On the conformal change of a special class of (α, β)-metrics

L.I. Pişcoran, C. Barbu and B. Najafi

Abstract. In this paper we will study the flag curvature; the E-curvature and some metric properties of a special class of (α, β)-metrics, obtained by conformal change. More precisely, starting with the (α, β)-metric introduced by us in [13], we will use the conformal change described in [5] to obtain a new $\alpha\phi(b^2, \frac{\beta}{\alpha})$-metric family. Then, we will investigate the above mentioned properties of this new class of metrics.

M.S.C. 2010: 53B40.
Key words: Finsler (α, β)-metric; conformal change; flag curvature; E-curvature.

1 Introduction

In Finsler geometry, one of the great difficulties is to find analogies with the results of Riemannian geometry. The conformal transformations in Riemannian and Finsler geometries play an important role, not just for this two types of geometries, but also for the process of geometrization of physical theories. As we know, the flag curvature, is a natural extension of the sectional curvature from Riemannian geometry, and play an important role in the theory of geodesics in Finsler geometry. Also, we know that the flag curvature for such a metric, arises from the second variation of arc length in Finsler geometry and for this reason, the study of the flag curvature is very important. In paper [13], we introduced the new (α, β)-metric:

$$F = \alpha(s^2 + s + a) \tag{1.1}$$

where $a \in (\frac{1}{4}, +\infty)$ is a positive scalar and $s = \frac{\beta}{\alpha}$. Furthermore, in [14], [15], [16], we have continued our investigation on this new class of (α, β)-metric and we have found more interesting properties for this class of metrics. In this paper we will use the Hashiguchi conformal change for our metric, obtaining in this way a new class of (α, β)-metric and then we will investigate its properties and we will find the flag curvature and E-curvature for this new class of metrics. The flag curvature is worth to be study in Finsler geometry because it is the generalisation of the sectional curvature

Balkan Journal of Geometry and Its Applications, Vol.25, No.1, 2020, pp. 104-116.

of Riemannian geometry, so is a very important geometric tool. The E-curvature is another unique Finsler quantities and is also called *mean Berwald curvature.*
In 1976, Hashiguchi, in [5], introduced the conformal change of Finsler metrics, given by:

$$\bar{F} = e^{\sigma(x)} F \tag{1.2}$$

Acording to [22], the conformal change of two Finsler manifolds can be given as follows:

Definition 1.1. Let (M, L) and $(M, \tilde{L})$, be two Finsler manifolds. The two associated metrics g and $\tilde{g}$, are said to be conformal if there exists a positive differential function $\sigma(x)$, such that $\tilde{g}(X, Y) = e^{2\sigma(x)} g(\bar{X}, \bar{Y})$. Equivalently, g and $\tilde{g}$ are conformal iff $\tilde{L}^2 = e^{2\sigma} L^2$. In this case, the transformation $L \to \tilde{L}$ is said to be a confromal transformation and the two Finsler manifolds (M, L) and $(M, \tilde{L})$ are said to be conformal related.

In 1984, Shibata, in [19], extended the notion of β-change to a general case in Finsler geometry. In the following lines we will recall some definitions and properties regarding the general (α, β)-metrics.

Definition 1.2. ([2]) Let F be a Finsler metric on a manifold M. F is called general (α, β)-metric, if F can be expressed on the form: $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$, $b^2 = \|\beta\|_\alpha^2$, where α is a Riemannian metric and $\beta = b_i(x)y^i$ is an one-form with $\|\beta\|_\alpha < b_0$; $s = \frac{\beta}{\alpha}$ and $\phi(\rho, s)$ is a C^∞ function.

Proposition 1.1. *([2]) For a general (α, β)-metric*

$$F = \alpha\phi(b^2, \frac{\beta}{\alpha}) \tag{1.3}$$

the fundamental tensor is given by:

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_1 (b_i \alpha_{y^j} + b_j \alpha_{y^i}) - s\rho_1 \alpha_{y^i} \alpha_{y^j},$$

where $\rho = \phi(\phi - s\phi_2)$, $\rho_0 = \phi\phi_{22} + \phi_2\phi_2$, $\rho_1 = \phi_2(\phi - s\phi_2) - s\phi\phi_{22}$.
Moreover,

$$det(g_{ij}) = \phi^{n+1}(\phi - s\phi_2)^{n-2}(\phi - s\phi_2 + (b^2 - s^2)\phi_{22}) det(a_{ij});$$

$$det(g^{ij}) = \rho^{-1}\left\{a^{ij} + \eta b^i b^j + \eta_0 \alpha^{-1}(b^i y^j + b^j y^i) + \eta_1 \alpha^{-2} y^i y^j\right\};$$

where $g^{ij} = (g_{ij})^{-1}$, $a^{ij} = (a_{ij})^{-1}$, $b^i = a^{ij} b_j$,

$$\eta = -\frac{\phi_{22}}{\phi - s\phi_2 + (b^2 - s^2)\phi_{22}}, \ \eta_0 = -\frac{(\phi - s\phi_2)\phi_2 - s\phi\phi_{22}}{\phi(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})},$$

$$\eta_1 = \frac{(s\phi + (b^2 - s^2)\phi_2)((\phi - s\phi_2)\phi_2 - s\phi\phi_2)}{\phi^2(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})}$$

Proposition 1.2. *([2]) Let M be an n-dimensional manifold, $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ is a Finsler metric on M for any Riemannian metric α and 1-form β, with $\|\beta\|_\alpha < b_0$, if and only if $\phi = \phi(b, s)$ is a positive C^∞ function, satisfying:*

$$\phi - s\phi_2 > 0, \ \phi - s\phi_2 + (b^2 - s^2)\phi_{22}$$

when $n \geq 3$ *or*

$$\phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0$$

when $n = 2$, *where* s *and* b *are arbitrary numbers with* $|s| \leq b < b_0$

Proposition 1.3. *([2]) For an general* (α, β)*-metric* $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ *its spray coefficients* G^i *are related to the spray coefficients* G^i_α, *of* α, *by*

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \left\{\Theta(-2\alpha Q s_0 + r_{00} + 2\alpha^2 Rr) + \alpha\Omega(r_0 + s_0)\right\}\frac{y^i}{\alpha} +$$

$$+\left\{\Psi(-2\alpha Q s_0 + r_{00} + 2\alpha^2 Rr) + \alpha\Pi(r_0 + s_0)\right\} b^i - \alpha^2 R(r^i + s^i)$$

where

$$Q = \frac{\phi_2}{\phi - s\phi_2},\ R = \frac{\phi_1}{\phi - s\phi_2};$$

$$\Theta = \frac{(\phi - s\phi_2)\phi_2 - s\phi\phi_{22}}{2\phi(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})},\ \Psi = \frac{\phi_{22}}{2(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})};$$

$$\Pi = \frac{(\phi - s\phi_2)\phi_{12} - s\phi_1\phi_{22}}{(\phi - s\phi_2)(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})}, \Omega = \frac{2\phi_1}{\phi} - \frac{s\phi + (b^2 - s^2)\phi_2}{\phi}\Pi \tag{1.4}$$

Remark 1.3. In order to compute the spray coefficients G^i, the authors of paper [2], obtained: $G^i = G^i_1 + G^i_2$, where

$$G^i_2 = G^i_\alpha + \alpha Q s^i_0 + \Theta\left\{-2\alpha Q s_0 + r_{00}\right\}\frac{y^i}{\alpha} + \Psi\left\{-2\alpha Q s_0 + r_{00}\right\} b^i \tag{1.5}$$

$$G^i_1 = g^{il}\left\{A y_l + B b_l + C(r_l + s_l)\right\} = \rho^{-1}\left\{D y^i + E b^i + F(r^i + s^i)\right\} \tag{1.6}$$

with

$$A = (2\phi\phi_1 - s\phi_1\phi_2 - s\phi\phi_{12})(r_0 + s_0);$$

$$B = \alpha(\phi_1\phi_2 + \phi\phi_{12})(r_0 + s_0),\ C = -\alpha^2\phi\phi_1;$$

$$D = A + (As + \alpha^{-1}Bb^2 + \alpha^{-1}Cr)\eta_0 + \left\{A + \alpha^{-1}Bs + \alpha^{-2}C(r_0 + s_0)\right\}\eta_1;$$

$$E = B + (\alpha As + Bb^2 + Cr)\eta + \left\{\alpha A + Bs + \alpha^{-1}C(r_0 + s_0)\right\}\eta_0;$$

$$F = C;$$

$$D = \left\{[2(\phi - s\phi_2) + s\phi_{22}\cdot\mu]\,\phi_1 - (\phi - s\phi_2)\cdot\mu\cdot\phi_{12}\right\}(r_0 + s_0)$$
$$+\frac{(\phi - s\phi_2)\phi_2 - s\phi\phi_{22}}{\phi - s\phi_2 + (b^2 - s^2)\phi_{22}}\phi_1\alpha r$$

$$E = \left\{\frac{\phi(\phi - s\phi_2)}{\phi - s\phi_2 + (b^2 - s^2)\phi_{22}}\phi_{12} - \frac{s\phi\phi_{22}}{\phi - s\phi_2 + (b^2 - s^2)\phi_{22}}\phi_1\right\}\alpha(r_0 + s_0)$$

$$+\frac{\phi\phi_{22}}{\phi - s\phi_2 + (b^2 - s^2)\phi_{22}}\phi_1\alpha^2 r,$$

where $\mu = \frac{s\phi+(b^2-s^2)\phi_2}{\phi-s\phi_2+(b^2-s^2)\phi_{22}}$. Also, we know:

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}); \quad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}); \quad s^i_j = a^{ih}s_{hj}$$
$$s_j = b_i s^i_j = s_{ij}b^i; \quad r_j = r_{ij}b^i;$$
$$r_0 = r_j y^j; \quad s_0 = s_j y^j, \ r_{00} = r_{ij}y^i y^j.$$

Here $b_{i|j}$ denotes the coefficients of the covariant derivative of β with respect to α. Let's recall now some properties about the flag curvature of the general (α, β)-metric, as are presented in [20]

Proposition 1.4. *([20]) Suppose general (α, β)-metric, $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ is a projectively flat Finsler metric, then its projectively factor P, is given by:*

$$P = \frac{2\alpha^{-1}(\phi - s\phi_2)G^m_\alpha y_m + \phi_2(2b_m G^2_\alpha + r_{00}) + 2\alpha\phi_1(r_0 + s_0)}{2F}$$

where G^m_α denotes the spray coefficients of α, $r_{00} = r_{ij}y^i y^j$, $r_0 = b^j r_{ij}y^i$, $s_0 = b^j s_{ij}y^i$

Remark 1.4. In computations, the authors of [20] used for this general (α, β)-metric, the following formulas:

$$P = \frac{F_{x^k}y^k}{2F} \tag{1.7}$$

for the projectively factor of this kind of metric, and

$$K = \frac{P^2 - P_{x^m}y^m}{F^2} \tag{1.8}$$

for the flag curvature of this (α, β)-metric.

Jacobi equation of the Finsler manifold $(M = \Omega_X, F)$ can be written in the scalar form as follows(see [1]):

$$\frac{d^2v}{ds^2} + Kv = 0 \tag{1.9}$$

where $\xi = v(s)\eta^i$ is a Jacobi field along $\gamma : x^i = x^i(s)$, η^i is the unit normal vector field along γ and K is the flag curvature of (M, F), which describes the shape of the space. According to ([1]), the flag curvature for the Finsler manifold (M, F), tell us how curved is the space at a point. The importance of the flag curvature in physics is huge. One of its applications is in the solving of the Jacobi equation how we already underlined in the above lines.
According to Z. Shen, ([17]) the E-curvature, is the most important non-Riemannian quantities in Finsler geometry. A lot of geometers investigates the E-curvature and they found interesting results in this respect. Some important research paper about the E-curvature are: ([17], [21], [3]).
The E-curvature is defined as follows (see [17]): $E_y = E_{ij}dx^i \otimes dx^j|_p : T_pM \otimes T_pM \to \mathbb{R}$

$$E_{ij} = \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j}\left[\frac{\partial G^m}{\partial y^m}\right] \tag{1.10}$$

The E-curvature is closely related with the flag curvature. For a two dimensional plane $P \subset T_pM$ and a non-zero vector $y \in T_pM$, the E-curvature is defined by:

$$E(P,y) = \frac{F^3(y)E_y(u,u)}{g_y(y,y)\cdot g_y(u,u) - g_y(y,u)^2}$$

with $P = span\,\{y,u\}$.
Considering that β is a closed and conformal 1-form, i.e.,

$$b_{i|j} = ca_{ij}, \tag{1.11}$$

M. Gabrani and B. Rezaei, found the following important theorem regarding the E-curvature for general (α,β)-metrics (see [4]):

Theorem 1.5. *([4]) Let $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$, be a general (α,β)-metric on an n-dimensional manifold M. Suppose that β, satisfies (1.11). Then, F is of isotropic E-curvature if and only if*

$$(n+1)(E - sE_2) + (b^2 - s^2)(H_2 - sH_{22}) = \rho(x)(n+1)(\phi - s\phi_2) \tag{1.12}$$

where $\rho(x) = \frac{k(x)}{c(x)}$, E and H, are defined by:

$$E = \frac{\phi_2 + 2s\phi_1}{2\phi} - H\frac{s\phi + (b^2 - s^2)\phi_2}{\phi} \tag{1.13}$$

$$H = \frac{\phi_{22} - 2(\phi_1 - s\phi_{12})}{2\left[\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\right]} \tag{1.14}$$

In this paper we will use the indices 1 respectively 2 in derivations, with respect to b^2, respectively s.
Finally, let's recall the following:

Theorem 1.6. *([4]) Let $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$, be a general (α,β)-metric on an n-dimensional manifold M. Suppose that β, satisfies (1.11). Then, the E-curvature of F is given by:*

$$\begin{aligned} E_{ij} = \frac{c}{2}\Big\{ &\frac{1}{\alpha}\left[(n+1)E_{22} + 2(H_2 - sH_{22}) + (b^2 - s^2)H_{222}\right] b_i b_j \\ &- \frac{s}{\alpha^2}\left[(n+1)E_{22} + 2(H_2 - sH_{22}) + (b^2 - s^2)H_{222}\right](b_i y_j + b_j y_i) \\ &+ \frac{1}{\alpha^3}\left[(n+1)s^2E_{22} - (n+1)(E - sE_2) + s^2(b^2 - s^2)H_{222} + (3s^2 - b^2)(H_2 - sH_{22})\right] y_i y_j \\ &+ \frac{1}{\alpha}\left[(n+1)(E - sE_2) + (b^2 - s^2)(H_2 - sH_{22})\right] a_{ij} \Big\} \end{aligned}$$

Corollary 1.7. *([4]) Let $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$, be a general (α,β)-metric on an n-dimensional manifold M. Suppose that β, satisfies (1.11). Then, F is of vanishing E-curvature if and only if:*

$$(n+1)(E - sE_2) + (b^2 - s^2)(H_2 - sH_{22}) = 0 \tag{1.15}$$

2 Preliminaries

The (α, β)-metrics were introduced in Finsler geometry by Matsumoto [9]. This kind of metrics are composed by a Riemannian metric α, a 1-form β and the C^∞ function $\phi(s)$, on a manifold M. Such an (α, β)-metric could be puted in the following form:

$$F = \alpha\phi(s);\ s = \frac{\beta}{\alpha}.$$

A classical example of this kind of metrics is the Funk metric, which can be expressed as follows:

$$F = \alpha\phi(b^2, s) = \alpha\frac{s + \sqrt{1-(b^2-s^2)}}{1-b^2}$$

where $b = |x|$; $\alpha = |y|$; $s = \frac{<x,y>}{|y|}$ and $\beta =< x, y >$. The Funk metric is a projectively flat Finsler metric with $G^i = Py^i$, where P represents its projective factor which is given by:

$$P = \frac{1}{2}\left\{\frac{\sqrt{|y| - (|x|^2|y|^2 - <x,y>^2)}}{1-|x|^2} + \frac{<x,y>}{1-|x|^2}\right\}.$$

Its flag curvature is $K = -\frac{1}{4}$.
Every Finsler metric F on M induces a spray $G = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$, as follows:

$$G^i(x,y) = \frac{1}{4}g^{il}(x,y)\left\{2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\right\}y^jy^k$$

where $g_{ij}(x,y) = \frac{1}{2}\left[F^2\right]_{y^iy^j}(x,y)$.
For a vector $y = y^i\frac{\partial}{\partial x^i}|_p \in T_pM$ set $Ry(u) := R^i_k u^k\frac{\partial}{\partial x^i}|_p$, where $u = u^i\frac{\partial}{\partial x^i}|_p$ and $R^i_k = R^i_k(x,y)$ is given by:

$$R^i_k = 2\frac{\partial G^i}{\partial x^k} - y^j\frac{\partial^2 G^i}{\partial x^j\partial y^k} + 2G^j\frac{\partial^2 G^i}{\partial y^j\partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}$$

It is easy to observe, that $R_y(y) = 0$.
Assuming that G is induced by a Finsler metric F, then R_y is self-adjoint with respect to g_y, i.e., $g_y(R_y(u), v) = g_y(u, R_y(v))$. For a tangent plane $P \subset T_pM$ and a vector $y \in P - \{0\}$, the flag curvature $\mathbf{K}(P, y)$ is defined by:

$$\mathbf{K}(P,y) = \frac{g_y(R_y(u),u)}{g_y(y,y)g_y(u,u) - g_y(y,u)g_y(y,u)}$$

where $u \in P$ such that $P = span\,\{y, u\}$. Interesting and important results regarding the flag curvature in Finsler geometry, were obtained in: [10], [11], [12], [17], [18].

3 Main Results

Starting now, with the metric (1.1), we will transform it using Hashiguchi conformal transform (1.2), in a $(b^2, \frac{\beta}{\alpha})$-type metric, where we will chose for the Hashiguchi transform, $\sigma(x) = b^2$. So, we get the following (α, β) general family of metrics:

$$\bar{F} = \alpha(s^2 + s + a)e^{b^2} \tag{3.1}$$

We are ready now, to give the following:

Proposition 3.1. *For the $(b^2, \frac{\alpha}{\beta})$-metric (3.1), the fundamental tensor is given by:*

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_1 (b_i \alpha_{y^j} + b_j \alpha_{y^i}) - s\rho_1 \alpha_{y^i} \alpha_{y^j},$$

where $\rho = e^{2b^2}(s^2+s+a)(a-s^2)$, $\rho_0 = e^{2b^2}(6s^2+6s+2a+1)$, $\rho_1 = e^{2b^2}(a-4s^3-3s^2)$. Moreover,

$$det(g_{ij}) = \phi^{n+1}(\phi - s\phi_2)^{n-2}(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})det(a_{ij});$$

$$det(g^{ij}) = \rho^{-1}\left\{a^{ij} + \eta b^i b^j + \eta_0 \alpha^{-1}(b^i y^j + b^j y^i) + \eta_1 \alpha^{-2} y^i y^j\right\};$$

where $g^{ij} = (g_{ij})^{-1}$, $a^{ij} = (a_{ij})^{-1}$, $b^i = a^{ij} b_j$,

$$\eta = \frac{2}{3s^2 - 2b^2 - a}, \; \eta_0 = \frac{4s^3 + 3s^2 - a}{(s^2 + s + a)(a - 3s^2 + 2b^2)},$$

$$\eta_1 = \frac{(-s^3 + 2b^2 s + b^2 + as)(a - 3s^2 - 4s^3)}{(s^2 + s + a)^2(-3s^2 + 2b^2 + a)}$$

Proof. After tedious computations, using Proposition 1.3, we get the desired result. □

Proposition 3.2. *The general (α, β)-metric (3.1) is a Finsler metric on M, for any Riemannian metric α and 1-form β, with $\|\beta\|_{\alpha_0} < b_0$, if and only if $\phi = \phi(b^2, s)$ is a positive C^∞ function, satisfying one of the following two conditions:*

$$1) - \sqrt{a} < s < \sqrt{a}$$

$$2) s < \sqrt{\frac{2b^2 + a}{3}}$$

for $|s| \leq b < b_0$.

Proof. We will impose the conditions from Proposition 1.4, namely

$$\phi - s\phi_2 > 0, \; \phi - s\phi_2 + (b^2 - s^2)\phi_{22}$$

when $n \geq 3$ or

$$\phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0$$

when $n = 2$, where s and b are arbitrary numbers with $|s| \leq b < b_0$. For the first condition, we get $e^{b^2}(a - s^2) > 0$ and for the second one, we get $-3s^2 + 2b^2 + a > 0$. Solving this two inequalities, we get the desired result. □

Proposition 3.3. *For the (α,β)-metric (3.1), the spray coefficients G^i, are related to the spray coefficients of α, by:*

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \left\{\Theta(-2\alpha Q s_0 + r_{00} + 2\alpha^2 R r) + \alpha\Omega(r_0+s_0)\right\}\frac{y^i}{\alpha} + \left\{\Psi(-2\alpha Q s_0 + r_{00} + 2\alpha^2 R r) + \alpha\Pi(r_0+s_0)\right\} b^i - \alpha^2 R(r^i+s^i)$$

where

$$(3.2)\qquad \begin{cases} Q = \frac{2s+1}{a-s^2},\ R = \frac{2b(s^2+s+a)}{a-s^2},; \\ \Theta = \frac{-4\,s^3-3\,s^2+a}{2(s^2+s+a)(-3\,s^2+a+2\,b^2)}, \\ \Psi = (-3\,s^2+a+2\,b^2)^{-1}, \\ \Pi = 2\,\frac{b(-4\,s^3-3\,s^2+a)}{(-s^2+a)(-3\,s^2+a+2\,b^2)}, \\ \Omega = 4\,b - \frac{(se^{b^2}(s^2+s+a)+b^2-s^2)\Pi}{e^{b^2}(s^2+s+a)}. \end{cases}$$

Proof. After computations, using Theorem 1.5, we get the result. □

Remark 3.1. For the metric (3.2), the spray coefficients can be obtained as follows: $G^i = G^i_1 + G^i_2$, where

$$G^i_2 = G^i_\alpha + \alpha(\frac{2s+1}{a-s^2})s^i_0 + \frac{-4\,s^3-3\,s^2+a}{2(s^2+s+a)(-3\,s^2+a+2\,b^2)}\left\{-2\alpha(\frac{2s+1}{a-s^2})s_0 + r_{00}\right\}\frac{y^i}{\alpha} + (-3\,s^2+a+2\,b^2)^{-1}\left\{-2\alpha(\frac{2s+1}{a-s^2})s_0 + r_{00}\right\}b^i$$

$$G^i_1 = g^{il}\left\{Ay_l + Bb_l + C(r_l+s_l)\right\} = \rho^{-1}\left\{Dy^i + Eb^i + F(r^i+s^i)\right\}$$

with

$$\begin{aligned} A &= 4e^{2b^2}b(-s^4-s^3+sa+a^2)(r_0+s_0), \\ B &= \alpha 4be^{2b^2}(s^2+s+a)(2s+1)(r_0+s_0), \\ C &= -\alpha^2 2(e^{b^2})^2(s^2+s+a)^2 b, \\ F &= C, \end{aligned}$$

$$D = \frac{2}{-3s^2+a+2b^2}\Big[e^{2b^2}b(2s^6+3s^5+(2a+4b^2)s^4+(6b^2-4a)s^3+(3b^2-6a^2)s^2 + (2ab^2+a^2)s - ab^2 + 2a^3 + 4a^2b^2\Big]\cdot(r_0+s_0) + \mu\cdot\alpha^2 r$$

$$E = \mu\cdot\alpha(r_0+s_0) + 4\frac{e^{2b^2}(s^2+s+a)^2 b}{-3s^2+a+2b^2}\alpha^2 r,$$

where $\mu = 2\frac{e^{2b^2}(-4s^3-3s^2+a)b(s^2+s+a)}{-3s^2+a+2b^2}$. These quantities were obtained using (1.5) and (1.6) from Remark 1.4 and using Maple 13, after tedious computations.

Proposition 3.4. *Suppose that the $(b^2, \frac{\alpha}{\beta})$-metric (3.1), is a projectively flat general Finsler metric. Then its projectively factor is given by:*

$$P = \frac{\alpha^{-1}(a-s^2)G^m_\alpha y_m + (s+\frac{1}{2})(2b_m G^m_\alpha + r_{00}) + 2\alpha b(s^2+s+a)(r_0+s_0)}{s^2+s+a} \tag{3.3}$$

where G^m_α, denotes the spray coefficients of α, $r_{00} = r_{ij}y^i y^j$; $r_0 = b^j r_{ij} y^i$, $s_0 = b^j s_{ij} y^i$

Proof. The proof is direct, using Proposition 1.4 for the general (α, β)-metric (3.1). $\square$

Theorem 3.5. *The flag curvature of the general (α, β)-metric (3.1), is given by:*

$$K = \frac{4\alpha^2(s^3+s^2-\frac{s}{\alpha}+s(a+1)+\frac{1}{2})(s^3+s^2+s(a+1)+\frac{1}{2})e^{2b^2} - \nu\cdot\alpha}{(s^2+s+a)^2}, \tag{3.4}$$

where $\nu = 2e^{b^2}(3s^2+2s+1+a)$.

Proof. First, we transform our metric (3.1), in the following way:

$$\bar{F} = \alpha(s^2+s+a)e^{b^2} = (\frac{<x,y>^2}{|y|} + <x,y> + a\,|y|)e^{|x|^2}$$

where $a \in (\frac{1}{4}, +\infty)$ is a scalar, and we made the following notations: $b = |x|$; $\alpha = |y|$; $s = \frac{<x,y>}{|y|}$ and $\beta = <x,y>$. Next, we compute:

$$F_{x^k}y^k = e^{|x|^2}(2<x,y>(a+1)\,|y| + |y|^2 + \frac{2<x,y>^3}{|y|} + 2<x,y>^2)$$

$$P_{x^k}y^k = 2e^{|x|^2} <x,y> (2<x,y>(a+1)\,|y| + |y|^2 + \frac{2<x,y>^3}{|y|} + 2<x,y>^2) +$$
$$e^{|x|^2}(2\,|y|^3(a+1) + 6y<x,y>^2 + 4<x,y>|y|^2)$$

Now, using $K = \frac{P^2 - P_{x^k}y^k}{F^2}$ we complete the proof, obtaining (3.4). $\square$

Theorem 3.6. *Let $\bar{F}$ be the general (α, β)-metric given in (3.1) on an n-dimensional manifold M. Suppose that β, satisfies (1.11). Then, $\bar{F}$ is of vanishing E-curvature if and only if:*

$$\begin{aligned}
&(n+1)\Bigg\{\frac{12bs^8 + (64ba+24+32b^3)s^7 + (39+36b^3+12ba)s^6 + \mu\cdot s^5}{2(-3s^2+a+2b^2)^2(s^2+s+a)^2}\\
&+\frac{(-5a-28ba^2-100b^3a+2b^2+8b^5)s^4 + \nu\cdot s^3}{2(-3s^2+a+2b^2)^2(s^2+s+a)^2}\\
&+\frac{1}{2(-3s^2+a+2b^2)^2(s^2+s+a)^2}\big[(4ba^3-4b^3a^2-3a^2+48b^5a+12ab^2)s^2\\
&+(8b^3a^2+2a^2+4ab^2+16b^5a)s + 4b^3a^3 + 2a^2b^2 + 8b^5a^2 + a^3\big]\Bigg\}+
\end{aligned}$$

$$+(b^2-s^2)(24\frac{s^3(4ba-4b^3-3)}{(-3s^2+a+2b^2)^3}) = 0, \tag{3.5}$$

where

$$\mu = (-64ba^2 - 128b^3a + 24b^3 + 18),$$
$$\nu = (16ab^2 - 12a + 64b^5a + 8a^2 - 48b^3a + 32b^3a^2).$$

Proof. Imposing the condition (1.15):

$$(n+1)(E - sE_2) + (b^2 - s^2)(H_2 - sH_{22}) = 0$$

after tedious computations in Maple software package, we get:

$$H = -\frac{-1 - 2bs^2 + 2ba}{-3s^2 + a + 2b^2}$$

$$E = \frac{1}{2}\frac{-8s^5b - 12bs^4 + (-16ba - 4)s^3 + (4b^3 - 3 + 4ba)s^2 + 8ab(a + 2b^2)s + a + 4b^3a}{(-3s^2 + a + 2b^2)(s^2 + s + a)}$$

and replacing in (1.15), we get the desired result. □

Using now Theorem 1.5, we are ready to give the following important result:

Theorem 3.7. *Let $\bar{F}$ be the general (α,β)-metric given in (3.1) on an n-dimensional manifold M. Suppose that β, satisfies (1.11). Then, the E-curvature of $\bar{F}$ is given by:*

$$E_{ij} = \frac{c}{2}\left\{\frac{1}{\alpha}\left[(n+1)E_{22} + 2(24\frac{s^3(4ba - 4b^3 - 3)}{(-3s^2 + a + 2b^2)^3}) + (b^2 - s^2) - \mu\right] b_ib_j\right.$$
$$-\frac{s}{\alpha^2}\left[(n+1)E_{22} + 2(24\frac{s^3(4ba - 4b^3 - 3)}{(-3s^2 + a + 2b^2)^3}) + (b^2 - s^2) - \mu\right]\cdot$$
$$\cdot(b_iy_j + b_jy_i) + \frac{1}{\alpha^3}\left[(n+1)s^2E_{22} - (n+1)(E - sE_2) + s^2(b^2 - s^2) - \mu\right.$$
$$\left.\left. + (3s^2 - b^2)(24\frac{s^3(4ba - 4b^3 - 3)}{(-3s^2 + a + 2b^2)^3})\right] y_iy_j + \frac{1}{\alpha}\left[(n+1)(E - sE_2) + (b^2 - s^2)\cdot\nu\right] a_{ij}\right\}$$

where $\mu = 72\frac{s(4ba - 4b^3 - 3)(a + 2b^2 + 3s^2)}{(-3s^2 + a + 2b^2)^4}$, $\nu = 24\frac{s^3(4ba - 4b^3 - 3)}{(-3s^2 + a + 2b^2)^3}$ *and*

$$E = \frac{1}{2}\frac{-8s^5b - 12bs^4 + (-16ba - 4)s^3 + (4b^3 - 3 + 4ba)s^2 + 8ab(a + 2b^2)s + a + 4b^3a}{(-3s^2 + a + 2b^2)(s^2 + s + a)};$$

$$E_2 = \frac{1}{2}\frac{-40bs^4 - 48bs^3 + 3(-16ba - 4)s^2 + 2(4b^3 - 3 + 4ba)s + 8ba(a + 2b^2)}{(-3s^2 + a + 2b^2)(s^2 + s + a)}$$
$$+3\frac{(-8s^5b - 12bs^4 + (-16ba - 4)s^3 + (4b^3 - 3 + 4ba)s^2 + 8ba(a + 2b^2)s + a + 4ab^3)s}{(-3s^2 + a + 2b^2)^2(s^2 + s + a)} -$$
$$-\frac{1}{2}\frac{(-8s^5b - 12bs^4 + (-16ba - 4)s^3 + \nu + 8ba(a + 2b^2)s + a + 4ab^3)(2s + 1)}{(-3s^2 + a + 2b^2)(s^2 + s + a)^2}$$

$$E_{22} = \frac{1}{2}\frac{-160bs^3 - 144bs^2 + 6(-16ba - 4)s + 8b^3 - 6 + 8ba}{(-3s^2 + a + 2b^2)(s^2 + s + a)} +$$

$$+6\frac{(-40bs^4-48bs^3+3(-16ba-4)s^2+2(4b^3-3+4ba)s+8ba(a+2b^2))s}{(-3s^2+a+2b^2)^2(s^2+s+a)}-$$

$$-\frac{(-40bs^4-48bs^3+3(-16ba-4)s^2+2(4b^3-3+4ba)s+8ba(a+2b^2))(2s+1)}{(-3s^2+a+2b^2)(s^2+s+a)^2}+$$

$$+36\frac{(-8s^5b-12bs^4+(-16ba-4)s^3+\nu+8ba(a+2b^2)s+a+4ab^3)s^2}{(-3s^2+a+2b^2)^3(s^2+s+a)}-$$

$$-6\frac{(-8s^5b-12bs^4+(-16ba-4)s^3+\nu+8ba(a+2b^2)s+a+4ab^3)s(2s+1)}{(-3s^2+a+2b^2)^2(s^2+s+a)^2}+$$

$$+3\frac{-8s^5b-12bs^4+(-16ba-4)s^3+(4b^3-3+4ba)s^2+8ba(a+2b^2)s+a+4ab^3}{(-3s^2+a+2b^2)^2(s^2+s+a)}+$$

$$+\frac{(-8s^5b-12bs^4+(-16ba-4)s^3+\nu+8ba(a+2b^2)s+a+4ab^3)(2s+1)^2}{(-3s^2+a+2b^2)(s^2+s+a)^3}-$$

$$-\frac{-8s^5b-12bs^4+(-16ba-4)s^3+(4b^3-3+4ba)s^2+8ba(a+2b^2)s+a+4ab^3}{(-3s^2+a+2b^2)(s^2+s+a)^2},$$

where $\nu=(4b^3-3+4ba)s^2$.

Proof. The proof can be done using Theorem 1.5 and making all the computations in Maple for the metric $\bar{F}$ given in (3.1). After tedious computations the results presented above can be obtained. □

4 Conclusion

In this paper we have continued our investigation on the new introduced (α,β)-metric (1), from [7]. Using the Hashiguchi transform we transform metric (1.1) and we investigate the new general (α,β)-metric obtained. We also find the flag curvature and the projective factor for this new metric. A special attention was given to the computation of E-curvature of this new general (α,β)-metric. The flag curvature and the E-curvature for a general (α,β)-metric worth to be studied because of their importance, not only in Finsler spaces theory, but also in mathematical physics. In our future paper we will investigate the flag curvature and E-curvature of another (α,β)-metrics families.

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Author's address:

Laurian-Ioan Pişcoran
Department of Mathematics and Computer Science,
North University Center of Baia Mare, Technical University of Cluj Napoca,
Victoriei 76, 430122 Baia Mare, Romania.
E-mail: plaurian@yahoo.com

Cătălin Barbu
"Vasile Alecsandri" College, Bacău,
str. Vasile Alecsandri nr. 37, 600011 Bacău, Romania.
E-mail: kafka_mate@yahoo.com

Behzad Najafi
Department of Mathematics and Computer Sciences,
Amirkabir University, Tehran, Iran
E-mail: behzad.najafi@aut.ac.ir