A rigidity result on Finsler surfaces

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Abstract. We introduce a new non-Riemannian quantity named mean stretch curvature. A Finsler metric with vanishing mean stretch curvature is called weakly stretch metric. We prove that every complete P-reducible weakly stretch metric with bounded Cartan torsion is a Landsberg metric. Then, we classify complete weakly stretch surfaces and show that every complete weakly stretch surface is Riemannian or Landsbergian. This provides a natural extension of Szabó's rigidity theorem on Berwald surfaces.

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1 Introduction

In Finsler geometry, the first non-Riemannian quantity called by Cartan torsion $\mathbf{C}$, was first introduced by Finsler and emphasized by Cartan. Other than the Cartan torsion, there are several important non-Riemannian quantities: the Berwald curvature $\mathbf{B}$, the Landsberg curvature $\mathbf{L}$, the mean Landsberg curvature $\mathbf{J}$, the stretch curvature $\mathbf{\Sigma}$, etc. Recently, some new interesting and meaningful non-Riemannian quantities $\mathbf{H}$-curvature, Ξ-curvature, χ-curvature and $\hat{C}$-curvature have been introduced in Finsler geometry (see [3], [8], [9]). They all vanish for Riemannian metrics, hence they are said to be non-Riemannian. These non-Riemannian geometric quantities describe the difference between Finsler geometry and Riemann geometry. The study of these quantities is benefit for us to make out their distinction and the nature of Finsler geometry.

Let (M, F) be a Finsler manifold. There are two basic tensors on Finsler manifolds: fundamental metric tensor $\mathbf{g}_y$ and the Cartan torsion $\mathbf{C}_y$, which are the second and the third order derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$, respectively. The rate of change of the Cartan torsion along geodesics, $\mathbf{L}_y$ is said to be Landsberg curvature. Taking trace with respect to g_y in first and second variables of $\mathbf{C}_y$ and $\mathbf{L}_y$ gives rise to the mean Cartan torsion $\mathbf{I}_y$ and to the mean Landsberg curvature $\mathbf{J}_y$, respectively. The

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mean Landsberg curvature is the rate of change of the mean Cartan torsion along geodesics.

In [2], L. Berwald introduced a non-Riemannian curvature, so-called *stretch curvature* and denoted this by $\mathbf{\Sigma}_y$. He showed that this tensor vanishes if and only if the length of a vector remains unchanged under the parallel displacement along an infinitesimal parallelogram. This curvature has been investigated by Shibata and Matsumoto [6]. A Finsler metric is said to be *stretch metric* if $\mathbf{\Sigma} = 0$. Taking trace with respect to $\mathbf{g}_y$ in the first and second variables of $\mathbf{\Sigma}_y$ gives rise to the *mean stretch curvature* $\bar{\mathbf{\Sigma}}_y$. A Finsler metric is said to be *weakly stretch metric* if $\bar{\mathbf{\Sigma}} = 0$. A Finsler metric F is called P-reducible if its Landsberg curvature is given by following

$$L_{ijk} = \frac{1}{n+1}\Big\{h_{ij}J_k + h_{jk}J_i + h_{ki}J_j\Big\},$$

where $h_{ij} := FF_{ij}$ is the angular metric. Every Randers metric $F = \alpha + \beta$ and Kropina metric $F = \alpha^2/\beta$ are P-reducible [7]. Then, we get the following.

Theorem 1.1. *Every complete P-reducible weakly stretch metric with bounded Cartan torsion is a Landsberg metric.*

In [10], Szabó considered Berwald surfaces and proved a rigidity theorem: any Berwald surface is either Riemannian. or locally Minkowskian. Berwald spaces have been classified by Szabó in [10] and explicitly constructed in [11] (for more details, see Chapter 10 in [1]). On the other hand, the class of Berwald metrics is a subclass of the class of weakly stretch metrics [12]. One might wonder if there exists non-Riemmannian and non-locally Minkowskian weakly stretch surfaces for which the Finsler structure is smooth and strongly convex on the slit tangent bundle. This motivates us to consider weakly stretch surfaces.

Theorem 1.2. *Every complete weakly stretch surface is Riemannian or Landsbergian.*

Throughout this paper, we use the Berwald connection on Finsler manifolds. The h- and v- covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively.

2 Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M. A *Finsler metric* on M is a function $F : TM \to [0, \infty)$ which has the following properties: (i) F is C^∞ on $TM_0 := TM \setminus \{0\}$; (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, and (iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(y + su + tv)\right]|_{s,t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\left[\mathbf{g}_{y+tw}(u, v)\right]|_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C_y}\}_{\mathbf{y}\in \mathbf{TM_0}}$ is called the Cartan torsion. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian. For $y \in T_xM_0$, define mean Cartan torsion $\mathbf{I}_y$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$. By Diecke Theorem, F is Riemannian if and only if $\mathbf{I}_y = 0$.

The horizontal covariant derivatives of $\mathbf{C}$ along geodesics give rise to the Landsberg curvature $\mathbf{L}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ defined by $\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^iv^jw^k$, where $L_{ijk} := C_{ijk|s}y^s$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y\in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L} = 0$ [4]. The horizontal covariant derivatives of $\mathbf{I}$ along geodesics give rise to the mean Landsberg curvature $\mathbf{J}_y(u) := J_i(y)u^i$, where $J_i := g^{jk}L_{ijk}$. A Finsler metric is said to be weakly Landsbergian if $\mathbf{J} = 0$.

Define the stretch curvature $\mathbf{\Sigma}_y : T_xM\otimes T_xM\otimes T_xM\otimes T_xM \to \mathbb{R}$ by $\mathbf{\Sigma}_y(u, v, w, z) := \Sigma_{ijkl}(y)u^iv^jw^kz^l$, where

$$\Sigma_{ijkl} := 2(L_{ijk|l} - L_{ijl|k}).$$

A Finsler metric is said to be *stretch metric* if $\mathbf{\Sigma} = 0$ [2]. Every Landsberg metric is a stretch metric. It is well known that $\mathbf{\Sigma} = 0$ if and only if the length of a vector remains unchanged under the parallel displacement along an infinitesimal parallelogram. Taking an average on two first indices of the stretch curvature, we get a new non-Riemannian curvature *mean stretch curvature*.

For a non-zero vector $y \in T_xM_0$, define $\bar{\mathbf{\Sigma}}_y : T_xM \otimes T_xM \to \mathbb{R}$ by $\bar{\mathbf{\Sigma}}_y(u, v) := \bar{\Sigma}_{ij}(y)u^iv^j$, where $\bar{\Sigma}_{ij} := g^{kl}\Sigma_{klij}$. A Finsler metric is said to be *weakly stretch metric* if $\bar{\mathbf{\Sigma}} = 0$. It is easy to see that every Landsberg metric or stretch Finsler metric is a weakly stretch metric.

Given a Finsler manifold (M, F), a global vector field $\mathbf{G}$ is induced by F on slit tangent bundle TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$, where

$$G^i(x, y) := \frac{1}{4}g^{il}\left\{\frac{\partial^2F^2}{\partial x^k\partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\right\}.$$

$\mathbf{G}$ is called the associated spray to (M, F). In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

3 Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. First, we prove the following.

Lemma 3.1. *Let (M, F) be P-reducible manifold. Suppose that F is weakly stretch metric. Then for any geodesic $c(t)$ and any parallel vector field $V(t)$ along c, the function $\mathbf{C}(t) := \mathbf{C}_{\dot{c}}(V(t))$ must be in the following form*

$$\mathbf{C}(t) = t\ \mathbf{L}(0) + \mathbf{C}(0). \tag{3.1}$$

Proof. F is P-reducible

$$L_{ijk} = \frac{1}{n+1}\Big\{h_{ij}J_k + h_{jk}J_i + h_{ki}J_j\Big\}. \tag{3.2}$$

Taking a horizontal derivative of (3.2) along Finslerian geodesics implies that

$$L_{ijk|s}y^s = \frac{1}{n+1}\Big\{h_{ij}J_{k|s}y^s + h_{jk}J_{i|s}y^s + h_{ki}J_{j|s}y^s\Big\}. \tag{3.3}$$

Since F is weakly stretch metric $J_{i|j} = J_{j|i}$, then by contracting it with y^j we get

$$J_{i|j}y^j = 0. \tag{3.4}$$

By (3.3) and (3.4), we get

$$L_{ijk|s}y^s = 0. \tag{3.5}$$

Let

$$\mathbf{C}(t) := \mathbf{C}_{\dot{c}}(V(t)), \qquad \mathbf{L}(t) := \mathbf{L}_{\dot{c}}(V(t)). \tag{3.6}$$

From our definition of $\mathbf{L}_y$, we have $\mathbf{L}(t) = \mathbf{C}^{'}(t)$. Then by (3.5) and (3.6), we obtain

$$\mathbf{C}^{''}(t) = \mathbf{L}^{'}(t) = L_{i|l}(\dot{c}(t))\dot{c}^l(t)V^i(t) = 0. \tag{3.7}$$

Then (3.1) follows. □

Remark 3.1. Let (M, F) be a Finsler space and $c : [a, b] \to M$ be a geodesic. For a parallel vector field $V(t)$ along c,

$$g_{\dot{c}}(V(t), V(t)) = constant. \tag{3.8}$$

Proof of Theorem 1.1: Let (M, F) be a complete Finsler manifold. Take an arbitrary unit vector $y \in T_xM$ and an arbitrary vector $v \in T_xM$. Let $c(t)$ be the geodesic with $c(0) = x$ and $\dot{c}(0) = y$ and $V(t)$ be the parallel vector field along c with $V(0) = v$. Then by Lemma 3.1, we get

$$\mathbf{C}(t) = t\ \mathbf{L}(0) + \mathbf{C}(0). \tag{3.9}$$

Suppose that $\mathbf{C}_y$ is bounded, i.e., there is a constant $B < \infty$ such that

$$||\mathbf{C}||_x := \sup_{y \in T_xM_0} \sup_{v \in T_xM} \frac{\mathbf{C}_y(v)}{[\mathbf{g}_y(v, v)]^{\frac{3}{2}}} \le B \tag{3.10}$$

By Remark 3.1, we have $|\mathbf{C}(t)| \le BT^{\frac{3}{2}} < \infty$ for some constant T. Therefore, $\mathbf{C}(t)$ is a bounded function on $[0, \infty)$. (3.9) implies that $\mathbf{L}_y(v) = \mathbf{L}(0) = 0$. Hence, F is a Landsberg metric. □

It is proved that every C-reducible metric with vanishing Landsberg curvature is a Berwald metric [5][6]. Then by the Theorem 1.1, we get the following.

Corollary 3.2. *Every complete C-reducible weakly stretch metric with bounded Cartan torsion is a Berwald metric.*

4 Finsler surfaces

Let (M,F) be a two-dimensional Finsler manifold. We refer to the Berwald's frame (ℓ^i, m^i) where $\ell^i = y^i/F(y)$, m^i is the unit vector with $\ell_i m^i = 0$ and $\ell_i = g_{ij}\ell^i$.

Theorem 4.1. *Every two-dimensional Finsler metric is a stretch metric if and only if it is a weakly stretch metric if and only if* $I_{|1|1} = 0$.

Proof. Since the Cartan torsion has no components in the direction ℓ^i, i.e., $C_{ijk}y^i = 0$, then it can be written in the frame (ℓ, m) as follows

$$FC_{ijk} = Im_i m_j m_k, \tag{4.1}$$

where the scalar field I is called the main scalar of F. By taking a horizontal derivation of (4.1), we get

$$FC_{ijk|l} = (I_{|1}\ell_l + I_{|2}m_l)m_i m_j m_k. \tag{4.2}$$

Contracting (4.2) with y^l yields

$$L_{ijk} = I_{|1}m_i m_j m_k. \tag{4.3}$$

Taking a horizontal derivation of (4.3) implies that

$$\begin{aligned}\Sigma_{ijkl} &= 2\big[L_{ijk|l} - L_{ijl|k}\big]\\ &= 2\Big[(I_{|1|1}\ell_l + I_{|1|2}m_l)m_k - (I_{|1|1}\ell_k + I_{|1|2}m_k)m_l\Big]m_i m_j\\ &= 2I_{|1|1}(\ell_l m_k - \ell_k m_l)m_i m_j. \end{aligned}\tag{4.4}$$

Since $g^{ij}m_i m_j = m^j m_j = 1$, then by multiplying (4.4) with g^{ij}, we have

$$\bar{\Sigma}_{kl} = 2I_{|1|1}(\ell_l m_k - \ell_k m_l). \tag{4.5}$$

By (4.4) and (4.5), we get the proof. □

Proof of Theorem 1.2: It follows that the Berwald curvature is given by

$$B^i{}_{jkl} = F^{-1}(-2I_{,1}\ell^i + I_2 m^i)m_j m_k m_l,$$

where I is the 0-homogeneous function called *the main scalar of* F and $I_2 = I_{,2} + I_{,1|2}$. Since the Cartan tensor of F is given by $C_{ijk} = F^{-1}Im_i m_j m_k$, then the Berwald curvature can be written as

$$B^i{}_{jkl} = \mu C_{jkl}\ell^i + \lambda(h^i_j h_{kl} + h^i_k h_{jl} + h^i_l h_{jk}), \tag{4.6}$$

where $h_{ij} := m_i m_j$ is the angular metric, $\mu = -2I_{,1}/I$, and $\lambda = I_2/3$. Contracting (4.6) with y_i, we infer

$$L_{ijk} = -\frac{1}{2}\mu F C_{ijk}. \tag{4.7}$$

Taking the trace of (4.7) yields

$$J_i = -\frac{1}{2}\mu F I_i. \tag{4.8}$$

Thus

$$J_{i|j} = -\frac{1}{2}F(\mu_j I_i + \mu I_{i|j}), \tag{4.9}$$

where $\mu_i := \mu_{|j}$. Therefore

$$\bar{\Sigma}_{ij} = -F\Big[\mu_j I_i - \mu_i I_j + \mu(I_{i|j} - I_{j|i})\Big]. \tag{4.10}$$

By the assumption and (4.10), we get

$$\mu_j I_i - \mu_i I_j = \mu(I_{j|i} - I_{i|j}). \tag{4.11}$$

Multiplying (4.11) with y^j yields

$$\mu' I_i = -\mu J_i. \tag{4.12}$$

Putting (4.8) in (4.12) implies that

$$[2\mu' - \mu^2 F]I_i = 0. \tag{4.13}$$

If $I_i = 0$, then F reduces to a Riemannian metric. Suppose that F is a non-Riemannian metric. Then we have

$$2\mu' - \mu^2 F = 0. \tag{4.14}$$

On a Finslerian geodesics, we have

$$2\mu' = \mu^2 \tag{4.15}$$

which its general solution is

$$\mu(t) = \frac{\mu(0)}{1 - t\mu(0)}. \tag{4.16}$$

By considering $||\mu(0)|| < \infty$, and letting $t \to +\infty$ implies that $\mu = 0$. By putting it in (4.7), it follows that F is a Landsberg metric.

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