifold, $dim M > 2$, and let G be a closed and connected subgroup of the isometries of M. If $Coh(M, G) = 2$ and $M^G \neq \varnothing$, then $\frac{M}{G}$ is homeomorphic to one of the following spaces:*

$$[0, +\infty) \times \mathbb{R}, S^1 \times [0, \infty), \mathbb{R}^2$$

Proof. Consider $\widetilde{M} = \mathbb{R}^n$ the universal Riemannian covering manifold of M, and use the symbols used in Fact 3.2. Since $M^G \neq \varnothing$ then by Fact 3.2(4), $\widetilde{M}^{\widetilde{G}} \neq \varnothing$. Put $L = \widetilde{M}^{\widetilde{G}}$ and let $m = \mathrm{dim} L$. By Lemma 3.4, we have $2 > m$, so $m = 0$ or $m = 1$.
If $m = 0$ then from the fact that $\widetilde{M}^{\widetilde{G}}$ is a (connected) totally geodesic submanifold

of $\mathbb{R}^n$, we get that $\widetilde{M}^{\widetilde{G}}$ is a one point set and by Fact 3.2(4), M is simply connected, so $M=\mathbb{R}^n$, $G=\widetilde{G}$. Then, by Theorem 3.5, $\frac{M}{G}=[0,\infty)\times\mathbb{R}$ or $\mathbb{R}^2$.
If $m=1$ and M is not simply connected, then L is a line in $\mathbb{R}^n$. Since the elements of $\widetilde{G}$ and Δ are commutative, then $\Delta(L)=L$. If $a\in L$, denote by W_a the hyperplane of $\mathbb{R}^n$ which is perpendicular to L at a. Without lose of generality we can suppose that $L=\{o\}\times\mathbb{R}\subset\mathbb{R}^{n-1}\times\mathbb{R}=\mathbb{R}^n$. Since $\widetilde{G}$ leaves L invariant point wisely, then $\widetilde{G}$ decomposes as $\widetilde{G}=\hat{G}\times\{I\}$, where $\hat{G}\subset SO(n-1)$ and I is the identity map on $\mathbb{R}$. So, for all $a\in L$ and all $x\in W_a$, $\widetilde{G}(x)\subset W_a$. Now, it is easy to show that the following map is a homeomorphism:

$$\begin{cases} \psi:\frac{\mathbb{R}^n}{\widetilde{G}}\to\frac{\mathbb{R}^{n-1}}{\hat{G}}\times\mathbb{R} \\ \psi(\widetilde{G}(x))=(\hat{G}(x_1),x_2) \quad , \quad x=(x_1,x_2)\in\mathbb{R}^{n-1}\times\mathbb{R} \end{cases}$$

Since $\mathrm{Coh}(\mathbb{R}^{n-1},\hat{G})=1$ then by Remark 2.2(3), $\frac{\mathbb{R}^{n-1}}{\hat{G}}=[0,\infty)$, so $\frac{\mathbb{R}^n}{\widetilde{G}}=[0,\infty)\times\mathbb{R}$. Since the members of Δ map $\widetilde{G}$-orbits to $\widetilde{G}$-orbits, then by curvture reasons, for each $(x_1,x_2)\in\mathbb{R}^{n-1}\times\mathbb{R}$, $\Delta(\hat{G}(x_1),x_2)=(\hat{G}(x_1),y_2)$ for some $y_2\in\mathbb{R}$. So, we get from $\Delta(L)=L$ that Δ decomposes as $\Delta=\{I\}\times\Gamma\subset\mathrm{Iso}(\mathbb{R}^{n-1})\times\mathrm{Iso}(L)$. Thus Δ can be considered as a discrete subgroup of the isometries of $L=\mathbb{R}$ without fixed point, then $\Delta=Z$, and we have

$$\frac{M}{G}=\frac{[0,\infty)\times\mathbb{R}}{\Delta}=[0,\infty)\times\frac{\mathbb{R}}{Z}=[0,\infty)\times S^1.$$

□

Remark 3.7.
(1) Let $E=\mathbb{R}^2$ or $[0,\infty)\times\mathbb{R}$, and Γ be a nontrivial discrete subgroup of the isometries of E such that $\Gamma(o)=o$, then $\frac{E}{\Gamma}$ is homeomorphic to $\mathbb{R}^2$ or $[0,\infty)\times\mathbb{R}$.
(2) If $\Gamma=Z$ and $E=[0,\infty)\times\mathbb{R}$, then $\frac{E}{\Gamma}=[0,\infty)\times S^1$.

Proof. (1) Let $E=\mathbb{R}^2$ and consider the circles $S^1(r)$ of radius $r>0$ around the origin of $\mathbb{R}^2$, and put $S^1(o)=o$. Since $\Gamma\subset O(2)$ is compact and discrete, it is finite. Consider a point $a\in S^1(1)$ and let $\Gamma(a)=\{a_1=a,a_2,...,a_n\}$ ordered in clockwise. Then, we have

$$\Gamma(ra)=\{ra_1,ra_2,...,ra_n\},\ ra\in S^1(r).$$

If b is the length of the arc $\widehat{a_1a_2}$ (clockwise arc) on $S^1(1)$ then the length of the arc $\widehat{ra_1ra_2}$ on $S^1(r)$ is equal to rb and we have $\frac{S^1(r)}{\Gamma}=S^1(rb)$. So,

$$\frac{\mathbb{R}^2}{\Gamma}=\bigcup_{r\geq0}\frac{S^1(r)}{\Gamma}=\bigcup_{rb\geq0}S^1(rb)=\mathbb{R}^2.$$

Now, let $E=[0,\infty)\times\mathbb{R}$. We know that the isometries of plane are combinations of three kind of isometries called rotations, reflections respect to lines, gelid reflections (see[3]). Since $\Gamma(E)=E$ and $\Gamma(o)=o$ then Γ can only contain a reflection respect to the line $[0,\infty)\times\{0\}$ and the identity, then $\frac{E}{\Gamma}$ is equal to $[0,\infty)\times[0,\infty)$, which is homeomorphic to $[0,\infty)\times\mathbb{R}$.
(2) Proof is similar to (1). □

4 Theorem B

Proof. Consider $\widetilde{M} = \mathbb{R}^n$ the universal covering manifold of M and use the symbols of Fact 3.2. Put

$$\Delta' = \{\delta \in \Delta : \delta(D) = D \ \ for \ \ all \ \ \widetilde{G} - orbits \ \ D \ \ in \ \ \mathbb{R}^n\}.$$

Since Δ' is normal in Δ, we can consider the quotient group $\widetilde{\Delta} = \frac{\Delta}{\Delta'}$. It is not hard to show that $\widetilde{\Delta}$ acts effectively on the orbit space $\widetilde{\Omega} = \frac{\mathbb{R}^n}{\widetilde{G}}$ and $\frac{M}{G} = \frac{\widetilde{\Omega}}{\widetilde{\Delta}}$. By Corollary 2.6, one of the following cases is true:
a) $\widetilde{G}$ is compact
b) $\widetilde{G}$ is helicoidical
c) $\widetilde{G}$ has compact factor
d) $\widetilde{G}$ has helicoidical factor
e) All orbits are Euclidean.

a) Since $\widetilde{G}$ is compact then $\widetilde{M}^{\widetilde{G}} \neq \varnothing$, so $M^G \neq \varnothing$ and we get the result from Theorem 3.6.

b) By Fact 3.1 and by suitable choice of ordinates, two cases may occur:
(1) $\widetilde{G}$ action is orbit equivalent to the action of a product $S \times T \subset So(d) \times \mathbb{R}^e$, $d + e = n$, on $\mathbb{R}^d \times \mathbb{R}^e$ such that each principal S-orbit in $\mathbb{R}^d$ is isometric to $S^{d-1}(r)$, $r > 0$, and T acts by cohomogeneity one on $\mathbb{R}^e$ such that all T-orbits are isometric to $\mathbb{R}^{e-1}$.
(2) Each principal $\widetilde{G}$-orbit is isometric to a helicoid around $S^{d-1}(r) \times \mathbb{R}^e$, $e > 1$, $r > 0$, and $\widetilde{G}$ acts transitively on $\{o\} \times \mathbb{R}^e = \mathbb{R}^e$.

In the case **(1)**, we have $\widetilde{\Omega} = \frac{\mathbb{R}^n}{\widetilde{G}} = \frac{\mathbb{R}^d}{S} \times \frac{\mathbb{R}^e}{T}$. Thus, by Remark 2.2 (3,1), $\widetilde{\Omega} = [0, \infty) \times \mathbb{R}$. If $x \in \{o\} \times \mathbb{R}^e$ then $dim\widetilde{G}(x) = e - 1$ and if $x \notin \{o\} \times \mathbb{R}^e$ then $dim\widetilde{G}(x) = d - 1 + e - 1 = d + e - 2$. Since $d > 1$, by dimensional reasons and the fact that each $\delta \in \Delta$ maps orbits to orbits, we get that $\Delta(\mathbb{R}^e) = \mathbb{R}^e$. Since $\widetilde{G}$ acts by cohomogeneity one on $\mathbb{R}^e$ and all orbits are Euclidean, then by Remark 2.2(1), and without lose of generality, we can suppose that each $\widetilde{G}$-orbit in $\mathbb{R}^e$ is equal to $\{b\} \times \mathbb{R}^{e-1}$ for some $b \in \mathbb{R}$ related to the orbit. Put

$$\Gamma = \{\delta \in \Delta : \delta(D) = D, \ \ for \ \ all \ \ orbits \ \ D \ \ in \ \ \mathbb{R}^e\}.$$

By Fact 3.3, we have $\frac{\Delta}{\Gamma} = Z$. It is not hard to show that $\Gamma = \Delta'$, so $\widetilde{\Delta} = \frac{\Delta}{\Gamma} = Z$. Then $\frac{M}{G} = \frac{\widetilde{\Omega}}{\widetilde{\Delta}} = \frac{\widetilde{\Omega}}{Z}$. Since $\widetilde{\Omega} = [0, \infty) \times \mathbb{R}$, then we get from Remark 3.7(2), that $\frac{M}{G} = [0, \infty) \times S^1$.

In the case **(2)**, First note that by Theorem 3.5, $\widetilde{\Omega} = \frac{\mathbb{R}^n}{\widetilde{G}} = \mathbb{R}^2$ or $[0, \infty) \times \mathbb{R}$. Since the elements of Δ are isometries which map orbits to orbits, then by curvature reasons, $\Delta(\{o\} \times \mathbb{R}^e) = \{o\} \times \mathbb{R}^e$. Without lose of generality we can suppose that the corresponding point of the orbit $\{o\} \times \mathbb{R}^e$ on the orbit space $\frac{\mathbb{R}^n}{\widetilde{G}}(= \mathbb{R}^2$ or $[0, \infty) \times \mathbb{R})$ is the point o the origin of $\mathbb{R}^2$ or $[0, \infty) \times \mathbb{R}$. Then o is a fixed point of the action of

$\widetilde{\Delta}$ on $\widetilde{\Omega}$. Then, by Remark 3.7(1), $\frac{M}{G} = \frac{\widetilde{\Omega}}{\widetilde{\Delta}} = [0, \infty) \times \mathbb{R}$ or $\mathbb{R}^2$.

c, d) If $\widetilde{G}$ has compact factor or helicoidical factor, then we have $\widetilde{G} = G_1 \times G_2$ and $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ such that G_1 is compact or helicoidical on $\mathbb{R}^{n_1}$ and G_2 acts transitively on $\mathbb{R}^{n_2}$. So, we have

$$\frac{\mathbb{R}^n}{\widetilde{G}} = \frac{\mathbb{R}^{n_1}}{G_1} \times \frac{\mathbb{R}^{n_2}}{G_2} = \frac{\mathbb{R}^{n_1}}{G_1}$$

The effective action of $\widetilde{\Delta}$ on $\frac{\mathbb{R}^n}{\widetilde{G}}$ induces an effective action of $\widetilde{\Delta}$ on $\frac{\mathbb{R}^{n_1}}{G_1}$ in the following way:
Each $\widetilde{G}$-orbit is in the form $D \times \mathbb{R}^{n_2}$ such that D is a G_1-orbit in $\mathbb{R}^{n_1}$. For each $\widetilde{\delta} \in \widetilde{\Delta}$, we have $\widetilde{\delta}(D \times \mathbb{R}^{n_2}) = D' \times \mathbb{R}^{n_2}$. Put $\widetilde{\delta}(D) = D'$. Then we get the from previous arguments that theorem is true in this case.

e) In this case all $\widetilde{G}$-orbits in $\mathbb{R}^n$ are isometric to $\mathbb{R}^{n-2}$ then each G-orbit is flat, which is contradiction by assumptions of the theorem. □

Acknowledgements. I am very grateful to the organizers of "2018 International Conference on Topology and Its Applications" to give an opportunity for useful discussions about the work.

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