Geometric bounds for δ-Casorati curvature in statistical submanifolds of statistical space forms

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Abstract. Lee et al. in [12] proved pinching theorem with normalized scalar curvature for statistical submanifolds of statistical manifolds of constant curvature. In this paper with a pair of conjugate connections $\overline{\nabla}$ and $\overline{\nabla}^*$, we generalize the result of [12] and derive bounds for generalized normalized δ-Casorati curvatures of statistical submanifolds in statistical manifold of constant curvature. The paper finishes with an application of divergence of Mean curvature vector field of statistical manifold.

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Key words: Statistical manifold; dual connection; Casorati curvature; generalized normalized δ-Casorati curvature; normalized scalar curvature.

1 Introduction

The theory of abstract generalization of statistical model as statistical manifold is a fast growing area of research in global differential geometry. In 1985, the concept of statistical manifolds (which was initiated from exploration of geometric structures on sets of certain probability distribtions) was introduced by Amari [1] which provide a setting for the field of information geometry and it also associate a dual connection (known as conjugate connection). The applications of statistical manifold attracts the attention of distinguished geometers due to its applications in the field of science and engineering. Many papers have been appeared in the literature of different submanifolds of different manifolds in the setting of statistical manifold (see [1, 2, 9, 14, 15, 17, 18]).

On the other hand, Casorati proposed the concept of an extrinsic invariant of a submanifold of Riemannian manifold, named as Casorati curvature is stated by the normalized square length of the second fundamental form [6]. The consideration of Casorati curvature widen the consideration of the principal direction of a hypersurface of a Riemannian manifold [10]. Its congruous's essence and influence have been examined by some well-known authors in a global differential geometry [3, 4, 5, 7, 8, 11, 13, 19].

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In the spirit of these quoted summary and stimulated by generalized normalized δ-Casorati curvatures, we have established the succeeding results.

Theorem 1.1. *Let $\mathcal{M}^m$ be a statistical submanifold of statistical manifold $N^n(c)$ of constant curvature c. Then, the generalized normalized δ-Casorati curvature $\delta_c^\circ(r,m-1)$ satisfy*

$$\rho \leq \frac{2\delta_c^\circ(r,m-1)}{m(m-1)} + \frac{\mathcal{C}^\circ}{m-1} - \frac{2m}{m-1}||H^\circ||^2 + \frac{m}{(m-1)}\overline{\mathfrak{g}}(H,H^*) + c, \tag{1.1}$$

where $2\delta_c^\circ(r,m-1) = \delta_c(r,m-1) + \delta_c^*(r,m-1)$.
This means, the normalized scalar curvature has a supremum by Casorati curvatures.

Theorem 1.2. *Let $\mathcal{M}^m$ be a statistical submanifold of statistical manifold $N^n(c)$ of constant curvature c. Then, the generalized normalized δ-Casorati curvature $\delta_c^\circ(r,m-1)$ satisfy*

$$\rho \geq -\frac{\delta_c^\circ(r,m-1)}{m(m-1)} + \frac{2m}{m-1}||H^\circ||^2 - \frac{2\mathcal{C}^\circ}{(m-1)} + c, \tag{1.2}$$

where $2\delta_c^\circ(r,m-1) = \delta_c(r,m-1) + \delta_c^*(r,m-1)$.
This means, the normalized scalar curvature has infimum given by Casorati curvatures.

2 Statistical Manifold

In this section, we collect certain couple of intrinsic analogues or terminologies in the setting of statistical manifold.

Definition 2.1. A Riemannian manifold $(\mathcal{N}^n, \overline{\mathfrak{g}}, \overline{\nabla})$ with a couple of torsionless affine connections $\overline{\nabla}$ and $\overline{\nabla}^*$ is statistical manifold if it fascinates [18]

$$(\overline{\nabla}_{\mathbb{X}}\overline{\mathfrak{g}})(\mathbb{Y},\mathbb{Z}) = (\overline{\nabla}_{\mathbb{Y}}\overline{\mathfrak{g}})(\mathbb{X},\mathbb{Z}), \tag{2.1}$$

$$\mathbb{X}\overline{\mathfrak{g}}(\mathbb{Y},\mathbb{Z}) = \overline{\mathfrak{g}}(\overline{\nabla}_{\mathbb{X}}\mathbb{Y},\mathbb{Z}) + \overline{\mathfrak{g}}(\mathbb{Y},\overline{\nabla}^*_{\mathbb{X}}\mathbb{Z}), \tag{2.2}$$

for any $\mathbb{X},\mathbb{Y},\mathbb{Z} \in \Gamma(T\mathcal{N})$. Then, $\overline{\nabla}$ and $\overline{\nabla}^*$ are called *dual (or conjugate) connections* and the pair $(\overline{\nabla}, \overline{\mathfrak{g}})$ is called statistical structure. Also, it is easily shown that $(\overline{\nabla}^*)^* = \overline{\nabla}$

Remark 2.2. [18] If $(\overline{\nabla}, \overline{\mathfrak{g}})$ is a statistical structure then so is $(\overline{\nabla}^*, \overline{\mathfrak{g}})$ where the dual connection $\overline{\nabla}^*$ is defined in terms of the Levi-Civita connection $\overline{\nabla}^\circ$ as

$$\overline{\nabla} + \overline{\nabla}^* = 2\overline{\nabla}^\circ. \tag{2.3}$$

Let us suppose that $\overline{R}$ and $\overline{R}^*$ be the curvature tensor fields of $\overline{\nabla}$ and $\overline{\nabla}^*$ respectively. A statistical structure $(\overline{\nabla}, \overline{\mathfrak{g}})$ is said to be of constant curvature c if it satisfy

$$\overline{R}(\mathbb{X},\mathbb{Y})\mathbb{Z} = c\{\overline{\mathfrak{g}}(\mathbb{Y},\mathbb{Z})\mathbb{X} - \overline{\mathfrak{g}}(\mathbb{X},\mathbb{Z})\mathbb{Y}\}, \tag{2.4}$$

for any $\mathbb{X}, \mathbb{Y}, \mathbb{Z} \in \Gamma(T\mathcal{N})$ where c is a real constant.

Now, we would first take a look on the definition of submanifold of statistical manifold and after defining this we will see some notations, general formulas.

Let us consider m-dimensional submanifold $\mathcal{M}^m$ in statistical manifold $(\mathcal{N}, \bar{\mathfrak{g}})$ with pairs of :

$$\begin{cases} induced\ connections, & \nabla, \nabla^*; \\ second\ fundamental\ forms, & \zeta, \zeta^*; \\ shape\ operators, & \Lambda, \Lambda^*; \\ normal\ connections, & \nabla^\perp, \nabla^{*\perp}. \end{cases}$$

Moreover, the induced metric $\mathfrak{g}$ is unique, $(\nabla, \mathfrak{g})$ and $(\nabla^*, \mathfrak{g})$ are induced dual statistical structures on the submanifolds. Then, the fundamental Gauss formulas are outlined by [18]

$$\overline{\nabla}_{\mathbb{X}}\mathbb{Y} = \nabla_{\mathbb{X}}\mathbb{Y} + \zeta(\mathbb{X}, \mathbb{Y}), \tag{2.5}$$

$$\overline{\nabla}^*_{\mathbb{X}}\mathbb{Y} = \nabla^*_{\mathbb{X}}\mathbb{Y} + \zeta^*(\mathbb{X}, \mathbb{Y}), \tag{2.6}$$

for $\mathbb{X}, \mathbb{Y} \in \Gamma(T\mathcal{M})$ whereas ζ and ζ^* are bilinear mapping from which bilinear transformations $\Lambda_{\mathbb{N}}$ and $\Lambda^*_{\mathbb{N}}$ are given by [18]

$$\mathfrak{g}(\Lambda_{\mathbb{N}}\mathbb{X}, \mathbb{Y}) = \mathfrak{g}(\zeta(\mathbb{X}, \mathbb{Y}), \mathbb{N}), \tag{2.7}$$

$$\mathfrak{g}(\Lambda^*_{\mathbb{N}}\mathbb{X}, \mathbb{Y}) = \mathfrak{g}(\zeta^*(\mathbb{X}, \mathbb{Y}), \mathbb{N}), \tag{2.8}$$

for any $\mathbb{N} \in \Gamma(T^\perp\mathcal{M})$. Furthermore, the fundamental Weingarten formulas are given by [18]

$$\overline{\nabla}_{\mathbb{X}}\mathbb{N} = -\Lambda^*_{\mathbb{N}}\mathbb{X} + \nabla^\perp_{\mathbb{X}}\mathbb{N}, \tag{2.9}$$

$$\overline{\nabla}^*_{\mathbb{X}}\mathbb{N} = -\Lambda_{\mathbb{N}}\mathbb{X} + \nabla^{*\perp}_{\mathbb{X}}\mathbb{N}, \tag{2.10}$$

for $\mathbb{N} \in \Gamma(T^\perp\mathcal{M})$ and $\mathbb{X} \in \Gamma(T\mathcal{M})$ whereas the normal dual connections $\nabla^\perp$ and ${\nabla^*}^\perp$ are the Riemannian dual connections on $\mathcal{M}^\perp$.

Let us denote R and R^* to be the curvature tensor field of ∇ and ∇^*. Then, the fundamental Gauss equation follows [18]

$$\begin{aligned} \bar{\mathfrak{g}}(\overline{R}(\mathbb{X}, \mathbb{Y})\mathbb{Z}, \mathbb{W}) = &\mathfrak{g}(R(\mathbb{X}, \mathbb{Y})\mathbb{Z}, \mathbb{W}) + \bar{\mathfrak{g}}(\zeta(\mathbb{X}, \mathbb{Z}), \zeta^*(\mathbb{Y}, \mathbb{W})) \\ &- \bar{\mathfrak{g}}(\zeta^*(\mathbb{X}, \mathbb{W}), \zeta(\mathbb{Y}, \mathbb{Z})). \end{aligned} \tag{2.11}$$

Now, let $\{e_i\}_1^m$ and $\{e_i\}_{m+1}^n$ be orthonormal basis of $T_p\mathcal{M}$ and $T_p^\perp\mathcal{M}$, respectively. Then, the Mean curvature vector fields H and H^* have the following forms [12]

$$H = \frac{1}{m}\sum_{i=1}^m \zeta(e_i, e_i) = \frac{1}{m}\sum_{\alpha=m+1}^n \left(\sum_{i=1}^m \zeta_{ii}^\alpha\right)e_\alpha, \tag{2.12}$$

$$H^* = \frac{1}{m}\sum_{i=1}^m \zeta^*(e_i, e_i) = \frac{1}{m}\sum_{\alpha=m+1}^n \left(\sum_{i=1}^m \zeta_{ii}^{*\alpha}\right)e_\alpha, \tag{2.13}$$

where $\zeta_{ij}^{\alpha} = \bar{\mathfrak{g}}\big(\zeta(e_i, e_j), e_\alpha\big)$ and $\zeta_{ij}^{*\alpha} = \bar{\mathfrak{g}}\big(\zeta^*(e_i, e_j), e_\alpha\big)$.
Moreover, the squared Mean curvatures are given by [12]

$$||H||^2 = \frac{1}{m^2} \sum_{\alpha=m+1}^{n} \left(\sum_{i=1}^{m} \zeta_{ii}^{\alpha} \right)^2, \quad ||H^*||^2 = \frac{1}{m^2} \sum_{\alpha=m+1}^{n} \left(\sum_{i=1}^{m} \zeta_{ii}^{*\alpha} \right)^2.$$

The scalar curvature τ at p is given by [12]

$$\tau(p) = \sum_{1 \leq i \leq j \leq m} \mathfrak{g}(R(e_i, e_j)e_j, e_i) \tag{2.14}$$

and the normalized scalar curvature ρ of $\mathcal{M}$ is defined by [12]

$$\rho = \frac{2\tau}{m(m-1)}.$$

The Casorati curvatures $\mathcal{C}$ and $\mathcal{C}^*$ of the submanifold $\mathcal{M}$ can be expressed as [12]

$$\mathcal{C} = \frac{1}{m} \sum_{\alpha=m+1}^{n} \sum_{i,j=1}^{m} (\zeta_{ij}^{\alpha})^2 = \frac{||\zeta||^2}{m}, \qquad \mathcal{C}^* = \frac{1}{m} \sum_{\alpha=m+1}^{n} \sum_{i,j=1}^{m} (\zeta_{ij}^{*\alpha})^2 = \frac{||\zeta^*||^2}{m}.$$

Now, let us denote a k-dimensional subspace of $T_p\mathcal{M}$ by $\mathcal{L}$, where $k > 2$ and $\{e_i\}_1^k$ as an orthonormal basis of $\mathcal{L}$. Then, $\mathcal{C}(\mathcal{L})$ and $\mathcal{C}^*(\mathcal{L})$ of $\mathcal{L}$ are defined as follows

$$\mathcal{C}(\mathcal{L}) = \frac{1}{k} \sum_{\alpha=m+1}^{n} \sum_{i,j=1}^{k} (\zeta_{ij}^{\alpha})^2, \qquad \mathcal{C}^*(\mathcal{L}) = \frac{1}{k} \sum_{\alpha=m+1}^{n} \sum_{i,j=1}^{k} (\zeta_{ij}^{*\alpha})^2.$$

We denote

$$\mathcal{B} = \{\mathcal{C}(\mathcal{L}) : \ \mathcal{L} \text{ is a hyperplane of } T_p\mathcal{M}\},$$
$$\mathcal{B}^* = \{\mathcal{C}^*(\mathcal{L}) : \ \mathcal{L} \text{ is a hyperplane of } T_p\mathcal{M}\}.$$

The normalized δ-Casorati curvatures $\delta_c(m-1)$ and $\tilde{\delta}_c(m-1)$ of $\mathcal{M}^m$ are given as follows [12]

$$\begin{aligned} [\delta_c(m-1)]_p &= \frac{1}{2}\mathcal{C}_p + \Big(\frac{m+1}{2m}\Big) \inf \mathcal{B}, \\ [\tilde{\delta}_c(m-1)]_p &= 2\mathcal{C}_p + \Big(\frac{2m-1}{2m}\Big) \sup \mathcal{B}. \end{aligned}$$

Moreover, the dual normalized δ^*-Casorati curvatures $\delta_c^*(m-1)$ and $\tilde{\delta}_c^*(m-1)$ of the submanifold $\mathcal{M}^m$ are given as [12]

$$\begin{aligned} [\delta_c^*(m-1)]_p &= \frac{1}{2}\mathcal{C}_p^* + \Big(\frac{m+1}{2m}\Big) \inf \mathcal{B}^*, \\ [\tilde{\delta}_c^*(m-1)]_p &= 2\mathcal{C}_p^* + \Big(\frac{2m-1}{2m}\Big) \sup \mathcal{B}^*. \end{aligned}$$

Then, the generalized normalized δ-Casorati curvatures $\delta_c(r; m-1)$ and $\tilde{\delta}_c(r; m-1)$ of $\mathcal{M}$ for $A(r, m-1) = \frac{(m-1)(m+r)[m^2-m-r]}{rm}$ are defined as [13]:

$$\begin{aligned} [\delta_c(r; m-1)](p) &= r\mathcal{C}(p) + A(r, m-1) \inf \mathcal{B}, \quad \text{if} \quad 0 < r < m(m-1), \\ [\tilde{\delta}_c(r; m-1)](p) &= r\mathcal{C}(p) + A(r, m-1) \sup \mathcal{B}, \quad \text{if} \quad r > m(m-1). \end{aligned}$$

Further, the dual generalized normalized δ^*-Casorati curvatures $\delta_c^*(r; m-1)$ and $\tilde{\delta}_c^*(r; m-1)$ of the submanifold $\mathcal{M}^m$ are defined as

$$\begin{aligned} [\delta_c^*(r; m-1)](p) &= r\mathcal{C}^*(p) + A(r, m-1) \inf \mathcal{B}^*, \quad \text{if} \quad 0 < r < m(m-1), \\ [\tilde{\delta}_c^*(r; m-1)](p) &= r\mathcal{C}^*(p) + A(r, m-1) \sup \mathcal{B}^*, \quad \text{if} \quad r > m(m-1). \end{aligned}$$

Here, one can note that $\delta_c(r; m-1)$ and $\tilde{\delta}_c(r; m-1)$ are the generalized versions of $\delta_c(m-1)$ and $\tilde{\delta}_c(m-1)$ respectively by substituting r to $\frac{m(m-1)}{2}$ as

$$\begin{aligned} \left[\delta_c\left(\frac{m(m-1)}{2}; m-1\right)\right](p) &= m(m-1)[\delta_c(m-1)](p) \quad \text{and} \\ \left[\tilde{\delta}_c\left(\frac{m(m-1)}{2}; m-1\right)\right](p) &= m(m-1)[\tilde{\delta}_c(m-1)](p), \end{aligned}$$

for $p \in \mathcal{M}$.

3 Proof of Main Results

First we need a lemma, which plays an important role in the proof of our main theorems.

Lemma 3.1. *[16] Let $\mathcal{S} = \{(x_1, x_2, ..., x_m) \in R^m : x_1 + x_2 + ... + x_m = k\}$ be a hyperplane of R^n and $f : R^m \to R$ a quadratic form stated as*

$$f(x_1, x_2, ..., x_m) = a \sum_{i=1}^{m-1} (x_i)^2 + b(x_m)^2 - 2 \sum_{1 \le i < j \le m} x_i x_j, a > 0, b > 0.$$

Then by the constrained extremum problem, f has a global solution given by

$$x_1 = x_2 = ... = x_{m-1} = \frac{k}{a+1}, x_n = \frac{k}{b+1} = (a - m + 2)\frac{k}{a+1},$$

where $b = \frac{m-1}{a-m+2}$.

Using (2.4) and (2.11) in (2.14), we get

$$2\tau = m(m-1)c + m^2 \mathfrak{g}(H, H^*) - \overline{\mathfrak{g}}(\zeta(e_i, e_j), \zeta^*(e_i, e_j)). \tag{3.1}$$

By the virtue of $2H^\circ = H + H^*$, we have $4||H^\circ||^2 = ||H||^2 + ||H^*||^2 + 2\overline{\mathfrak{g}}(H, H^*)$ which yields

$$m(m-1)c = 2\tau - 2m^2||H^\circ||^2 + \frac{m^2}{2}\left(||H||^2 + ||H^*||^2\right) + 2m\mathcal{C}^\circ - \frac{m}{2}(\mathcal{C} + \mathcal{C}^*). \tag{3.2}$$

Proof of the Theorem 1.1

Consider the quadratic polynomial P given by

$$\begin{aligned} P &= 2r\mathcal{C}^\circ + \frac{2(m-1)(m+r)(m^2-m-r)}{rm}\mathcal{C}^\circ(\mathcal{L}) - 2\tau \\ &\quad -\frac{m^2}{2}\left(||H||^2 + ||H^*||^2\right) + \frac{m}{2}(\mathcal{C}+\mathcal{C}^*) + m(m-1)c. \end{aligned} \tag{3.3}$$

Using (3.2) and writing the expression in the indices form, we derive

$$\begin{aligned} P &= \sum_{\alpha=m+1}^{n}\Bigg[\frac{2r}{m}\sum_{i,j=1}^{m}(\zeta_{ij}^{\circ^\alpha})^2 + \sum_{i,j=1}^{m-1}\frac{2(m+r)(m^2-m-r)}{rm}(\zeta_{ij}^{\circ^\alpha})^2 \\ &\quad + \frac{1}{2}\sum_{i,j=1}^{m}\left((\zeta_{ij}^{\alpha})^2 + (\zeta_{ij}^{*^\alpha})^2\right)\Bigg] - 2m^2||H^\circ||^2 + 2m\mathcal{C}^\circ - \frac{m}{2}(\mathcal{C}+\mathcal{C}^*) \\ &= \sum_{\alpha=m+1}^{n}\Bigg[2\left(\frac{(m-1)(m+r)}{r} - 1\right)\sum_{i=1}^{m-1}(\zeta_{ii}^{\circ^\alpha})^2 + \frac{4(m-1)(m+r)}{r}\sum_{1=i<j}^{m-1}(\zeta_{ij}^{\circ^\alpha})^2 \\ &\quad + 4\left(\frac{r}{m}+1\right)\sum_{i=1}^{m-1}(\zeta_{im}^{\circ^\alpha})^2 + \frac{2r}{m}(\zeta_{mm}^{\circ^\alpha})^2 - 4\sum_{1\le i<j\le}^{m}\zeta_{ii}^{\circ^\alpha}\zeta_{jj}^{\circ^\alpha}\Bigg] \\ \frac{P}{2} &\geq \sum_{\alpha=m+1}^{n}\Bigg[\left(\frac{(m-1)(m+r)}{r} - 1\right)\sum_{i=1}^{m-1}(\zeta_{ii}^{\circ^\alpha})^2 + \frac{r}{m}(\zeta_{mm}^{\circ^\alpha})^2 - 2\sum_{1\le i<j\le}^{m}\zeta_{ii}^{\circ^\alpha}\zeta_{jj}^{\circ^\alpha}\Bigg]. \end{aligned}$$

Now, we consider a real valued function $\mathcal{F}_\alpha : R^n \to R$ given by

$$\mathcal{F}_\alpha(\zeta_{11}^{\alpha}, ..., \zeta_{mm}^{\alpha}) = \left(\frac{(m-1)(m+r)}{r} - 1\right)\sum_{i=1}^{m-1}(\zeta_{ii}^{\circ^\alpha})^2 + \frac{r}{m}(\zeta_{mm}^{\circ^\alpha})^2 - 2\sum_{1\le i<j\le}^{m}\zeta_{ii}^{\circ^\alpha}\zeta_{jj}^{\circ^\alpha}.$$

We start with the optimization dilemma for invariant real constant $\mathcal{K}^\alpha$

$$\min \mathcal{F}_\alpha$$
$$\text{subjected to } P : \zeta_{11}^{\circ^\alpha} + \zeta_{22}^{\circ^\alpha} + ... + \zeta_{mm}^{\circ^\alpha} = \mathcal{K}^\alpha$$

By comparing this optimization problem with the Lemma 3.1, we get

$$a = \frac{(m-1)(m+r)}{r} - 1, b = \frac{r}{m}$$

Next, using simple calculations the partial derivative of $\mathcal{F}_\alpha$ for $i \in \{1, 2, ..., m-1\}$ are given as

$$\begin{cases} \frac{\partial \mathcal{F}_\alpha}{\partial \zeta_{ii}^{\circ^\alpha}} = \frac{2(m+r)(m-1)}{r}\zeta_{ii}^{\circ^\alpha} - 2\sum_{k=1}^{m}\zeta_{kk}^{\circ^\alpha}, \\ \frac{\partial \mathcal{F}_\alpha}{\partial \zeta_{mm}^{\circ^\alpha}} = \frac{2r}{m}\zeta_{mm}^{\circ^\alpha} - 2\sum_{k=1}^{m-1}\zeta_{kk}^{\circ^\alpha}. \end{cases} \tag{3.4}$$

Now, to get an extremum solution $(\zeta_{11}^{\circ^\alpha}, \zeta_{22}^{\circ^\alpha}, ..., \zeta_{mm}^{\circ^\alpha})$ of the constraint P, the vector grad$\mathcal{F}_\alpha \in T^\perp\mathcal{M}$ at $\mathcal{F}_\alpha$. From system of equation (3.4), the critical point of the optimized problem is outlined by

$$(\zeta_{11}^{\circ^\alpha}, \zeta_{22}^{\circ^\alpha}, ..., \zeta_{mm}^{\circ^\alpha}) = (\frac{r\lambda}{m(m-1)}, \frac{r\lambda}{m(m-1)}, \ \cdots, \frac{r\lambda}{m(m-1)}, \ \lambda) \tag{3.5}$$

Since $\sum_{i=1}^{m}\zeta_{ii}^{\circ^\alpha}=\mathcal{K}^\alpha$, (3.5) implies that $\frac{(r+m)\lambda}{m}=\mathcal{K}^\alpha$ or $\lambda=\frac{m\mathcal{K}^\alpha}{r+m}$. Thus, finally we have

$$\begin{aligned}\zeta_{ii}^{\circ^\alpha} &= \frac{r\mathcal{K}^\alpha}{(r+m)(m-1)}=\frac{\mathcal{K}^\alpha}{a+1};\quad \text{for } 1\le i\le m-1,\\ \zeta_{mm}^{\circ^\alpha} &= \frac{m\mathcal{K}^\alpha}{m+r}=\frac{\mathcal{K}^\alpha}{b+1}.\end{aligned}$$

Thus, we have $P\ge 0$ which yields

$$\begin{aligned}2\tau(p)\le 2r\mathcal{C}^\circ &+\frac{2(m-1)(m+r)(m^2-m-r)}{rm}\mathcal{C}^\circ(\mathcal{L})-\frac{m^2}{2}\big(||H||^2+||H^*||^2\big)\\ &+\frac{m}{2}(\mathcal{C}+\mathcal{C}^*)+m(m-1)c\end{aligned}$$

or

$$\begin{aligned}\rho &\le \frac{2\delta_c^\circ(r,m-1)}{m(m-1)}+c-\frac{m}{m-1}\big(2||H^\circ||^2-\overline{g}(H,H^*)\big)+\frac{1}{2(m-1)}(\mathcal{C}+\mathcal{C}^*)\\ &= \frac{2\delta_c^\circ(r,m-1)}{m(m-1)}+c-\frac{2m}{m-1}||H^\circ||^2+\frac{m}{m-1}\overline{\mathfrak{g}}(H,H^*)+\frac{\mathcal{C}^\circ}{m-1}\end{aligned}$$

where $2\mathcal{C}^\circ=\mathcal{C}+\mathcal{C}^*$. This completes the proof of Theorem 1.1.

Proof of the Theorem 1.2

We consider the quadratic polynomial Q by

$$\begin{aligned}Q=&-\frac{r}{2}(\mathcal{C}+\mathcal{C}^*)-\frac{(m-1)(m+r)(m^2-m-r)}{2rn}\big(\mathcal{C}(\mathcal{L})+\mathcal{C}^*(\mathcal{L})\big)-2\tau(p)+2m^2||H^\circ||^2\\ &-2m\mathcal{C}^\circ+m(m-1)c\\ =&-\sum_{\alpha=m+1}^{n}\Big[\frac{r}{2m}\sum_{i,j=1}^{m}\big((\zeta_{ij}^\alpha)^2+(\zeta_{ij}^{*^\alpha})^2\big)+\frac{(m+r)(m^2-m-r)}{2rm}\sum_{i,j=1}^{m-1}\big((\zeta_{ij}^\alpha)^2+(\zeta_{ij}^{*^\alpha})^2\big)\Big]\\ &+\frac{m^2}{2}\big(||H||^2+||H^*||^2\big)-\frac{1}{2}\sum_{\alpha=m+1}^{n}\sum_{i,j=1}^{m}\big((\zeta_{ij}^\alpha)^2+(\zeta_{ij}^{*^\alpha})^2\big)\\ =&\sum_{\alpha=m+1}^{n}\Big[-\Big(\frac{(m+r)(m-1)}{2r}-\frac{1}{2}\Big)\sum_{i=1}^{m-1}\big((\zeta_{ii}^\alpha)^2+(\zeta_{ii}^{*^\alpha})^2\big)-\frac{r}{2m}\big((\zeta_{mm}^\alpha)^2+(\zeta_{mm}^{*^\alpha})^2\big)\\ &-\frac{(m+r)(m-1)}{r}\sum_{1=i<j}^{m-1}\big((\zeta_{ij}^\alpha)^2+(\zeta_{ij}^{*^\alpha})^2\big)+\sum_{1\le i<j\le}^{m}(\zeta_{ii}^\alpha\zeta_{jj}^\alpha+\zeta_{ii}^{*^\alpha}\zeta_{jj}^{*^\alpha})\\ &-\Big(\frac{r}{m}+1\Big)\sum_{i=1}^{m-1}\Big((\zeta_{im}^\alpha)^2+(\zeta_{im}^{*^\alpha})^2\Big)\Big]\end{aligned}$$

On multiplying by -2, above relation reduced to

$$
\begin{aligned}
-2Q = &\sum_{\alpha=m+1}^{n}\Bigg[\Big(\frac{(m+r)(m-1)}{r}-1\Big)\sum_{i=1}^{m-1}\Big((\zeta_{ii}^{\alpha})^2+(\zeta_{ii}^{*^{\alpha}})^2\Big)+\frac{r}{m}\Big((\zeta_{mm}^{\alpha})^2+(\zeta_{mm}^{*^{\alpha}})^2\Big)\\
&+\frac{2(m+r)(m-1)}{r}\sum_{1=i<j}^{m-1}\Big((\zeta_{ij}^{\alpha})^2+(\zeta_{ij}^{*^{\alpha}})^2\Big)-2\sum_{1\le i<j}^{m}(\zeta_{ii}^{\alpha}\zeta_{jj}^{\alpha}+\zeta_{ii}^{*^{\alpha}}\zeta_{jj}^{*^{\alpha}})\\
&+2\Big(\frac{r}{m}+1\Big)\sum_{i=1}^{m-1}\Big((\zeta_{im}^{\alpha})^2+(\zeta_{im}^{*^{\alpha}})^2\Big)\Bigg]\\
\ge &\sum_{\alpha=m+1}^{n}\Bigg[\Big(\frac{(m+r)(m-1)}{r}-1\Big)\sum_{i=1}^{m-1}\Big((\zeta_{ii}^{\alpha})^2+(\zeta_{ii}^{*^{\alpha}})^2\Big)+\frac{r}{m}\Big((\zeta_{mm}^{\alpha})^2+(\zeta_{mm}^{*^{\alpha}})^2\Big)\\
&-2\sum_{1\le i<j}^{m}(\zeta_{ii}^{\alpha}\zeta_{jj}^{\alpha}+\zeta_{ii}^{*^{\alpha}}\zeta_{jj}^{*^{\alpha}})\Bigg]
\end{aligned}
$$

For $\alpha = m+1, ..., n$, consider a real valued function $\mathcal{G}_\alpha : R^{2m} \to R$ given by

$$
\begin{aligned}
\mathcal{G}_\alpha(\zeta_{11}^{\alpha},...,\zeta_{mm}^{\alpha},\zeta_{11}^{*^{\alpha}},...,\zeta_{mm}^{*^{\alpha}}) = &\Big(\frac{(m+r)(m-1)}{r}-1\Big)\sum_{i=1}^{n-1}\Big((\zeta_{ii}^{\alpha})^2+(\zeta_{ii}^{*^{\alpha}})^2\Big)\\
&-2\sum_{1\le i<j\le}^{m}(\zeta_{ii}^{\alpha}\zeta_{jj}^{\alpha}+\zeta_{ii}^{*^{\alpha}}\zeta_{jj}^{*^{\alpha}})+\frac{r}{m}\Big((\zeta_{mm}^{\alpha})^2+(\zeta_{mm}^{*^{\alpha}})^2\Big)
\end{aligned}
$$

and optimization dilemma for invariant real constants t^α and l^α

$$
\begin{aligned}
&\min \mathcal{G}_\alpha\\
&\text{subjected to } Q_\alpha = \zeta_{11}^{\alpha}+...+\zeta_{mm}^{\alpha} = t^\alpha \quad\text{and}\\
&\zeta_{11}^{*^{\alpha}}+...\zeta_{mm}^{*^{\alpha}} = l^\alpha.
\end{aligned}
$$

Now, with the virtue of some simple computations, the partial derivative of $\mathcal{G}_\alpha$ for $i \in \{1, 2, ..., m-1\}$ are given by

$$
\begin{cases}
\frac{\partial \mathcal{G}_\alpha}{\partial \zeta_{ii}^{\alpha}} = \frac{2(m+r)(m-1)}{r}\zeta_{ii}^{\alpha} - 2\sum_{k=1}^{m}\zeta_{kk}^{\alpha},\\
\frac{\partial \mathcal{G}_\alpha}{\partial \zeta_{ii}^{*^{\alpha}}} = \frac{2(m+r)(m-1)}{r}\zeta_{ii}^{*^{\alpha}} - 2\sum_{k=1}^{m}\zeta_{kk}^{*^{\alpha}},\\
\frac{\partial \mathcal{G}_\alpha}{\partial \zeta_{mm}^{\alpha}} = \frac{2r}{m}\zeta_{mm}^{\alpha} - 2\sum_{k=1}^{m-1}\zeta_{kk}^{\alpha},\\
\frac{\partial \mathcal{G}_\alpha}{\partial \zeta_{mm}^{*^{\alpha}}} = \frac{2r}{m}\zeta_{mm}^{*^{\alpha}} - 2\sum_{k=1}^{m-1}\zeta_{kk}^{*^{\alpha}},
\end{cases} \tag{3.6}
$$

From system of equations (3.6), the critical point of the optimized problem outlined by

$$
\begin{aligned}
(\zeta_{11}^{\alpha},...,\zeta_{mm}^{\alpha},\zeta_{11}^{*\alpha},...,\zeta_{mm}^{*\alpha}) = &\Big(\frac{r\lambda}{m(m-1)},\frac{r\lambda}{m(m-1)},\ ..\ ,\frac{r\lambda}{m(m-1)},\ \lambda,\\
&\frac{r\lambda^*}{m(m-1)},\frac{r\lambda^*}{m(m-1)},\ ..\ ,\frac{r\lambda^*}{m(m-1)},\ \lambda^*\Big)
\end{aligned} \tag{3.7}
$$

Since $\sum_{i=1}^{m}\zeta_{ii}^{\alpha}=\mathcal{K}^{\alpha}$ and $\sum_{i=1}^{m}\zeta_{ii}^{*^{\alpha}}=l^{\alpha}$, (3.7) implies that $\frac{(r+m)\lambda}{m}=\mathcal{K}^{\alpha}$ and $\frac{(r+m)\lambda^*}{m}=l^{\alpha}$ for $\lambda=\frac{m\mathcal{K}^{\alpha}}{r+m}$ and $\lambda^*=\frac{ml^{\alpha}}{r+m}$ respectively. Thus, we have the critical points as follows

$$\begin{aligned}\zeta_{ii}^{\alpha} &= \frac{rt^{\alpha}}{(m-1)(m+r)}=\frac{t^{\alpha}}{a+1}, \zeta_{ii}^{*^{\alpha}}=\frac{rl^{\alpha}}{(m-1)(m+r)}=\frac{l^{\alpha}}{a+1};\ 1\le i\le m-1,\\ \zeta_{mm}^{\alpha} &= \frac{mt^{\alpha}}{m+r}=\frac{t^{\alpha}}{b+1}, \zeta_{mm}^{*^{\alpha}}=\frac{ml^{\alpha}}{m+r}=\frac{l^{\alpha}}{b+1},\end{aligned}$$

where a and b have the following forms

$$a=\frac{(m-1)(m+r)}{r}-1,\ \ b=\frac{r}{m},$$

such that $\mathcal{G}_{\alpha}(\zeta_{11}^{\alpha},...,\zeta_{mm}^{\alpha},\zeta_{11}^{*^{\alpha}},...,\zeta_{mm}^{*^{\alpha}})=0$. Hence, we have $-2Q\ge 0$ or, $Q\le 0$. From this, we deduce that

$$\begin{aligned}2\tau(p)\ge &-\frac{r}{2}(\mathcal{C}+\mathcal{C}^*)-\frac{(m-1)(m+r)(m^2-m-r)}{2rm}\big(\mathcal{C}(\mathcal{L})+\mathcal{C}^*(\mathcal{L})\big)\\ &+2m^2||H^{\circ}||^2-2m\mathcal{C}^{\circ}+m(m-1)c\\ =&-\frac{\delta_c(r,m-1)}{2}-\frac{\delta_c^*(r,m-1)}{2}+2m^2||H^{\circ}||^2-2m\mathcal{C}^{\circ}+m(m-1)c\end{aligned}$$

which yields

$$\rho\ge-\frac{\delta_c(r,m-1)}{2m(m-1)}-\frac{\delta_c^*(r,m-1)}{2m(m-1)}+\frac{2m}{m-1}||H^{\circ}||^2-\frac{2}{m-1}\mathcal{C}^{\circ}+c$$

or

$$\rho\ge-\frac{\delta_c^{\circ}(r,m-1)}{m(m-1)}+\frac{2m}{m-1}||H^{\circ}||^2-\frac{2}{m-1}\mathcal{C}^{\circ}+c$$

where $2\delta_c^{\circ}(r,m-1)=\delta_c(r,m-1)+\delta_c^*(r,m-1)$.
This completes the proof of Theorem 1.2.

4 Glimpse of an Application: Divergence of Mean curvature vector field of statistical manifold

In this section, we deliberate an immediate application of obtained result using the relation of divergence of Mean curvature vector field with their inner product.

Proposition 4.1. *Let $\mathcal{M}^m$ be a statistical submanifold of statistical manifold $N^n(c)$ of constant curvature c. Then, we have*

$$\rho\le\frac{2\delta_c^{\circ}(r,m-1)}{m(m-1)}+\frac{\mathcal{C}^{\circ}}{m-1}-\frac{2m}{m-1}||H^{\circ}||^2-\frac{divH_p}{(m-1)}+c, \tag{4.1}$$

where $2\delta_c^{\circ}(r,m-1)=\delta_c(r,m-1)+\delta_c^*(r,m-1)$ and $divH_p$ denotes the divergence of the Mean curvature vector field H_p at a point $p\in\mathcal{M}$.

Proof. For an orthonormal basis $\{e_i\}_1^m$ of $T_p\mathcal{M}$, we know the divergence of H_p associated to the connection ∇ is given by

$$\begin{aligned} divH_p =& \sum_{i=1}^{m} \overline{\mathfrak{g}}(\nabla_{e_i} H, e_i) \\ =& \frac{1}{m} \sum_{i,j=1}^{m} \overline{\mathfrak{g}}(\nabla_{e_i} \zeta(e_j, e_j), e_i). \end{aligned} \tag{4.2}$$

Since $\overline{\mathfrak{g}}(\zeta(e_j, e_j), e_i) = 0$, it implies

$$\begin{aligned} \overline{\mathfrak{g}}(\nabla_{e_i} \zeta(e_j, e_j), e_i) =& - \overline{\mathfrak{g}}(\zeta(e_j, e_j), \nabla^*_{e_i} e_i) \\ =& - \overline{\mathfrak{g}}(\zeta(e_j, e_j), \zeta^*(e_j, e_j)) \\ =& - m^2 \overline{\mathfrak{g}}(H, H^*) \end{aligned} \tag{4.3}$$

Using (4.3) in (4.2), we arrive

$$divH_p = -m\overline{\mathfrak{g}}(H, H^*). \tag{4.4}$$

Using above relation in Theorem 1.1, we get our desired inequality (4.1) and this completes the proof. □

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